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# EIGENSTRUCTURE ASSIGNMENT FOR REDUCED ORDER OBSERVER DESIGN

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## ABSTRACT

A reduced order state observer design procedure for multivariable linear time-invariant systems is developed. The procedure is based on the eigenstructure assignment approach. A compact form for the obsever gain matrix is derived according to any arbitrary prespecified set of eigenvalues and eigenvectors.

A set of free (design) parameter vectors is required upon which the eigenvector assignability is determined. In case of having observer dynamics of order equal to or less than the number of system outputs, complete assignability can be achieved.

Having such a reduced order observer in a feedback loop, the choice of the free parameters in order to reduce the impact of the initial mismatch between the system states and the corresponding estimates is derived. The approach developed is applied to minimize the effect of external gust on the lateral axis of L-1011 aircraft at cruise flight condition .

## I. INTRODUCTION.

The state observer is a dynamical system whose basic function is to give an estimate for the inaccessible states of any observable system [1]. having the states or their estimates, it allows, for example, for early detection of critical working conditions or implemetation of high quality feedback control (such as optimal, robust, pole assignment, ...etc). In case of insignificant noise level for the measurable states, Luenberger introduced the reduced order observer. The eigenstructure assigment approach is the most popular approach for designing such type of observers. Based on the separation principle [4], an observer based control can be designed assuming that the system states are available for measurement. The observer's eigenvalues should be more negative than those for the system under consideration to catch the system states as fast as possible. The mismatch between the states and their estimates is due to unknown system initial conditions. In order to alleviate the impact of the initial mismatch, an optimal choice of the observer eigenstructure has to be considered knowing that this structure is parametrized by the observer gain matrix L.

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## II. Problem Formulation

Consider the linear time-invariant, controllable, observable multivariable system described by:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$
$$y(t) = Cx(t) \quad (2)$$

where $x \in R^n$, $y \in R^m$, and $u \in R^r$. Assuming that the state vector $x(t)$ is accessible, a feedback gain matrix $K$ is already obtained to satisfy a prespecified design objective (such as pole assignment, optimal control,..., etc). In order to realize this control using the accessible output vector $y(t)$, an estimate of the state vector should be employed using a reduced order state obsever whose dynamics can be expressed by:

$$\dot{z}(t) = Dz(t) + Ey(t) + Gu(t) \quad (3)$$
$$x(t) = Py(t) + Qz(t) \quad (4)$$

where $z(t) \in R^{n-m}$, is the observer state and $\hat{x}(t) \in R^n$ is the asymptotic estimate of the inaccessible state $x(t)$. The matrices D, E, G, P, and Q are of compatible dimensions and to be specified. Without loss of generality, assume that:

$$C = [\ I_m\ ,\ 0_{n-m}\ ] \quad (5)$$

and decompose the system states into $x'(t) = [\ x'_1(t),\ x'_2(t)\ ]$ where the prime denotes transposition and $x_1(t) \in R^m$. Having equation (5), the observer matrices can be expressed as [4]:

$$D = A_{22} + LA_{12} \quad (6)$$

$$E = A_{21} + LA_{11} - A_{22}L - LA_{12}L \quad (7)$$

$$G = B_2 + LB_1 \quad (8)$$

$$P = \begin{bmatrix} I_m \\ -L \end{bmatrix} , \qquad Q = \begin{bmatrix} 0_{m,n-m} \\ I_{n-m} \end{bmatrix} \quad (9)$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} , \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad (10)$$

From the above partitioning, it is evident that the observer matrices are parametrized by only one gain matrix L of dimension (n-m)xm. The observer error, $\epsilon(t)$, is given by:

$$\epsilon(t) = z(t) - Tx(t)$$

$$= \exp(Dt)\,\epsilon(0) \qquad (11)$$

where:

$$T = [\,L,\ I_{n-m}\,] \quad \text{and} \quad \epsilon(0) = z(0) - x_2(0) - Ly(0) \qquad (12)$$

and $\epsilon(0)$ has resulted due to the mismatch between the system and observer initial states.

Having system (1), (2) and observer (3), (4), the overall augmented system can be described by:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} A + BK & -BK_2 \\ 0 & D \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} \qquad (13)$$

when:

$$u(t) = K\hat{x}(t) \qquad (14)$$

Using the modal decomposition technique [2], the solution of (13) can be expressed by:

$$\begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} = \sum_{i=1}^{n} p_i \exp(s_i t)\bar{p}_i x(0) + \sum_{j=1}^{n-m} q_j \exp(\mu_j t)\bar{q}_j \epsilon(0) \qquad (15)$$

where;

$$p_i = \begin{bmatrix} w_i \\ 0 \end{bmatrix} \qquad \bar{p}_i = \begin{bmatrix} \bar{w}_i \\ 0 \end{bmatrix} \qquad q_i = \begin{bmatrix} G(\mu_j)v_j \\ v_j \end{bmatrix} \quad \bar{q}_i = \begin{bmatrix} G(\mu_j)\bar{v}_j \\ \bar{v}_j \end{bmatrix}$$

and $w_i$, $v_j$ are the right and $\bar{w}_i$, $\bar{v}_j$ are the left eigenvectors, $\{s_i\}$ and $\{\mu_j\}$ are the eigenvalues of $A_c = A + BK$ and D, respectively and $G(\mu_j) = (\mu_j I - A)^{-1}$.

Expression (15) reflects (in a feedback loop) the effect of the observer error $\epsilon(0)$ on the behavior of the state $x(t)$ by a factor depending on the chosen set of $\{\mu_j\}$, $\{v_j\}$ and $\{v_j\}$ of the observer matrix D. In order to minimize the impact of the initial mismatch, the above sets should be chosen in an optimal sense.

The problem, at hand, to be treated is, what would be the best choice of the observer gain matrix L such that the impact of any initial mismatch could be alleviated?. The paper is arranged as follows, in section III, an explicit form for L as a function of the design parameters is derived and in section IV, an objective function for minimization is defined. Using gradient techniques, an optimal set for the free parameters is derived and upon which the best set of observer matrices are deduced.

## III. Observer Design

The design technique depends on the approach derived in [5] where the observer characteristic equation is given by:

$$| s_i I_{n-m} - D | = 0 \qquad (17)$$

where |.| denotes the determinant of a matrix. Using equation (6) then:

$$f'_i A_{12} \emptyset(s_i) L = f'_i \qquad (18)$$

for some m-dimensional vector $f_i$ and $\emptyset(s_i) =: (s_i I - A_{22})^{-1}$. Collecting (n-m) vector equations, for i=1,2,...,n-m, equation (18) produces the form:

$$VL = F \qquad (19)$$

in which:

$$V = \begin{bmatrix} v'_1 \\ v'_2 \\ . \\ . \\ v'_{n-m} \end{bmatrix} = \begin{bmatrix} f'_1 A_{12} \emptyset(s_1) \\ f'_2 A_{12} \emptyset(s_2) \\ . \\ . \\ f'_{n-m} A_{12} \emptyset(s_{n-m}) \end{bmatrix}, \quad F = \begin{bmatrix} f'_1 \\ f'_2 \\ . \\ . \\ f'_{n-m} \end{bmatrix}$$

and $v_i$, i=1,...,n-m are the right eigenvectors associated with the observer matrix D. Since V is the modal matrix of the matrix D, then:

$$L = V^{-1} F \qquad (20)$$

Obviously, L is an explicit function of both the n-m observer eigenvalues $s_1, s_2, \ldots, s_{n-m}$ and of the free (n-m) parameter vectors $f_1, f_2, \ldots, f_{n-m}$. Consequently, they in common represent the free parameters in observer design and have to be chosen according to the requirements imposed on the observer. The only restrictions on these parameters are that $s_i$ should be at most stable, self-conjugate and $v_i$ in addition be linearly independent.

From equation (19), the assignability of a given eigenvector set is determined by the relative value of m and n-m according to the set of equations:

$$v'_i = f'_i A_{12} \emptyset(s_i) \qquad (21)$$

where $f_i \in R^m$ and $v_i \in R^{n-m}$. In case of having $m \geq n-m$, complete assignability can be realized. However, in case of

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$m < n-m$ ( similar to the control problem ) an approximate assignability can be achieved in a minimum norm sense [6].

## V. CHOICE OF FREE PARAMETERS:

In order to minimize the impact of the initial mismatch and to guarantee the boundedness of the observer matrices, an objective function of the form;

$$J = \delta_1 J_1 + \delta_2 J_2 \tag{22}$$

will be adopted with

$$J_1 = \tfrac{1}{2} \int_0^\infty \epsilon'(t) Q \epsilon(t)\, dt \quad , \quad Q \geq 0 \tag{23}$$

and

$$J_2 = \tfrac{1}{2} \sum_{i,j} L^2_{ij}\, q_{ij} \quad , \quad q_{ij} \geq 0 \tag{24}$$

$$\text{for} \quad \delta_1,\ \delta_2 \geq 0 \tag{25}$$

For asymptotic stable observer, $\epsilon(t)$ tends to zero as the time tends to infinity. This results in [7];

$$J_1 = \tfrac{1}{2}\, \epsilon'(0) P \epsilon(0) \tag{26}$$

where P is the solution of the Lyapunov equation;

$$D'P + PD + Q = 0 \tag{27}$$

From equation (6) and (27), the index J explicitly depends on the observer matrix L and consequently on the free parameters $\{ s_i \}$ and $\{ f_i \}$ i.e.

$$J = J( s_1, \ldots, s_{n-m}; f_1, \ldots, f_{n-m}) \tag{28}$$

To minimize that index, the best set of free parameters has to be defined. To search for, a gradient technique is to be considered as follows;

a) Gradients of $J_1$:-

From equations (26) and (27) we get;

$$\delta J_1/\delta s_r = \epsilon'(0)[\delta P/\delta s_r]\epsilon(0) \tag{29}$$

and

$$D'\delta P/\delta s_r + \delta P/\delta s_r D + Q = 0 \tag{30}$$

where;

$$Q = [\ \delta D/\delta s_r\ ]'P + P[\ \delta D/\delta s_r\ ] \tag{31}$$

From (6), we get

$$\delta D/\delta s_r = [\delta L/\delta s_r]\ A_{12} \tag{32}$$

and

$$Q = A'_{12}[\delta L'/\delta s_r]P + P[\delta L/\delta s_r]A_{12} \tag{33}$$

It follows that

$$\delta J_1/\delta s_r = \tfrac{1}{2} \int_0^\infty \epsilon'(t)Q\epsilon(t)\ dt$$

$$= \tfrac{1}{2}\ tr[\ \int_0^\infty \epsilon(t)\epsilon'(t)dt\ Q] \tag{34}$$

with $\epsilon(t) = \exp(Dt)\epsilon(0)$, then

$$\delta J_1/\delta s_r = \tfrac{1}{2}\ tr[\ \int_0^\infty \exp(Dt)\epsilon(0)\epsilon'(0)\exp(D't)dt\ Q]$$

$$= \tfrac{1}{2}\ tr[\ RQ\ ] \tag{35}$$

where R is the solution of the Lyapunov equation;

$$D'R + RD + \epsilon(0)\epsilon'(0) = 0 \tag{36}$$

Substituting for Q using (33) in (35), this leads to;

$$\delta J_1/\delta s_r = tr\ [\ RP\ (\delta L/\delta s_r)\ A_{12}\ ] \tag{37}$$

From (20) and (21), we get;

$$\delta L/\delta s_r = \delta V^{-1}/\delta s_r\ F$$

$$= -\ V^{-1}[\delta V/\delta s_r]V^{-1}F$$

$$= -\ V^{-1}[\delta V/\delta s_r]L \tag{38}$$

and

$$\delta V/\delta s_r = \delta v'_r/\delta s_r\ e_r = v'_r(s_r I - A_{22})^{-1}\ e_r \tag{39}$$

where $e'_r =: (0,\ldots,0,1,0,\ldots)$
....... r

Therefore;

$$\delta J_1/\delta s_r = \delta L/\delta s_r A_{12} RP$$

$$= -V^{-1}[\delta V/\delta s_r]LA_{12}RP$$

$$= -\delta V/\delta s_r L A_{12} R P V^{-1}$$

$$= \{-v'_r (s_r I - A_{22})^{-1} L\}\ A_{12} R P V^{-1}\ e_r \qquad (40)$$

where R is derived from (36) and P is derived from (27). In the same way, we can get;

$$\delta J_1/\delta f_r = \{\ I - A_{12}(s_r I - A_{22})^{-1} L\ \}\ A_{12} R P V^{-1}\ e_r \qquad (41)$$

b) Gradients of J2:-

Taking the partial derivatives of $J_2$ with respect to $s_r$, we get;

$$\delta J_2/\delta s_r = \delta/\delta s_r \sum_i \sum_j L^2_{ij}\ q_{ij}$$

$$= \sum_i \sum_j q_{ij}\ L_{ij}\ \delta L_{ij}/\delta s_r$$

$$= \sum_i \sum_j q_{ij}\ L_{ij}\ e'_i\ \delta L/\delta s_r\ e_j$$

$$= \{\ -\ v'_r (s_r I - A_{22})^{-1} L\}[\ \Sigma\ \Sigma\ q\ \ L\ \ e\ e'\ ]$$

$$= \{\ -\ v'_r (s_r I - A_{22})^{-1} L\}\ G_2 V^{-1} e_r \qquad (42)$$

In the same way, we get;

$$\delta J_2/\delta f_r = [\ I - A_{12}(s_r I - A_{22})^{-1} L\ ]\ G_2 V^{-1} e_r \qquad (43)$$

c) Gradients of J:-

Using the results of a) and b) we get;

$$\delta J/\delta s_r = \{\ -v'_r (s_r I - A_{22})^{-1} L\ \}\ G V^{-1}\ e_r \qquad (44)$$

$$\delta J/\delta f_r = \{\ I - A_{12}(s_r I - A_{22})^{-1} L\ \}\ G V^{-1}\ e_r \qquad (45)$$

where $r \in \{\ 1, \ldots, n-m\ \}$ and;

$$G = \delta_1\ A_{12} R P + \delta_2\ G_2 \qquad (46)$$

Therefore numerical search techniques can be adopted for minimization.

## VI NUMERICAL EXAMPLE

The following example [3] is used to demonstrate the effectiveness of the proposed approach. We consider the lateral axis equations of the rigid body model of the L-1011 aircraft at cruise flight condition. The state vector for this problem is given as;

$$x = \begin{bmatrix} \phi \\ p \\ r \\ \beta \end{bmatrix} \quad \begin{matrix} \text{- bank angle} \\ \text{- roll rate} \\ \text{- yaw rate} \\ \text{- slideslip angle} \end{matrix}$$

The system matrices are;

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.89 & 0.39 & -5.555 \\ 0 & -0.034 & -2.98 & 2.43 \\ 0.035 & -0.0011 & -0.99 & -0.21 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0.76 & -1.6 \\ -0.95 & -0.032 \\ 0.03 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The input vector u is given by

$$u = \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} \quad \begin{matrix} \text{- rudder deflection} \\ \text{- aileron deflection} \end{matrix}$$

From our definition of state, the output variables are the bank angle and the roll rate. A feedback gain matrix K is derived for pole assignment assuming state accessibility. The value of this gain matrix is;

$$K = \begin{bmatrix} 0 & 0 & -0.689 & 4.56 \\ -13.1 & -3.13 & 0 & 0 \end{bmatrix}$$

Since only p and $\phi$ are measured, we employ a reduced order observer to obtain asymptotically estimates of r, $\beta$. For the example, a starting value for each of the two eigenvalues is taken as;

$$\mu_1 = -3.0$$
$$\mu_2 = -3.5$$

To reflect gust perturbations, the initial state vector x(0) is given by $x'(0) = (0, 0, 0, 1)$. Assuming initial eigenvectors of the form;

$$V = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Then the observer gain would be of the form;

$$L = \begin{bmatrix} 1 & 0.435 \\ 0 & 0.608 \end{bmatrix}$$

and the free parameter matrix is;

$$F = \begin{bmatrix} 1 & 0.435 \\ 1 & 1.043 \end{bmatrix}$$

with;

$$J_1 = 0.5 \qquad\qquad J_2 = 1.559$$

Assuming that $q_{i,j} = 1$ and $Q = \text{diag}(1, 1)$

Starting with these values, the numerical minimzation yields the optimal gain matrix;

$$D = \begin{bmatrix} -3.377 & 8.086 \\ -0.643 & -5.153 \end{bmatrix}$$

at

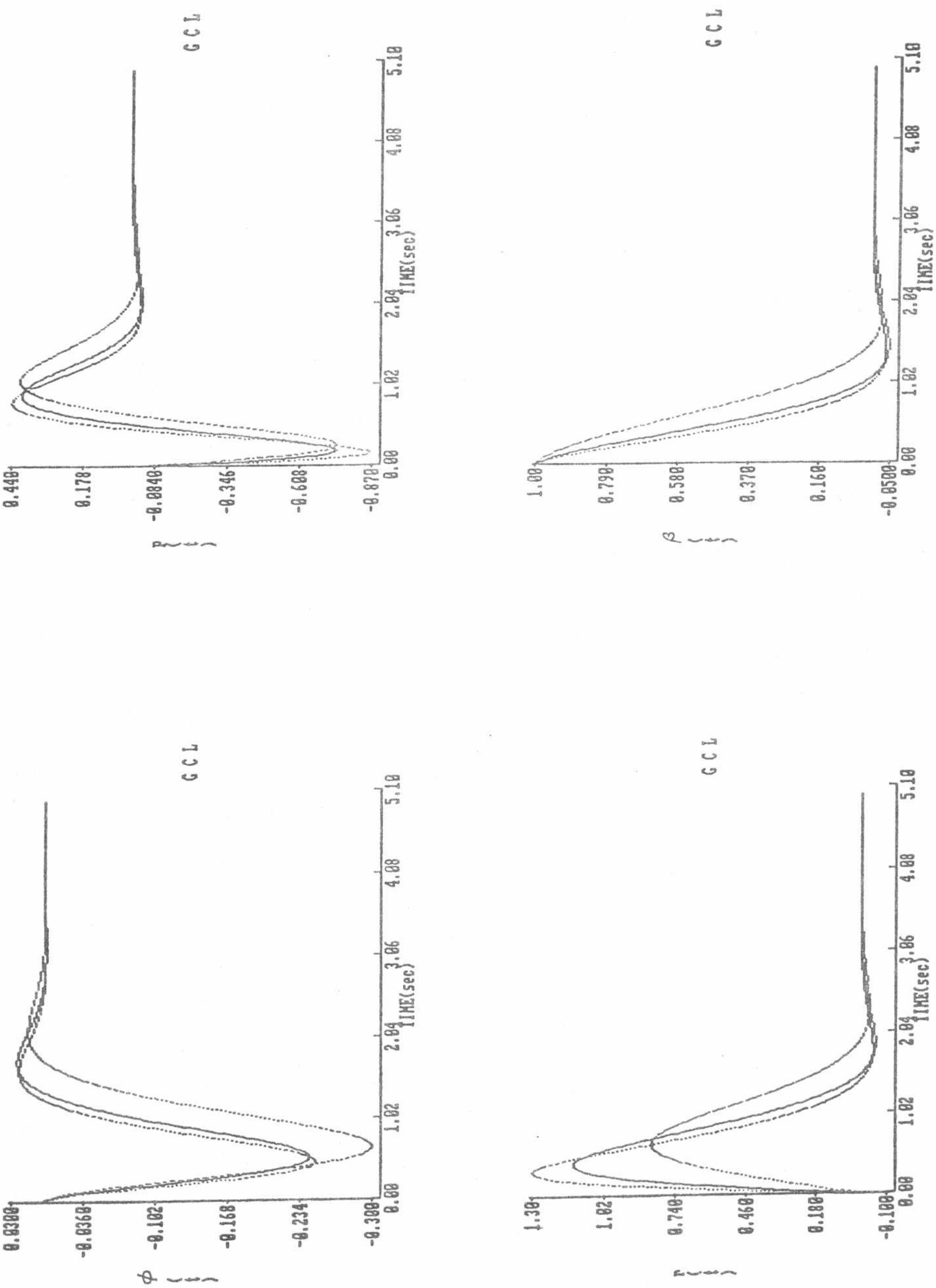
$$J_1 = 7.615643E-02 \qquad\qquad J_2 = 0.822706$$

and the eigenvalues are $\mu_1 = -5.012993$, $\mu_2 = -5.327345$

The time responses of the system states in a feedback with and without observer are shown in Fig.1. The observer-based control employed once the full-order observer given in [3] and the proposed optimal reduced-order once more. From Figures, it can be concluded that the reduced order optimal observer is better in following the system states than the full-order one.

## VI CONCLUSION

A new method for designing an optimal reduced-order observer based on a set of eigenvectors and eigenvalues has been suggested. The approach shows the superiority of the reduced order observer over the full order one suggested in [3] in two aspects either the alleviation of the impact of the initial mismatch on the behavior of the system states or the reduction of implementation cost by having less instrumentations.


G C L
TIME(sec)
0.00
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3.06
4.08
5.10
G C L
TIME(sec)
G C L
TIME(sec)
G C L
TIME(sec)

FIG. 1

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