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A Review on Some New Bivariate and Univariate Discrete Distributions

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ABSTRACT **:A three-parameter new discrete analog of Alpha-power Weibull distribution (DAPW) is presented. Some of its fundamental distributional and reliability properties are established. Two datasets are used showing the flexibility of the proposed model. An attempt to introduce a new lifetime model as a discrete version of the continuous exponentiated exponential distribution which is called discrete exponentiated exponential distribution (DEE). a discrete analog of the continuous generalized inverted exponential distribution denoted by discrete generalized inverted exponential distribution (DGIE). The model with two real data sets is also examined. In addition, the developed DGIE is applied as color image segmentation which aims to cluster the pixels into their groups. To evaluate the performance of DGIE, a set of six color images is used, as well as it is compared with other image segmentation methods including Gaussian mixture model (GMM), K-means, and Fuzzy subspace clustering (FSC). The DGIE provides high performance than other competitive methods.**

KEYWORDS: Failure rate function; Order Statistics; Maximum likelihood estimator; characteristic function.

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1. INTRODUCTION

Many models have been developed to describe lifetime data using continuous lifetime distributionsin many disciplines of life testing experiment and reliability analysis, for example, Kapur and Lamberson (1977), Lawless (2003), Sinha (1986), Gnedenko and Ushakou (1995). Estimating the life length of a device using continuous distributions poses derivational challenges as closed forms for integration may not exist. There is a lot of technology is used to determine how long someone lives. Even more so for a continuous procedure including a continuous measurement of longevity, a discrete model, with recordings made at periodic time points, may be more appropriate.We propose a discrete Alpha-power Weibull (DAPW) with the same reliance as its continuous cousin and the same data-analysis features.

The presented distribution discrete generalized inverted exponential (DGIE) is constructed from a generalized inverted exponential distribution. Parameters are estimated using two methods, namely moments and maximum likelihood. The consistency of the estimated parameters is illustrated using simulation. Based on two data sets the proposed distribution is more convenient to analyze the given data than competitive distributions. The proposed distribution is applied in color segmentation which helps in clustering the pixels into their groups. The DGIE provides higher performance than other competitive methods.

The main contribution of the current study can be summarized as follows:

1. Present a new distribution discrete generalized inverted exponential (DGIE) to avoid the limitations of other distributions.

2. Compute the basic distributional properties, moments, probability function, reliability indices, characteristic function, and the order statistics of DGIE.
3. Evaluate the applicability of DGIE by using it to improve the color segmentation.

2. Discrete alpha-power Weibull distribution:

Roy (1993) introduced a discretization method using the reliability function of the model concerned

$$P(X = x) = S(x) - S(x + 1) \quad \text{when } x = 0,1,2, \dots \dots \qquad (2.1)$$

Roy (1993) applied this method for discretizing Geometric distribution, S(x) being the survival function of the Exponential random variable.

The discrete Alpha-Power Weibull distribution may be defined as a non-negative integer-valued distribution with PMF, p(x), using the method of discretizing, as

$$p(x) = \frac{\alpha}{\alpha - 1}\left[\left(1 - \alpha^{-\theta^{x^{\beta}}}\right) - \left(1 - \alpha^{-\theta^{(x+1)^{\beta}}}\right)\right] \quad x = 0,1,2, \dots \qquad (2.2)$$

Where$\theta = e^{-\lambda}$, $0 < \theta < 1$, $\alpha, \beta > 0$ and $\alpha \neq 1$. We denote this distribution as DAPW (α, θ, β).

2.Results and discussion

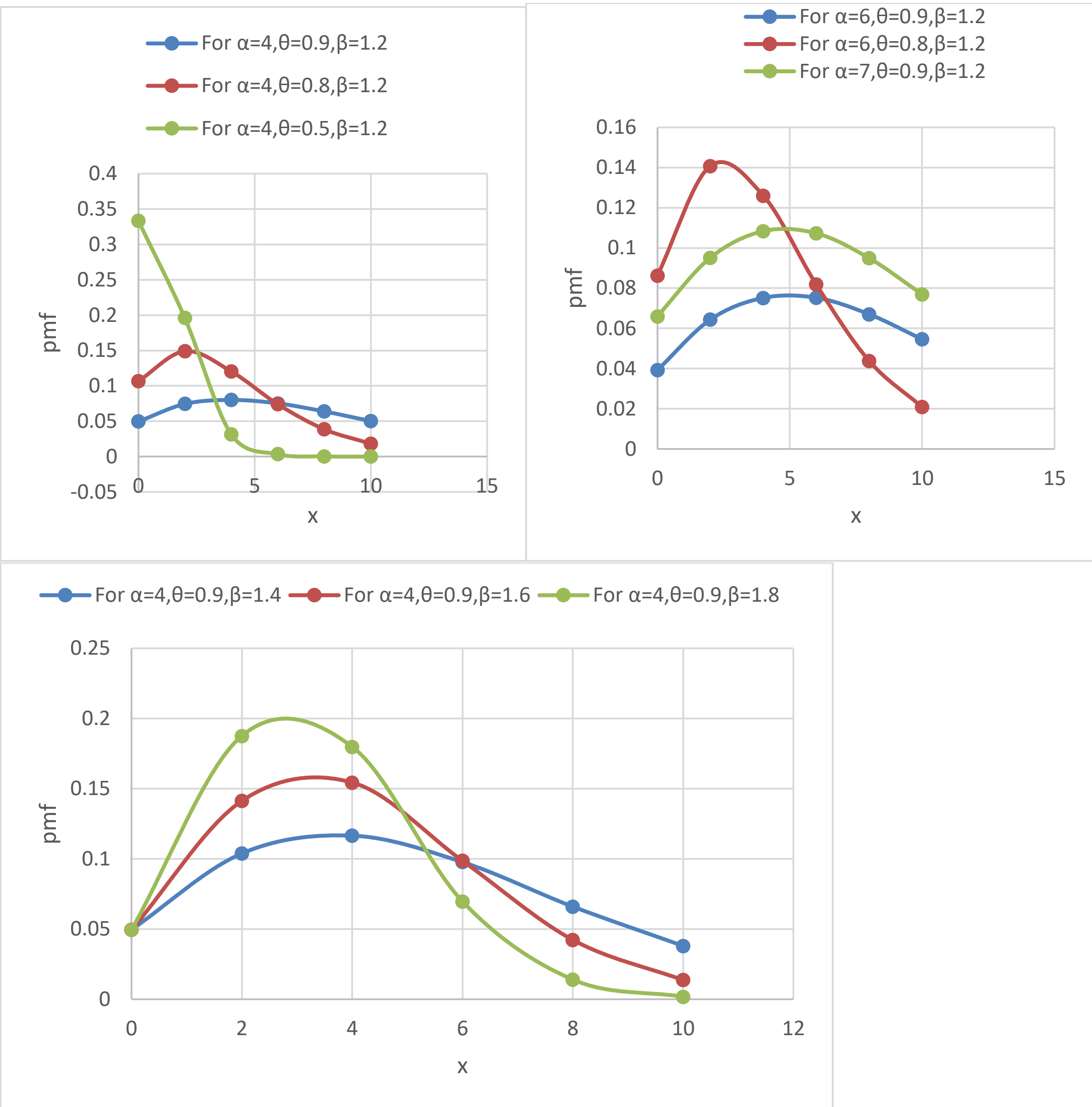


Figure 2.1. The PMF plots of DAPW (α, θ, β) for different values of (α, θ, β).

Table 2.2 The averages bias and averages MSE in parenthesis for simulated results of ML estimates.

(α, θ, β)	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
n=10			
(0.5, 0.2, 1)	-0.477(0.694)	-0.266 (0.127)	0.354 (1.839)
(0.5, 0.1, 1.5)	-0.387 (0.173)	-0.095 (0.012)	0.134 (0.783)
(0.5, 0.2, 2)	-0.437 (0.207)	-0.25 (0.073)	-1.273 (2.938)
n=15			
(0.5, 0.2, 1)	-0.494 (0.263)	-0.25 (0.071)	0.13 (0.914)
(0.5, 0.1, 1.5)	-0.401 (0.18)	-0.098 (0.012)	0.076 (0.616)
(0.5, 0.2, 2)	-0.447 (0.215)	-0.257 (0.076)	-1.248 (2.837)
n=30			
(0.5, 0.2, 1)	-0.501 (0.256)	-0.227 (0.055)	-0.118 (0.228)
(0.5, 0.1, 1.5)	-0.394 (0.166)	-0.092 (0.009246)	-0.162 (0.2)
(0.5, 0.2, 2)	-0.446 (0.217)	-0.274 (0.089)	-1.192 (2.74)
n=50			
(0.5, 0.2, 1)	-0.497 (0.249)	-0.224 (0.052)	-0.17 (0.103)
(0.5, 0.1, 1.5)	-0.397 (0.162)	-0.091 (0.00847)	-0.264 (0.14)
(0.5, 0.2, 2)	-0.439 (0.24)	-0.286 (0.102)	-1.257 (2.667)

The empirical results are given in Table 2.2
From Table 2, the following observations can be noted:

- The magnitude of the bias always decreases to zero as $n \longrightarrow \infty$.
- The MSEs always decrease to zero as $n \longrightarrow \infty$. This shows the consistency of the estimators.

Data application

- Here, we illustrate the superiority of the discrete alpha-power Weibull distribution over traditional distributions (Poisson and Geometric) beside new models (discrete Gamma, discrete Weibull, Discrete Logistic, and Discrete Lindley).
- The data set given in table 2.3 consists of the 2003 final examination marks of 48 slow space students in mathematics in the Indian Institute of Technology at Kanpur. The data set is taken from Gupta and Kundu (2009).

Table 2.3. Data set 1.

29	25	50	15	13	27
15	18	7	7	8	19
12	18	5	21	15	86
21	15	14	39	15	14
70	44	6	23	58	19
50	23	11	6	34	18
28	34	12	37	4	60
20	23	40	65	19	31

The MLE, of α, μ, β, and λ values in all these cases have been computed. The Kolmogorov-Smirnov (K-S) distance between the empirical cumulative distribution function and the fitted distribution function in each case and the associated P-value are computed. The result is reported in table 2.4.

Table 2.4. Fitted estimates for data set 1.

Distribution	p(x)	Parameter Estimates	p − value	K-S statistics
Discrete Alpha-power Weibull Distribution	(2.2)	$\alpha = 4.677, \beta = 1.248, \theta = .978$	$.339044$	$.133417096$
Poisson	$\lambda^x e^{-\lambda}/x!$	$\lambda = 25.8958$	2.4013×10^{-7}	$.3998$
Geometric	$p(1-p)^x$	$p = .0372$	$.0145$	$.2223$
Discrete Weibull	$q^{x^\beta} - q^{(x+1)^\beta}$	$q = .6488, \beta = .6758$	2.9221×10^{-24}	$.7419$
Discrete Gamma	$\frac{\gamma(\alpha, \beta(x+1))}{\Gamma(\alpha)} - \frac{\gamma(\alpha, \beta(x))}{\Gamma(\alpha)}$	$\alpha = .8098, \beta = .0350$	2.6082×10^{-4}	$.2993$

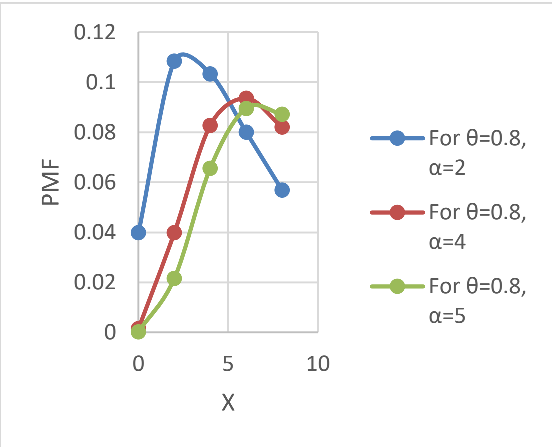
3. Discrete Exponentiated Exponential Distribution

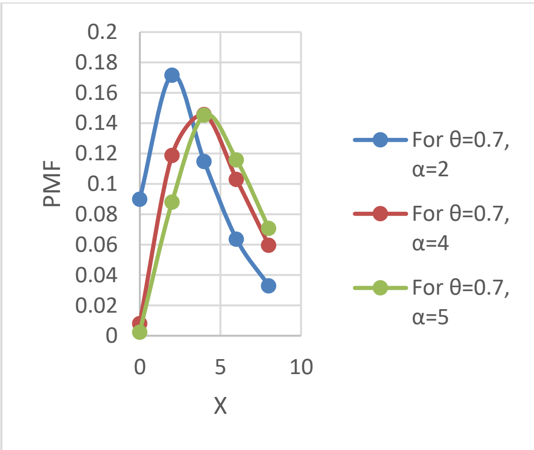
Definition. A random variable X is said to have a discrete exponentiated exponential distribution with parameter α ($\alpha > \mathbf{0}$) and $\boldsymbol{\theta} = \boldsymbol{e}^{-\boldsymbol{\lambda}}$, $\mathbf{0} < \boldsymbol{\theta} < \mathbf{1}$, if its pmf has the form:

$$\boldsymbol{P}(\boldsymbol{X} = \boldsymbol{x}) = \left(\mathbf{1} - \boldsymbol{\theta}^{(\boldsymbol{x}+\mathbf{1})}\right)^{\boldsymbol{\alpha}} - (\mathbf{1} - \boldsymbol{\theta}^{\boldsymbol{x}})^{\boldsymbol{\alpha}}; \quad \boldsymbol{x} \in \mathbb{N}_{\mathbf{0}}. \qquad (3.1)$$

We denote this distribution as $\boldsymbol{DEE}(\boldsymbol{\alpha}, \boldsymbol{\theta})$.

It is observed that at α=1, we have the geometric distribution as a special case.





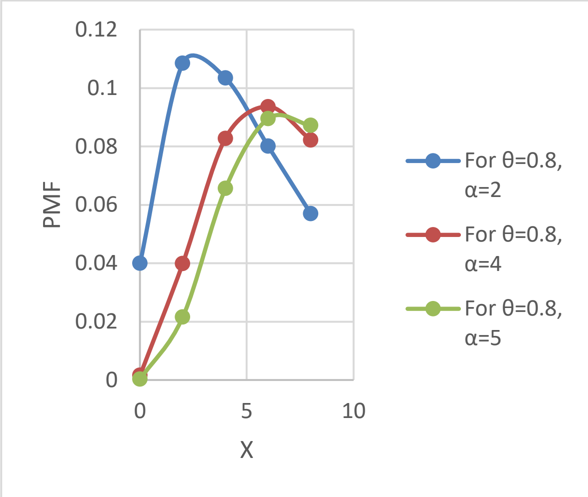


Figure 3.1 illustrates several examples of the probability mass function of DEE (α, θ) distribution for different values of α and θ.

Table 3.1 The averages bias and averages MSE for simulated results of ML estimates

Sample Size	α=2.73					
	θ=.93		θ=.95		θ=.99	
	Bias	MSE	Bias	MSE	Bias	MSE
10	$5.762.10^{-5}$	$3.32.10^{-9}$	$-1.285.10^{-4}$	$1.651.10^{-8}$	$-8.95.10^{-5}$	$-8.01.10^{-9}$
70	$-8.37.10^{-4}$	$7.005.10^{-7}$	$-8.287.10^{-4}$	$6.867.10^{-7}$	$-2.588.10^{-4}$	$6.698.10^{-8}$
130	$-4.019.10^{-3}$	$1.615.10^{-5}$	$-3.129.10^{-3}$	$9.788.10^{-6}$	$-7.456.10^{-4}$	$5.559.10^{-7}$
210	$-4.245.10^{-3}$	$1.802.10^{-5}$	$-3.313.10^{-3}$	$1.098.10^{-5}$	$-7.815.10^{-4}$	$6.108.10^{-7}$
Sample Size	θ=.93					
	α=2.69		α=2.7		α=2.73	
	Bias	MSE	Bias	MSE	Bias	MSE
10	-0.39	0.152	-0.378	0.143	$5.762.10^{-5}$	$3.32.10^{-9}$
70	0.078	$6.011.10^{-3}$	0.055	3.10^{-3}	$-8.37.10^{-4}$	$7.005.10^{-7}$
130	0.279	0.078	0.234	0.055	$-4.019.10^{-3}$	$1.615.10^{-5}$
210	0.015	$2.258.10^{-4}$	$-6.965.10^{-3}$	$4.851.10^{-5}$	$-4.245.10^{-3}$	$1.802.10^{-5}$

The empirical results are given in Table 3.1

From Table 3.1, the following observations can be noted:

- The magnitude of the bias always decreases to zero as $n \longrightarrow \infty$.
- The MSEs always decrease to zero as $n \longrightarrow \infty$. This shows the consistency of the estimators.

Data application

Here, we illustrate the superiority of a discrete exponentiated exponential distribution over traditional distributions (Poisson and Geometric) beside new models (discrete Gamma, discrete Weibull, Discrete Logistic, and Discrete Lindley).

We use three real data sets. The first data are given in Table3.4 consists of survival times in days of 72 guinea pigs. These data are taken from table 6 in (1960). The data have been analyzed by Alshunnar et al. (2010) and Ghitany et al. (2011). The data are discrete by definition.

Table 3.2 Data set 1.

12	15	22	24	24	32	32	33	34	38	38	43	44	48
52	53	54	54	55	56	57	58	58	59	60	60	60	60
61	62	63	65	65	67	68	70	70	72	73	75	76	76
81	83	84	85	87	91	95	96	98	99	109	110	121	127
129	131	143	146	146	175	175	211	233	258	258	263	297	341
341	376												

The MLE of (α, θ) values in all these cases has been computed. The Kolmogorov-Smirnov (K-S) distance between the empirical cumulative distribution function and the fitted distribution function in each case and the associated P-value are computed. The result is reported in table 3.3.

Table 3.3 Fitted estimates for data set 1.

Distribution	p(x)	Parameter Estimates	p − value	K-S statistics
Discrete Exponentiated Exponential Distribution	(3)	$\alpha = 2.739, \theta = .983$	$.0524$	$.1567$
Poisson	$\lambda^x e^{-\lambda}/x!$	$\lambda = 99.8194$	9.5313×10^{-22}	$.5755$
Geometric	$p(1-p)^x$	$p = .0099$	$.0020$	$.2160$
Discrete Weibull	$q^{x^\beta} - q^{(x+1)^\beta}$	$q = .9532, \beta = .9020$	1.0821×10^{-24}	$.6140$
Discrete Gamma	$\frac{\gamma(\alpha, \beta(x+1))}{\Gamma(\alpha)} - \frac{\gamma(\alpha, \beta(x))}{\Gamma(\alpha)}$	$\alpha = .9853, \beta = .0125$	4.1283×10^{-6}	$.2966$

4. Discrete Generalized Inverted Exponential distribution

Definition. A random variable X is said to have a discrete generalized inverted exponential distribution with parameter β (β> 0) and $\theta = e^{-\lambda}$, $0 < \theta < 1$, if its probability mass function (pmf) has the form:

$$P(X = x) = \left(1 - \theta^{\frac{1}{x}}\right)^{\beta} - \left(1 - \theta^{\frac{1}{(x+1)}}\right)^{\beta} ; \quad x \in \mathbb{N}_0$$

Results and discussion

The comparison between the developed color image segmentation method (i.e., DGIEMM) and the other methods is given in Table 11. It can be noticed from these results the high ability of the developed method to cluster the images into their objects overall the other methods. For example, according to the results in terms of accuracy, it can be seen from these values that the DGIEMM has a high ability to assign each pixel into its true label (i.e., the object that contains it). The FSC and GMM provide results better than K-means, and this observation can be noticed from Figure 7 (a) which shows the average overall of the tested six images.

In terms of AR, it can be seen that the DGIEMM still provides results better than other methods. The same observations are noticed in the other three measures (i.e., RI, NMI, and Hubert); also, Figure 7 (c)-(E) shows the superiority of DGIEMM.

To justify the superiority of DGIEMM, the non-parametric Friedman test is used. In general, this test is applied to make a decision about the difference between the DGIEMM and other methods is significant or not. There are two hypotheses; the first one is named null, and it is assumed that there is no difference between the tested methods. In contrast, the second hypothesis, called alternative, is considered there is a difference between the method. We accept the alternative hypothesis when the obtained value is less than significant level 0.05.

5. Conclusions

- **Conclusions for Discrete alpha-power Weibull distribution**

A new discrete distribution, the Alpha-power Weibull distribution, is presented. After obtaining it, the probabilistic properties of the parameter are studied and its parameters are estimated. The reliability function was also studied using the stress-strength model based on the new distribution, and during the review of the steps, we studied the statistics arranged for the APW distribution. It was also applied to the stress-strength model using data produced from a simulation model by the Monte-Carlo method. Then we applied the proposed distribution to two sets of real data and found that the distribution would be a strong competitor to known discrete distributions. It can be used in applications for materials that the environment such as coal or in general, it can be used to study natural disasters, and the univariate case has been presented. It is more important to show how it can be generalized to the multivariate case. More work is required in this area in the future.

- **Conclusions for Discrete Exponentiated Exponential Distribution**

A new two-parameter for lifetime modeling which is organized from continuous exponentiated exponential distribution is introduced, so-called discrete exponentiated exponential distribution $\boldsymbol{DEE(\alpha,\theta)}$ distribution. The proposed distribution contains the geometric distribution as a special case. The failure rate of the new model is decreasing. Some important probabilistic properties of this distribution are studied in detail. The unknown parameters of the DEE distribution are estimated using two methods, namely, the moment's method and the maximum likelihood method. The flexibility of the DEE distribution has been empirically proven by using three real-life data sets. The DEE distribution has proven to show efficiency in fitting data better than some existing distributions. Finally, we believe that the presented distribution will benefit a wide range of applications including reliability, physics, and so on.

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