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On Properties of Certain Subclasses Harmonic Functions Defined by Using the Quantum Derivative

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ABSTRACT **: By using the q-difference Operator we investigate some properties of certain subclasses of harmonic functions defined by using the quantum calculus. We obtain the coefficient estimates and partial sums inequalities of these subclasses. By using the q-difference Operator we investigate some properties of certain subclasses of harmonic functions defined by using the quantum calculus. We obtain the coefficient estimates and partial sums inequalities of these subclasses. By using the q-difference Operator we investigate some properties of certain subclasses of harmonic functions defined by using the quantum calculus. We obtain the coefficient estimates and partial sums inequalities of these subclasses. By using the q-difference Operator we investigate some properties of certain subclasses of harmonic functions defined by using the quantum calculus. We obtain the coefficient estimates and partial sums inequalities of these subclasses. By using the q-difference Operator we investigate some properties of certain subclasses of harmonic functions defined by using the quantum calculus. We obtain the coefficient estimates and partial sums inequalities of these subclasses**.

KEYWORDS: **analytic; univalent; starlike; convex; harmonic; q-difference. 2020 Mathematics Subject Classification: 30C45; 30D30**

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I. INTRODUCTION

Let $\mathcal{H}$ refers to the class of complex-valued harmonic functions on the unit disc $U := \{z: |z| < 1\}$, then $f \in \mathcal{H}$ if $f = h + \bar{g}$, where h, g are functions analytic in U. Let $\mathcal{H}_0$ be the class of functions $f \in \mathcal{H}$ with the following normalization:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ \ (a_k \in \mathbb{C}), \tag{1}$$

and let $S_{\mathcal{H}}$ denote the class of functions $f \in \mathcal{H}_0$, which are orientation preserving and univalent in U.

For functions $f_1, f_2 \in \mathcal{H}$ of the form:

$$f_n(z) = z - \sum_{k=0}^{\infty} a_{n,k}\, z^k + \sum_{k=1}^{\infty} b_{n,k}\, z^k \ \ (z \in U, n \in \{1,2\}), \tag{2}$$

by $f_1 * f_2 \in \mathcal{H}$ we denote the Hadamard product or convolution of f_1 and f_2, defined by:

$$(f_1 * f_2)(z) = \sum_{k=0}^{\infty} a_{1,k} a_{2,k} z^k + \sum_{k=1}^{\infty} \overline{b_{1,k}, b_{2,k} z^k} \ \ (z \in U).$$

We say that a function $f: U \to C$ is subordinate to a function $F: U \to C$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if there exists a complex-valued function ω which maps U into oneself with $\omega(0) = 0$, such that $f = F \circ \omega$. In particular, if F is univalent in U, we have the following equivalence:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{and} f(U) \subset F(U)$$

In 1956 Sakaguchi [1] introduced the class S^{**} of analytic univalent functions in U which are starlike with respect to symmetrical points. An analytic function f is said to be starlike with respect to symmetric points if :

$$Re\{\frac{zf'(z)}{f(z)}\} > 0 \ \ (z \in U) \tag{3}$$

If $f \in S^{**}$ then the angular velocity of $f(z)$ about the point $f(-z)$ is positive as z traverses the circle $|z| = r$ in a positive direction.

Let A and B be two distinct complex parameters and let $0 \leq \alpha < 1$. In [2] (see also [3]) it is defined the class $S^{*}_{\mathcal{H}}(A, B)$ of Janowski harmonic starlike functions$f \in S_{\mathcal{H}}$such that:

$$\frac{D_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \tag{4}$$

where,

$$D_H f(z := zh'(z) - \overline{zg'(z)}(Z \in U)$$

The classes $SH^{*}(\alpha) := S^{**}_{\mathcal{H}}(2\alpha - 1,1)$ and $S^{c}_{\mathcal{H}}(\alpha) := S^{c}_{\mathcal{H}}(2\alpha - 1,1)$ are studied by Jahangiri [4] (see also [5]). In particular, Cho and Dziok [6] obtained the classes $S^{c}_{\mathcal{H}} := S^{c}_{\mathcal{H}}(0)$ and $S^{*}_{\mathcal{H}}$ of function $f \in S_{\mathcal{H}}$ which are convex in $U(r)$ or starlike in $U(r)$, respectively, for any $r \in (0,1]$. Motivated by Sakaguchi [1], Cho and Dziok [6] defined the class $S^{**}_{\mathcal{H}}(A, B)$ of function $f \in \mathcal{H}_0$ such that:

$$\frac{2D_{\mathcal{H}} f(z)}{f(z) - f(-z)} \prec \frac{1+Az}{1+Bz}, \tag{5}$$

In particular, the class $SH^{*}(\alpha) := S^{**}_{\mathcal{H}}(2\alpha - 1,1)$ was introduced by Ahuja and Jahangiri [7] (see also [8, 9]). The class $HS^{*}_{S}(b, \alpha) := S^{**}_{\mathcal{H}}(2(\alpha - 1) + 1,1)$ was investigated by Janteng and Halim [10].

Let $0 < q < 1$ we define the class $S^{**}_{\mathcal{H}}(A, B, q)$ of function $f \in \mathcal{H}_0$ such that:

$$\frac{2D_{\mathcal{H},q} f(z)}{f(z) - f(-z)} \prec \frac{1+Az}{1+Bz}, \tag{6}$$

For $q > 0$, the q −differential operator of a function f, analytic in is defined for a function $f \in H$ is, by definition given as follows:

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}. \tag{7}$$

From (1), we have

$$\mathcal{D}_p f(z) = 1 + \sum_{n=2}^{\infty} [n]_q \alpha_n z^{n-1}, \tag{8}$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \quad [0]_q = 0. \tag{9}$$

One can easily verify that $D_q f(z) \to f'(z)$ as $p \to 1^{-}$. For details on $q-$ calculus and (p, q) −calculus, one can refer to [11] and also references cited therein. Recently for $f \in \mathcal{A}$, Govindaraj and Sivasubramanian [12] defined Sălăgean q-differential operator and further Kanas and Răducanu [13] defined and discussed Sălăgean q-differential operator as follows:

$$\mathcal{D}_q^0 f(z) = f(z),$$ (10)

$$\mathcal{D}_q^1 f(z) = z\mathcal{D}_q f(z),$$

$$\mathcal{D}_q^m f(z) = z\mathcal{D}_q^m(\mathcal{D}_q^{m-1} f(z)),$$

$$\mathcal{D}_q^m f(z) = z + \sum_{n=2}^{\infty} [n]_q \alpha_n z^{n-1} \quad (m \in N_0, z \in D).$$

II. MATERIALS AND METHODS

Theorem 2.1. Let $Tf := f(z) - f(-z)$.If $f \in S_{\mathcal{H}}^{**}(A,B,q)$, then $Tf \in S_{\mathcal{H}}^{*}(A,B,q)$

Proof. Let $f \in S_{\mathcal{H}}^{**}(A,B,q)$ and $H(z) := \frac{1+Az}{1+Bz}$. Then:

$$\frac{2D_{\mathcal{H},q} f(z)}{f(z) - f(-z)} \prec H(z).$$

and

$$\frac{2D_{\mathcal{H},q}(-f)(z)}{f(z) - f(-z)} = \frac{2D_{\mathcal{H},q} f(-z)}{f(-z) - f(z)} \prec H(-z) \prec H(z).$$

Thus, we have:

$$\frac{2D_{\mathcal{H},q} f(z)}{Tf(z)} \in H(U) \text{and} \frac{2D_{\mathcal{H},q}(-f)(z)}{Tf(z)} \in H(U)(z \in U).$$

Since H is the convex function in U, we have:

$$\frac{1}{2}\frac{2D_{\mathcal{H},q} f(z)}{Tf(z)} + \frac{1}{2}\frac{2D_{\mathcal{H},q}(-f)(z)}{Tf(z)} = \frac{2D_{\mathcal{H},q}(Tf)(z)}{Tf(z)} \in H(U)(z \in U).$$

or equivalently:

$$\frac{D_{\mathcal{H},q}(Tf)(z)}{Tf(z)} \prec H(z),$$

which implies that:

$$f \in S_{\mathcal{H}}^{**}(A,B,q)$$

Let $\mathcal{V} \subset \mathcal{H}, U_0 := U \backslash \{0\}$. Due to Ruscheweyh [14] we define the dual set of $\mathcal{V}$ by:

$$\mathcal{V}^* := \{f \in \mathcal{H}_0 : \bigwedge_{q \in V} (f * q)(z) \neq 0 (z \in U_0)\}$$

Theorem 2.2. We have:

$$S_{\mathcal{H}}^{**}(A,B,q) = \{\psi_\xi : |\xi| = 1\}^*,$$

where,

$$\psi_\xi(z):=z\frac{\xi(B-A)+(2+A\xi+B\xi)qz}{(1-qz)(1+qz)(1-z)} \tag{11}$$

$$-z\frac{2+(A+B)\xi-(B-A)\xi q\overline{z}}{(1-q\overline{z})(1+q\overline{z})(1-\overline{z})}(z\in U)$$

Proof. Let $f\in\mathcal{H}_0$ be of the form (1). Then $f\in S_{\mathcal{H}}^{**}(A,B,q)$ if and only if it satisfies Equation (5), or equivalently:

$$\frac{2D_{\mathcal{H},q}f(z)}{f(z)-f(-z)}\neq\frac{1+A\xi}{1+B\xi}(z\in U_0,|\xi|=1). \tag{12}$$

Since,

$$D_{\mathcal{H},q}h(z)=h(z)*\frac{z}{(1-qz)(1-z)},$$

$$\frac{h(z)-h(-z)}{2}=h(z)*\frac{z}{(1+qz)(1-z)}$$

the above inequality yields:

$$(1+B\xi)D_{\mathcal{H},q}f(z)-(1+A\xi)\frac{f(z)-f(-z)}{2}$$

$$=(1+B\xi)D_{\mathcal{H},q}h(z)-(1+A\xi)\frac{h(z)-h(-z)}{2}$$

$$-\left\{(1+B\xi)\overline{D_{\mathcal{H},q}g(z)}+(1+A\xi)\overline{\frac{g(z)-g(-z)}{2}}\right\}$$

$$=h(z)*\left(\frac{(1+B\xi)z}{(1-qz)(1-z)}-\frac{(1+A\xi)z}{(1+qz)(1-z)}\right)$$

$$-g(z)*\left(\frac{(1+B\xi)\overline{z}}{(1-q\overline{z})(1-\overline{z})}-\frac{(1+A\xi)\overline{z}}{(1+q\overline{z})(1-\overline{z})}\right)$$

$$=f(z)*\psi_\xi(z)\neq0(z\in U_0,|\xi|=1).$$

Thus, $f\in S_{\mathcal{H}}^{**}(A,B,q)$ if and only if $f(z)*\psi\xi(z)\neq0$ for $z\in U_0,|\xi|=1$, i.e. $S_{\mathcal{H}}^{**}(A,B,q)=\{\psi_\xi:|\xi|=1\}^*$.

Theorem 2.3. If a function $f\in H$ of the form (1) satisfies the condition:

$$\sum_{k=2}^{\infty}(|\alpha_k||a_k|+|\beta_k||b_k|)\leq B-A, \tag{13}$$

where $-B\leq A<B\leq1$ and

$$\alpha_k=k(1+B)-(1+A)\big(1-(-1)^k\big)/2, \tag{14}$$

$$\beta_k=k(1+B)+(1+A)\big(1-(-1)^k\big)/2,$$

then $f\in S_{\mathcal{H}}^{**}(A,B,q)$.

Proof. The result of Lewy [15] gives that the f is orientation preserving and locally univalent if

$$|h'(z)| > |g'(z)|(z \in U). \tag{15}$$

By Equation (14) we have:

$$|a_k|/(B-A) \geq k, |\beta_k|/(B-A) \geq k \quad (k = 2,3,). \tag{16}$$

Therefore, by Equation (13) we obtain:

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1 \tag{17}$$

and

$$|h'(z)| - |g'(z)| \geq 1 - \sum_{n=2}^{\infty} k|a_k||z|^k - \sum_{n=2}^{\infty} |z|^n \geq 1 - |z| \sum_{n=2}^{\infty} (k|a_k| + k|b_k|)$$

$$k|b_k| \geq 1 - \frac{|z|}{B-A} \sum_{k=2}^{\infty} (|\alpha_k||a_k| + |\beta_k||bk|) \geq 1 - |z| > 0(z \in U).$$

$$|h'(z)| - |g'(z)| \geq 1 - \sum_{n=2}^{\infty} k|a_k||z|^k - \sum_{n=2}^{\infty} |z|^n \geq 1 - |z| \sum_{n=2}^{\infty} (k|a_k| + k|b_k|)$$

$$k|b_k| \geq 1 - \frac{|z|}{B-A} \sum_{k=2}^{\infty} (|\alpha_k||a_k| + |\beta_k||bk|) \geq 1 - |z| > 0(z \in U).$$

Therefore, by Equation (10) the function f is locally univalent and sense-preserving in U. Moreover, if $z_1, z_2 \in U, z_1 \neq z_2$, then:

$$\left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| = \left|\sum_{l}^{k}{}_1 z_1^{l-1} z_2^{k-1}\right| \leq \sum_{l}^{k}{}_1 |z_1|^{l-1} |z_2|^{k-1} < k \quad (k = 2,3,).$$

Let $f \in \mathcal{H}_0$ be a function of the form (1). Without loss of generality, we can assume that f is not an identity function. Then there exist $n \in \mathbb{N}_2$ such that $a_n \neq 0$ or $b_n \neq 0$. Thus, by Equation (12) we get:

$$|f(z_1) - f(z_2)| \geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|$$

$$= \left|z_1 - z_2 - \sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k\right)\right| - \left|\sum_{k=2}^{\infty} \overline{b_k \left(z_1^k - zk\right)}\right|$$

$$\geq |z_1 - z_2| - \sum_{k=2}^{\infty} |a_k| \left|z_1^k - z_2^k\right| - \sum_{k=2}^{\infty} |b_k| \left|z_1^k - z_2^k\right|$$

$$= |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} |a_k| \left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| - \sum_{k=2}^{\infty} |b_k| \left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right|\right)$$

$$> |z_1 - z_2| (1 - \sum_{k=2}^{\infty} k|a_k| - \sum_{k=2}^{\infty} k|b_k|) \geq 0.$$

This leads to the univalence of f , i.e., $f \in S_{\mathcal{H}}$. Therefore, $f \in S_{\mathcal{H}}^{**}(A, B, q))$ if and only if there exists a complex-valued function ω, $\omega(0) = 0$, $|\omega(z)| < 1(z \in U)$ such that:

$$\frac{2D_{\mathcal{H},q}f(z)}{f(z)-f(-z)}=\frac{1+A\omega(z)}{1+B\omega(z)}(z\in U),$$

or equivalently:

$$\left|\frac{2D_{\mathcal{H},q}F(z)-f(z)+f(-z)}{2BD_{\mathcal{H},q}F(z)-A(f(z)-f(-z))}\right|<1(z\in U). \tag{18}$$

Thus for $z\in U\backslash\{0\}$ it suffices to show that:

$$\left|D_{\mathcal{H},q}f(z)-\frac{f(z)-f(-z)}{2}\right|-\left|BD_{\mathcal{H},q}f(z)-\frac{f(z)-f(-z)}{2}\right|<0.$$

Indeed, letting $|z|=r\ (0<r<1)$ we have:

$$\left|D_{\mathcal{H},q}f(z)-\frac{f(z)-f(-z)}{2}\right|-\left|BD_{\mathcal{H},q}f(z)-A\frac{f(z)-f(-z)}{2}\right|$$

$$=\left|\sum_{k=2}^{\infty}\left([k]_q-\frac{1-(-1)^k}{2}\right)a_kz^k+\sum_{k=2}^{\infty}\left([k]_q+\frac{1-(-1)^k}{2}\right)\overline{b_kz^k}\right|$$

$$-\left|(B-A)z+\sum_{k=2}^{\infty}\left(B[k]_q-A\frac{1-(-1)^k}{2}\right)a_kz^k+\sum_{k=2}^{\infty}\left(B[k]_q+A\frac{1-(-1)^k}{2}\right)\overline{b_kz^k}\right|$$

$$\leq\sum_{n=2}^{\infty}\left([n]_q-\frac{1-(-1)^n}{2}\right)|a_n|r^n+\sum_{n=2}^{\infty}\left([n]_q+\frac{1-(-1)^n}{2}\right)|b_n|r^n-(B-A)r$$

$$+\sum_{n=2}^{\infty}\left(B[n]_q-A\frac{1-(-1)^n}{2}\right)|a_n|r^n+\sum_{n=2}^{\infty}\left(B[n]_q+A\frac{1-(-1)^n}{2}\right)|b_n|r^n$$

$$\leq r\{\sum_{n=2}^{\infty}(|\alpha_n||a_n|+|\beta_n||b_n|)r^{n-1}-(B-A)\}<0.$$

Hence $f\in S_{\mathcal{H}}^{**}(A,B,q)$.

Motivated by Silverman [15] we denote by $\mathcal{T}$ the class of functions $f\in\mathcal{H}_0$ of the form (1) such that $a_n=-|a_n|, b_n=|b_n|(n=2,3,\ldots)$, i.e.,

$$f=h+\bar{g}, h(z)=z-\sum_{n=2}^{\infty}|a_n|z^n, g(z)=\sum_{n=2}^{\infty}|b_n|\bar{z}^n(z\in U) \tag{19}$$

Moreover, let us define:

$$S_{\mathcal{T}}^{**}(A,B,q):=\mathcal{T}\cap S_{\mathcal{H}}^{**}(A,B,q), -B\leq A<B\leq 1.$$

Now, we show that the condition (8) is also the sufficient condition for a function $f\in\mathcal{T}$ to be in the class $S_{\mathcal{T}}^{**}(A,B,q)$.

Theorem 2.4. Let $f\in\mathcal{T}$ be a function of the form (19). Then $f\in S_{\mathcal{T}}^{**}(A,B,q)$ if and only if condition (13) holds true.

Proof. In view of Theorem 2.3 we need only show that each function $f\in S_{\mathcal{T}}^{**}(A,B,q)$ satisfies the coefficient inequality of Equation (13). If $f\in S_{\mathcal{T}}^{**}(A,B,q)$, then it satisfies Equation (18) or equivalently:

$$\left|\frac{\sum_{n=2}^{\infty}\left\{\left([n]_q-\frac{1-(-1)^n}{2}\right)|a_n|z^n+\left([n]_q+\frac{1-(-1)^n}{2}\right)|b_n|\bar{z}^n\right\}}{(B-A)z-\sum_{n=2}^{\infty}\left\{\left(B[n]_q-A\frac{1-(-1)^n}{2}\right)|a_n|z^n+\left(B[n]_q+A\frac{1-(-1)^n}{2}\right)|b_n|\bar{z}^n\right\}}\right|<1(Z\in U)$$

It is clear that the denominator of the left hand side cannot vanish for$r \in 0,1)$. Moreover, it is positive for $r=0$, and in consequence for $r \in \langle 0,1)$. Thus, by Equation(13) we have:

$$\sum_{n=2}^{\infty}(\alpha_n|a_n|+\beta_n|b_n|)r^{n-1}<(B-A) \quad (0\le r<1). \tag{20}$$

The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty}(\alpha_n|a_n|+\beta_n|b_n|)$ is non-decreasing sequence. Moreover, by Equation (20) it is bounded by $B-A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty}(\alpha_n|a_n|+\beta_n|b_n|)=\lim_{n\to\infty}S_n\le B-A$$

which yields the assertion (13).

Example 1. For the function:

$$f(z)=z-\sum_{n=2}^{\infty}\frac{B-A}{2^n\alpha_n}z^n-\sum_{n=2}^{\infty}\frac{B-A}{2^n\beta_n}\bar{z}^n \ \ (Z\in U). \tag{21}$$

we have,

$$\sum_{n=2}^{\infty}(\alpha_n|a_n|+\beta_n|b_n|)=\sum_{n=2}^{\infty}\frac{B-A}{2^n}+\sum_{n=2}^{\infty}\frac{B-A}{2^n}. \tag{22}$$

$$=(B-A)\sum_{n=1}^{\infty}\frac{1}{2^n}=B-A.$$

Thus, $f\in S_{\mathcal{J}}^{**}(A,B,q)$.

3 Partial sums

`Several authors studied the partial sums of analytic univalent functions, yet analogous results on partial sums on harmonic univalent functions have not been so far explored. Motivated with the work of Silverman [16] (see also [17, 18, 19]) an attempt has been made to systematically study on the ratio of starlike harmonic univalent function to its sequences of partial sums. We let the sequences of partial sums of functions of the form (1) with $b_1=0$ are

$$f_m(z)=z+\sum_{k=2}^{m}a_kz^k+\sum_{k=2}^{\infty}\overline{b_kz^k}, \tag{23}$$

$$f_n(z)=z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=2}^{n}\overline{b_kz^k}, \tag{24}$$

$$f_{m,n}(z)=z+\sum_{k=2}^{m}a_kz^k+\sum_{k=2}^{n}\overline{b_kz^k} \tag{25}$$

when the coefficients of f are sufficiently small to satisfy the condition (8).

In the present paper, we determine sharp lower bounds for $Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\}$, $Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\}$,

$$Re\left\{\frac{f'(z)}{f'_{m,n}(z)}\right\} \text{ and } Re\left\{\frac{f'_{m,n}(z)}{f'(z)}\right\} \text{ where } f'(z) = \frac{\partial}{\partial\theta} f(re^{i\theta}).$$

In our first theorem, we determine sharp lower bounds for $Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\}$.

Theorem 3.1. If f of the form (1) with $b_1 = 0$, satisfies condition (13), then

$$Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\} \geq \frac{d_{m+1}-(B-A)}{d_{m+1}}, (z \in U) \tag{26}$$

where

$$d_n = \min(\alpha_n, \beta_n) \tag{27}$$

$$d_n = \begin{cases} B-A & for \quad n = 2,3,4,\dots\dots\dots \\ d_{k+1} & for \quad n = k+1, k+2, \dots\dots. \end{cases} \tag{28}$$

Proof. In order to show (20), let us write

$$\psi(z) = \frac{d_{m+1}}{B-A}\{\frac{f(z)}{f_{m,n}(z)} - (1 - \frac{(B-A)}{d_{m+1}})\} \tag{29}$$

$$= 1 + \frac{\frac{d_{m+1}}{B-A}(\sum_{k=m+1}^{\infty} a_k z^k + \sum_{k=n+1}^{\infty} \overline{b_k z^k})}{z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{n} \overline{b_k z^k}}. \tag{30}$$

The result can be obtained if we can prove $Re(w(z)) > 0$ and for this we need to prove the below inequality:

$$\left|\frac{\psi(z)-1}{\psi(z)+1}\right| \leqslant 1 \tag{31}$$

In other words

$$\left|\frac{\psi(z)-1}{\psi(z)+1}\right| \leqslant \frac{\frac{d_{m+1}}{B-A}(\sum_{k=m+1}^{\infty} a_k z^k + \sum_{k=n+1}^{\infty} \overline{b_k z^k})}{2-2(\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{n} |b_k|) - \frac{d_{m+1}}{B-A}(\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k|)} \tag{32}$$

since from the use of (13), we observe that the denominator of the last inequality is positive.

Thus the right hand side of the last inequality is bounded above by one if and only if the following inequality hold

$$\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{n} |b_k|) + \frac{d_{m+1}}{B-A}(\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k|) \leqslant 1 \tag{33}$$

Finally ,to prove this inequality in (20) ,it suffices to show that the left hand side of (27) is bounded by

$$\sum_{k=2}^{\infty} \frac{d_m}{B-A} |a_k| + \sum_{k=2}^{\infty} \frac{d_m}{B-A} |b_k|, \tag{34}$$

which is equivalent to

$$\sum_{k=2}^{m} \frac{d_m-(B-A)}{B-A}|a_k| + \sum_{k=2}^{n} \frac{d_m-(B-A)}{B-A}|b_k|) + \frac{d_m-d_{m+1}}{B-A}\left(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right) \geqslant 0 \quad (35)$$

and the last inequality holds true. For the sharpness, let's consider

$$f(z) = z + \frac{B-A}{d_{k+1}} z^k, \quad (36)$$

which provides the best result. We note for $z = re^{i\frac{\pi}{k}}$ that

$$\frac{f(z)}{f_{m,n}(z)} = 1 + \frac{B-A}{d_{k+1}} z^k \longrightarrow 1 - \frac{B-A}{d_{k+1}} r^k, \quad (37)$$

$$= \frac{d_{k+1}-B+A}{d_{k+1}}.$$

Theorem 3.2. If f of the form (1) with $b_1 = 0$, satisfies condition (13), then

$$Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\} \geq \frac{d_{k+1}}{d_{k+1}+(B-A)} \quad (38)$$

The proof is identical to that of theorem 3.1 proof and is thus excluded.

Theorem 3.3. If f of the form (1.1) with $b_1 = 0$, satisfies condition (13), then

$$Re\left\{\frac{D_{H,q}f(z)}{D_{\mathcal{H},q}f_{m,n}(z)}\right\} \geq \frac{d_{n+1-([n]_q+1)(B-A)}}{d_{n+1}}, (z \in U) \quad (39)$$

Proof. we may write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\frac{D_{H,q}f(re^{i\theta})}{D_{\mathcal{H},q}f_{m,n}(re^{i\theta})} - \frac{d_{n+1-([n]_q+1)(B-A)}}{d_{n+1}}\right]$$

$$= \frac{1+\sum_{k=2}^{m}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=2}^{n}[k]_q \overline{b_k} r^{k-1}e^{-i(k-1)\theta} + \frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty}[k]_q \overline{b_k} r^{k-1}e^{-i(k-1)\theta}\right]}{1+\sum_{k=2}^{m}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=2}^{n}[k]_q \overline{b_k} r^{k-1}e^{-i(k-1)\theta}}$$

So that

$$\omega(z) = \frac{\frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty}[k]_q \overline{b}_k r^{k-1}e^{-i(k-1)\theta}\right]}{2+2\left(\sum_{k=2}^{m}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=2}^{n}[k]_q \overline{b}_k r^{k-1}e^{-i(k-1)\theta}\right) + \frac{d_{n+1}}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty}[k]_q \overline{b}_k r^{k-1}e^{-i(k-1)\theta}\right)}$$

Then

$$|\omega(z)| \leq \frac{\frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty}[k]_q|a_k| + \sum_{k=n+1}^{\infty}[k]_q|b_k|\right]}{2-2\left(\sum_{k=2}^{m}[k]_q|a_k| + \sum_{k=2}^{n}[k]_q|b_k|\right) - \frac{d_{n+1}}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty}[k]_q|a_k| + \sum_{k=n+1}^{\infty}[k]_q|b_k|\right)}$$

This last expression is bounded above by 1 if and only if $\sum_{k=2}^{m}[k]_q|a_k| + \sum_{k=2}^{n}[k]_q|b_k| + \frac{d_{n+1}}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty}[k]_q|a_k| + \sum_{k=n+1}^{\infty}[k]_q|b_k|\right) \leq 1.$ (40)

It suffices to show that L. H. S. of (40) is bounded above by $\sum_{k=2}^{\infty}\frac{d_{n+1}}{B-A}|a_k|+\sum_{k=2}^{\infty}\frac{d_{n+1}}{B-A}|b_k|$, which is equivalent to

$$\sum_{k=2}^{m}\frac{\alpha_k-[k]_q(B-A)}{B-A}|a_k|+\sum_{k=2}^{n}\frac{\beta_k-[k]_q(B-A)}{B-A}|b_k|$$

$$+\sum_{k=m+1}^{\infty}\frac{([n]_q+1)\alpha_k-d_{n+1}[k]_q}{([n]_q+1)(B-A)}|a_k|+\sum_{k=n+1}^{\infty}\frac{([n]_q+1)d_k-d_{n+1}[k]_q}{([n]_q+1)(B-A)}|b_k|\geqslant 0$$

To see that $f(z)=z+\frac{B-A}{d_{k+1}}\bar{z}^{n+1}$ gives the sharp result, we observe that for $z=re^{i\frac{\pi}{n+2}}$ that

$$\frac{D_{H,q}f(z)}{D_{\mathcal{H},q}f_{m,n}(z)}=1+\frac{([n]_q+1)(B-A)}{d_{n+1}}r^n e^{-i(n+2)\frac{\pi}{n+2}}\to 1-\frac{([n]_q+1)(B-A)}{d_{n+1}}$$

$$=\frac{d_{n+1}-([n]_q+1)(B-A)}{d_{n+1}},$$

when $r\to 1^-$.

We next determine bounds for $Re\left\{\frac{D_{\mathcal{H},q}f_{m,n}(z)}{D_{H,q}f(z)}\right\}$,

Theorem 3.4. If f of the form (1) with $b_1=0$, satisfies condition (13), then

$$Re\left\{\frac{D_{\mathcal{H},q}f_{m,n}(z)}{D_{H,q}f(z)}\right\}\geq\frac{d_{k+1}}{d_{k+1}+(B-A)} \quad (41)$$

we write

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{d_{n+1}+([n]_q+1)(B-A)}{([n]_q+1)(B-A)}\left[\frac{D_{\mathcal{H},q}f_{m,n}(z)}{D_{H,q}f(z)}-\frac{d_{n+1}}{d_{n+1}+([n]_q+1)(B-A)}\right]$$

$$1+\sum_{k=2}^{m}[k]_q a_k r^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{n}[k]_q\overline{b_k}r^{k-1}e^{-i(k-1)\theta}$$

$$=\frac{-\frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty}[k]_q a_k r^{k-1}e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty}[k]_q\overline{b_k}r^{k-1}e^{-i(k-1)\theta}\right]}{1+\sum_{k=2}^{m}[k]_q a_k r^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{n}[k]_q\overline{b_k}r^{k-1}e^{-i(k-1)\theta}}$$

so that

$$|\omega(z)|\leq\frac{\frac{d_{n+1}+([n]_q+1)(B-A)}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty}[k]_q|a_k|+\sum_{k=n+1}^{\infty}[k]_q|b_k|\right]}{2-2\left(\sum_{k=2}^{m}[k]_q|a_k|+\sum_{k=2}^{n}[k]_q|b_k|\right)-\frac{d_{n+1}-([n]_q+1)(B-A)}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty}[k]_q|a_k|+\sum_{k=n+1}^{\infty}[k]_q|b_k|\right)}\leqslant 1,$$

if $\sum_{k=2}^{m}[k]_q|a_k|+\sum_{k=2}^{n}[k]_q|b_k|+\frac{d_{n+1}}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty}[k]_q|a_k|+\sum_{k=n+1}^{\infty}[k]_q|b_k|\right)\leq 1,\quad (42)$

then L.H.S. of (42) is bounded above by $\sum_{k=2}^{\infty}\frac{\alpha_n}{B-A}|a_k|+\sum_{k=2}^{\infty}\frac{\beta_n}{B-A}|b_k|$, the proof is completed.

Theorem 2.1. Let $Tf:=f(z)-f(-z)$.If $f\in S_{\mathcal{H}}^{**}(A,B,q)$, then $Tf\in S_{\mathcal{H}}^{*}(A,B,q)$

Proof. Let $f \in S_{\mathcal{H}}^{**}(A,B,q)$ and $H(z) := \frac{1+Az}{1+Bz}$. Then:

$$\frac{2D_{\mathcal{H},q}f(z)}{f(z)-f(-z)} \prec H(z).$$

and

$$\frac{2D_{\mathcal{H},q}(-f)(z)}{f(z)-f(-z)} = \frac{2D_{\mathcal{H},q}f(-z)}{f(-z)-f(z)} \prec H(-z) \prec H(z).$$

Thus, we have:

$$\frac{2D_{\mathcal{H},q}f(z)}{Tf(z)} \in H(U) \text{and} \frac{2D_{\mathcal{H},q}(-f)(z)}{Tf(z)} \in H(U)(z \in U).$$

Since H is the convex function in U, we have:

$$\frac{1}{2}\frac{2D_{\mathcal{H},q}f(z)}{Tf(z)} + \frac{1}{2}\frac{2D_{\mathcal{H},q}(-f)(z)}{Tf(z)} = \frac{2D_{\mathcal{H},q}(Tf)(z)}{Tf(z)} \in H(U)(z \in U).$$

or equivalently:

$$\frac{D_{\mathcal{H},q}(Tf)(z)}{Tf(z)} \prec H(z),$$

which implies that:

$$f \in S_{\mathcal{H}}^{**}(A,B,q)$$

Let $\mathcal{V} \subset \mathcal{H}, U_0 := U \backslash \{0\}$. Due to Ruscheweyh [14] we define the dual set of $\mathcal{V}$ by:

$$\mathcal{V}^* := \{f \in \mathcal{H}_0 : \bigwedge_{q \in V} (f * q)(z) \neq 0 (z \in U_0)\}$$

Theorem 2.2. We have:

$$S_{\mathcal{H}}^{**}(A,B,q) = \{\psi_\xi : |\xi| = 1\}^*,$$

where,

$$\psi_\xi(z) := z\frac{\xi(B-A)+(2+A\xi+B\xi)qz}{(1-qz)(1+qz)(1-z)} \quad (11)$$

$$-z\frac{2+(A+B)\xi-(B-A)\xi q\bar{z}}{(1-q\bar{z})(1+q\bar{z})(1-\bar{z})}(z \in U)$$

Proof. Let $f \in \mathcal{H}_0$ be of the form (1). Then $f \in S_{\mathcal{H}}^{**}(A,B,q)$ if and only if it satisfies Equation (5), or equivalently:

$$\frac{2D_{\mathcal{H},q}f(z)}{f(z)-f(-z)} \neq \frac{1+A\xi}{1+B\xi}(z \in U_0, |\xi| = 1). \quad (12)$$

Since,

$$D_{\mathcal{H},q}h(z) = h(z) * \frac{z}{(1-qz)(1-z)},$$

$$\frac{h(z)-h(-z)}{2} = h(z) * \frac{z}{(1+qz)(1-z)}$$

the above inequality yields:

$$(1+B\xi)D_{\mathcal{H},q}f(z) - (1+A\xi)\frac{f(z)-f(-z)}{2}$$

$$= (1+B\xi)D_{\mathcal{H},q}h(z) - (1+A\xi)\frac{h(z)-h(-z)}{2}$$

$$-\left\{(1+B\xi)\overline{D_{\mathcal{H},q}g(z)} + (1+A\xi)\overline{\frac{g(z)-g(-z)}{2}}\right\}$$

$$= h(z) * \left(\frac{(1+B\xi)z}{(1-qz)(1-z)} - \frac{(1+A\xi)z}{(1+qz)(1-z)}\right)$$

$$-g(z) * \left(\frac{(1+B\xi)\bar{z}}{(1-q\bar{z})(1-\bar{z})} - \frac{(1+A\xi)\bar{z}}{(1+q\bar{z})(1-\bar{z})}\right)$$

$$= f(z) * \psi_\xi(z) \neq 0 (z \in U_0, |\xi| = 1).$$

Thus, $f \in S^{**}_{\mathcal{H}}(A,B,q)$ if and only if$f(z) * \psi\xi(z) \neq 0$ for$z \in U_0, |\xi| = 1$, i.e.$S^{**}_{\mathcal{H}}(A,B,q) = \{\psi_\xi : |\xi| = 1\}^*$.

Theorem 2.3. If a function$f \in H$ of the form (1) satisfies the condition:

$$\sum_{k=2}^{\infty} (|\alpha_k||a_k| + |\beta_k||b_k|) \leq B - A, \tag{13}$$

where$-B \leq A < B \leq 1$ and

$$\alpha_k = k(1+B) - (1+A)\big(1-(-1)^k\big)/2, \tag{14}$$

$$\beta_k = k(1+B) + (1+A)\big(1-(-1)^k\big)/2,$$

then $f \in S^{**}_{\mathcal{H}}(A,B,q)$.

Proof. The result of Lewy [15] gives that the$f$ is orientation preserving and locally univalent if

$$|h'(z)| > |g'(z)| (z \in U). \tag{15}$$

By Equation (14) we have:

$$|a_k|/(B-A) \geq k, |\beta_k|/(B-A) \geq k \quad (k = 2,3,). \tag{16}$$

Therefore, by Equation (13) we obtain:

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1 \tag{17}$$

and

$$|h'(z)| - |g'(z)| \geq 1 - \sum_{n=2}^{\infty} k|a_k||z|^k - \sum_{n=2}^{\infty} |z|^n \geq 1 - |z| \sum_{n=2}^{\infty} (k|a_k| + k|b_k|)$$

$$k|b_k| \geq 1 - \frac{|z|}{B-A} \sum_{k=2}^{\infty} (|\alpha_k||a_k| + |\beta_k||bk|) \geq 1 - |z| > 0 (z \in U).$$

$$|h'(z)| - |g'(z)| \geq 1 - \sum_{n=2}^{\infty} k|a_k||z|^k - \sum_{n=2}^{\infty} |z|^n \geq 1 - |z| \sum_{n=2}^{\infty} (k|a_k| + k|b_k|)$$

$$k|b_k| \geq 1 - \frac{|z|}{B-A} \sum_{k=2}^{\infty} (|\alpha_k||a_k| + |\beta_k||bk|) \geq 1 - |z| > 0 (z \in U).$$

Therefore, by Equation (10) the function f is locally univalent and sense-preserving in U. Moreover, if $z_1, z_2 \in U, z_1 \neq z_2$, then:

$$\left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| = \left|\sum_{l}^{k} {}_1\, z_1^{l-1} z_2^{k-1}\right| \leq \sum_{l}^{k} {}_1\, |z_1|^{l-1} |z_2|^{k-1} < k \quad (k = 2,3,).$$

Let $f \in \mathcal{H}_0$ be a function of the form (1). Without loss of generality, we can assume that f is not an identity function. Then there exist $n \in \mathbb{N}_2$such that $a_n \neq 0$ or $b_n \neq 0$. Thus, by Equation (12) we get:

$$|f(z_1) - f(z_2)| \geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|$$

$$= \left|z_1 - z_2 - \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)\right| - \left|\sum_{k=2}^{\infty} \overline{b_k (z_1^k - zk)}\right|$$

$$\geq |z_1 - z_2| - \sum_{k=2}^{\infty} |a_k||z_1^k - z_2^k| - \sum_{k=2}^{\infty} |b_k||z_1^k - z_2^k|$$

$$= |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} |a_k| \left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| - \sum_{k=2}^{\infty} |b_k| \left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right|\right)$$

$$> |z_1 - z_2| (1 - \sum_{k=2}^{\infty} k|a_k| - \sum_{k=2}^{\infty} k|b_k|) \geq 0.$$

This leads to the univalence of f , i.e., $f \in S_{\mathcal{H}}$. Therefore,$f \in S_{\mathcal{H}}^{**}(A, B, q))$ if and only if there exists a complex-valued function ω, $\omega(0) = 0$, $|\omega(z)| < 1 (z \in U)$such that:

$$\frac{2D_{\mathcal{H},q} f(z)}{f(z) - f(-z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} (z \in U),$$

or equivalently:

$$\left|\frac{2D_{\mathcal{H},q} F(z) - f(z) + f(-z)}{2BD_{\mathcal{H},q} F(z) - A(f(z) - f(-z))}\right| < 1 (z \in U). \tag{18}$$

Thus for$z \in U \backslash \{0\}$ it suffices to show that:

$$\left|D_{\mathcal{H},q} f(z) - \frac{f(z) - f(-z)}{2}\right| - \left|BD_{\mathcal{H},q} f(z) - \frac{f(z) - f(-z)}{2}\right| < 0.$$

Indeed, letting $|z| = r\ (0 < r < 1)$ we have:

$$\left|D_{\mathcal{H},q}f(z) - \frac{f(z)-f(-z)}{2}\right| - \left|BD_{\mathcal{H},q}f(z) - A\frac{f(z)-f(-z)}{2}\right|$$

$$= \left|\sum_{k=2}^{\infty}\left([k]_q - \frac{1-(-1)^k}{2}\right)a_k z^k + \sum_{k=2}^{\infty}\left([k]_q + \frac{1-(-1)^k}{2}\right)\overline{b_k z^k}\right|$$

$$-\left|(B-A)z + \sum_{k=2}^{\infty}\left(B[k]_q - A\frac{1-(-1)^k}{2}\right)a_k z^k + \sum_{k=2}^{\infty}\left(B[k]_q + A\frac{1-(-1)^k}{2}\right)\overline{b_k z^k}\right|$$

$$\leq \sum_{n=2}^{\infty}\left([n]_q - \frac{1-(-1)^n}{2}\right)|a_n|r^n + \sum_{n=2}^{\infty}\left([n]_q + \frac{1-(-1)^n}{2}\right)|b_n|r^n - (B-A)r$$

$$+\sum_{n=2}^{\infty}\left(B[n]_q - A\frac{1-(-1)^n}{2}\right)|a_n|r^n + \sum_{n=2}^{\infty}\left(B[n]_q + A\frac{1-(-1)^n}{2}\right)|b_n|r^n$$

$$\leq r\{\sum_{n=2}^{\infty}(|\alpha_n||a_n| + |\beta_n||b_n|)r^{n-1} - (B-A)\} < 0.$$

Hence $f \in S_{\mathcal{H}}^{**}(A,B,q)$.

Motivated by Silverman [15] we denote by $\mathcal{T}$ the class of functions $f \in \mathcal{H}_0$ of the form (1) such that $a_n = -|a_n|, b_n = |b_n| (n = 2,3,\ldots)$, i.e.,

$$f = h + \bar{g}, h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n, g(z) = \sum_{n=2}^{\infty}|b_n|\bar{z}^n (z \in U) \tag{19}$$

Moreover, let us define:

$$S_{\mathcal{T}}^{**}(A,B,q) := \mathcal{T} \cap S_{\mathcal{H}}^{**}(A,B,q), -B \leq A < B \leq 1.$$

Now, we show that the condition (8) is also the sufficient condition for a function $f \in \mathcal{T}$ to be in the class $S_{\mathcal{T}}^{**}(A,B,q)$.

Theorem 2.4. Let $f \in \mathcal{T}$ be a function of the form (19). Then$f \in S_{\mathcal{T}}^{**}(A,B,q)$ if and only if condition (13) holds true.

Proof. In view of Theorem 2.3 we need only show that each function $f \in S_{\mathcal{T}}^{**}(A,B,q)$ satisfies the coefficient inequality of Equation (13). If $f \in S_{\mathcal{T}}^{**}(A,B,q)$, then it satisfies Equation (18) or equivalently:

$$\left|\frac{\sum_{n=2}^{\infty}\left\{\left([n]_q - \frac{1-(-1)^n}{2}\right)|a_n|z^n + \left([n]_q + \frac{1-(-1)^n}{2}\right)|b_n|\bar{z}^n\right\}}{(B-A)z - \sum_{n=2}^{\infty}\left\{\left(B[n]_q - A\frac{1-(-1)^n}{2}\right)|a_n|z^n + \left(B[n]_q + A\frac{1-(-1)^n}{2}\right)|b_n|\bar{z}^n\right\}}\right| < 1 (Z \in U)$$

It is clear that the denominator of the left hand side cannot vanish for$r \in 0,1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in \langle 0,1)$. Thus, by Equation(13) we have:

$$\sum_{n=2}^{\infty}(\alpha_n|a_n| + \beta_n|b_n|)r^{n-1} < (B-A) \quad (0 \leq r < 1). \tag{20}$$

The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty}(\alpha_n|a_n| + \beta_n|b_n|)$ is non-decreasing sequence. Moreover, by Equation (20) it is bounded by $B - A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty}(\alpha_n|a_n|+\beta_n|b_n|)=\lim_{n\to\infty}S_n\le B-A$$

which yields the assertion (13).

Example 1. For the function:

$$f(z)=z-\sum_{n=2}^{\infty}\frac{B-A}{2^n\alpha_n}z^n-\sum_{n=2}^{\infty}\frac{B-A}{2^n\beta_n}\bar{z}^n\ \ (Z\in U). \tag{21}$$

we have,

$$\sum_{n=2}^{\infty}(\alpha_n|a_n|+\beta_n|b_n|)=\sum_{n=2}^{\infty}\frac{B-A}{2^n}+\sum_{n=2}^{\infty}\frac{B-A}{2^n}. \tag{22}$$

$$=(B-A)\sum_{n=1}^{\infty}\frac{1}{2^n}=B-A.$$

Thus, $f\in S_{\mathcal{J}}^{**}(A,B,q)$.

3 Partial sums

Several authors studied the partial sums of analytic univalent functions, yet analogous results on partial sums on harmonic univalent functions have not been so far explored. Motivated with the work of Silverman [16] (see also [17, 18, 19]) an attempt has been made to systematically study on the ratio of starlike harmonic univalent function to its sequences of partial sums. We let the sequences of partial sums of functions of the form (1) with $b_1=0$ are

$$f_m(z)=z+\sum_{k=2}^{m}a_kz^k+\sum_{k=2}^{\infty}\overline{b_kz^k}, \tag{23}$$

$$f_n(z)=z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=2}^{n}\overline{b_kz^k}, \tag{24}$$

$$f_{m,n}(z)=z+\sum_{k=2}^{m}a_kz^k+\sum_{k=2}^{n}\overline{b_kz^k} \tag{25}$$

when the coefficients of f are sufficiently small to satisfy the condition (8).

In the present paper, we determine sharp lower bounds for $Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\}$, $Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\}$, $Re\left\{\frac{f'(z)}{f'_{m,n}(z)}\right\}$ and $Re\left\{\frac{f'_{m,n}(z)}{f'(z)}\right\}$ where $f'(z)=\frac{\partial}{\partial\theta}f(re^{i\theta})$.

In our first theorem, we determine sharp lower bounds for $Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\}$.

Theorem 3.1. If f of the form (1) with $b_1=0$, satisfies condition (13), then

$$Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\}\ge\frac{d_{m+1}-(B-A)}{d_{m+1}},(z\in U) \tag{26}$$

where

$$d_n=\min(\alpha_n,\beta_n) \tag{27}$$

$$d_n = \begin{cases} B-A & for \quad n = 2,3,4,\dots\dots\dots \\ d_{k+1} & for \quad n = k+1, k+2,\dots\dots\dots \end{cases} \tag{28}$$

Proof. In order to show (20), let us write

$$\psi(z) = \frac{d_{m+1}}{B-A}\{\frac{f(z)}{f_{m,n}(z)} - (1 - \frac{(B-A)}{d_{m+1}})\} \tag{29}$$

$$= 1 + \frac{\frac{d_{m+1}}{B-A}(\sum_{k=m+1}^{\infty} a_k z^k + \sum_{k=n+1}^{\infty} \overline{b_k z^k})}{z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{n} \overline{b_k z^k}}. \tag{30}$$

The result can be obtained if we can prove $Re(w(z)) > 0$ and for this we need to prove the below inequality:

$$\left|\frac{\psi(z)-1}{\psi(z)+1}\right| \leqslant 1 \tag{31}$$

In other words

$$\left|\frac{\psi(z)-1}{\psi(z)+1}\right| \leqslant \frac{\frac{d_{m+1}}{B-A}(\sum_{k=m+1}^{\infty} a_k z^k + \sum_{k=n+1}^{\infty} \overline{b_k z^k})}{2 - 2(\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{n} |b_k|) - \frac{d_{m+1}}{B-A}(\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k|)} \tag{32}$$

since from the use of (13), we observe that the denominator of the last inequality is positive.

Thus the right hand side of the last inequality is bounded above by one if and only if the following inequality hold

$$\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{n} |b_k|) + \frac{d_{m+1}}{B-A}(\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k|) \leqslant 1 \tag{33}$$

Finally ,to prove this inequality in (20) ,it suffices to show that the left hand side of (27) is bounded by

$$\sum_{k=2}^{\infty} \frac{d_m}{B-A}|a_k| + \sum_{k=2}^{\infty} \frac{d_m}{B-A}|b_k|, \tag{34}$$

which is equivalent to

$$\sum_{k=2}^{m} \frac{d_m-(B-A)}{B-A}|a_k| + \sum_{k=2}^{n} \frac{d_m-(B-A)}{B-A}|b_k|) + \frac{d_m-d_{m+1}}{B-A}(\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k|) \geqslant 0 \tag{35}$$

and the last inequality holds true. For the sharpness, let's consider

$$f(z) = z + \frac{B-A}{d_{k+1}} z^k, \tag{36}$$

which provides the best result. We note for $z = re^{i\frac{\pi}{k}}$ that

$$\frac{f(z)}{f_{m,n}(z)} = \ 1 + \frac{B-A}{d_{k+1}} z^k \longrightarrow 1 - \frac{B-A}{d_{k+1}} r^k, \tag{37}$$

$$= \frac{d_{k+1} - B + A}{d_{k+1}}.$$

Theorem 3.2. If f of the form (1) with $b_1 = 0$, satisfies condition (13), then

$$Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\} \geq \frac{d_{k+1}}{d_{k+1} + (B - A)} \qquad (38)$$

The proof is identical to that of theorem 3.1 proof and is thus excluded.

Theorem 3.3. If f of the form (1.1) with $b_1 = 0$, satisfies condition (13), then

$$Re\left\{\frac{D_{H,q}f(z)}{D_{\mathcal{H},q}f_{m,n}(z)}\right\} \geq \frac{d_{n+1-([n]_q+1)(B-A)}}{d_{n+1}}, (z \in U) \qquad (39)$$

Proof. we may write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\frac{D_{H,q}f(re^{i\theta})}{D_{\mathcal{H},q}f_{m,n}(re^{i\theta})} - \frac{d_{n+1-([n]_q+1)(B-A)}}{d_{n+1}}\right]$$

$$= \frac{1+\sum_{k=2}^{m}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=2}^{n}[k]_q \overline{b_k} r^{k-1}e^{-i(k-1)\theta} + \frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty}[k]_q \overline{b_k} r^{k-1}e^{-i(k-1)\theta}\right]}{1+\sum_{k=2}^{m}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=2}^{n}[k]_q \overline{b_k} r^{k-1}e^{-i(k-1)\theta}}$$

So that

$$\omega(z) = \frac{\frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty}[k]_q \overline{b}_k r^{k-1}e^{-i(k-1)\theta}\right]}{2+2\left(\sum_{k=2}^{m}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=2}^{n}[k]_q \overline{b}_k r^{k-1}e^{-i(k-1)\theta}\right) + \frac{d_{n+1}}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty}[k]_q a_k r^{k-1}e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty}[k]_q \overline{b}_k r^{k-1}e^{-i(k-1)\theta}\right)}$$

Then

$$|\omega(z)| \leq \frac{\frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty}[k]_q|a_k| + \sum_{k=n+1}^{\infty}[k]_q|b_k|\right]}{2-2\left(\sum_{k=2}^{m}[k]_q|a_k| + \sum_{k=2}^{n}[k]_q|b_k|\right) - \frac{d_{n+1}}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty}[k]_q|a_k| + \sum_{k=n+1}^{\infty}[k]_q|b_k|\right)}$$

This last expression is bounded above by 1 if and only if

$$\sum_{k=2}^{m}[k]_q|a_k| + \sum_{k=2}^{n}[k]_q|b_k| + \frac{d_{n+1}}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty}[k]_q|a_k| + \sum_{k=n+1}^{\infty}[k]_q|b_k|\right) \leq 1. \qquad (40)$$

It suffices to show that L. H. S. of (40) is bounded above by $\sum_{k=2}^{\infty}\frac{d_{n+1}}{B-A}|a_k| + \sum_{k=2}^{\infty}\frac{d_{n+1}}{B-A}|b_k|$, which is equivalent to

$$\sum_{k=2}^{m}\frac{\alpha_k - [k]_q(B-A)}{B-A}|a_k| + \sum_{k=2}^{n}\frac{\beta_k - [k]_q(B-A)}{B-A}|b_k|$$

$$+\sum_{k=m+1}^{\infty}\frac{([n]_q+1)\alpha_k - d_{n+1}[k]_q}{([n]_q+1)(B-A)}|a_k| + \sum_{k=n+1}^{\infty}\frac{([n]_q+1)d_k - d_{n+1}[k]_q}{([n]_q+1)(B-A)}|b_k| \geqslant 0$$

To see that $f(z) = z + \frac{B-A}{d_{k+1}}\bar{z}^{n+1}$ gives the sharp result, we observe that for $z = re^{i\frac{\pi}{n+2}}$ that

$$\frac{D_{H,q}f(z)}{D_{\mathcal{H},q}f_{m,n}(z)} = 1 + \frac{([n]_q+1)(B-A)}{d_{n+1}} r^n e^{-i(n+2)\frac{\pi}{n+2}} \to 1 - \frac{([n]_q+1)(B-A)}{d_{n+1}}$$

$$= \frac{d_{n+1}-([n]_q+1)(B-A)}{d_{n+1}},$$

when $r \to 1^-$.

We next determine bounds for $Re\left\{\frac{D_{\mathcal{H},q}f_{m,n}(z)}{D_{H,q}f(z)}\right\}$,

Theorem 3.4. If f of the form (1) with $b_1 = 0$, satisfies condition (13), then

$$Re\left\{\frac{D_{\mathcal{H},q}f_{m,n}(z)}{D_{H,q}f(z)}\right\} \geq \frac{d_{k+1}}{d_{k+1}+(B-A)} \tag{41}$$

we write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{d_{n+1}+([n]_q+1)(B-A)}{([n]_q+1)(B-A)}\left[\frac{D_{\mathcal{H},q}f_{m,n}(z)}{D_{H,q}f(z)} - \frac{d_{n+1}}{d_{n+1}+([n]_q+1)(B-A)}\right]$$

$$1 + \sum_{k=2}^{m} [k]_q a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} [k]_q \overline{b_k} r^{k-1} e^{-i(k-1)\theta}$$

$$= \frac{-\frac{d_{n+1}}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty} [k]_q a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} [k]_q \overline{b_k} r^{k-1} e^{-i(k-1)\theta}\right]}{1+\sum_{k=2}^{m} [k]_q a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} [k]_q \overline{b_k} r^{k-1} e^{-i(k-1)\theta}}$$

so that

$$|\omega(z)| \leq \frac{\frac{d_{n+1}+([n]_q+1)(B-A)}{([n]_q+1)(B-A)}\left[\sum_{k=m+1}^{\infty} [k]_q |a_k| + \sum_{k=n+1}^{\infty} [k]_q |b_k|\right]}{2-2\left(\sum_{k=2}^{m} [k]_q |a_k| + \sum_{k=2}^{n} [k]_q |b_k|\right) - \frac{d_{n+1}-([n]_q+1)(B-A)}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty} [k]_q |a_k| + \sum_{k=n+1}^{\infty} [k]_q |b_k|\right)} \leqslant 1,$$

if $\sum_{k=2}^{m} [k]_q |a_k| + \sum_{k=2}^{n} [k]_q |b_k| + \frac{d_{n+1}}{([n]_q+1)(B-A)}\left(\sum_{k=m+1}^{\infty} [k]_q |a_k| + \sum_{k=n+1}^{\infty} [k]_q |b_k|\right) \leq 1,$ (42)

then L.H.S. of (42) is bounded above by $\sum_{k=2}^{\infty} \frac{\alpha_n}{B-A} |a_k| + \sum_{k=2}^{\infty} \frac{\beta_n}{B-A} |b_k|$, the proof is completed.

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