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ASAT

MILITARY TECHNICAL COLLEGE

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"LIQUID SLOSH DUE TO A PULSATING SOURCE"

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ABSTRACT

The problem of the slosh of an inviscid liquid partially filling a rigid cylindrical tank is studied theoretically. The tank is connected - through an outlet at its base - with a pump that causes a velocity fluctuation at the outlet due to its pressure. The nonlinear governing equations have been constructed and their solution accompanied by the stability is given. It has been found that the results reduce to the results of previous investigations in proper limits.

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1. INTRODUCTION

In the present paper, we study the slosh phenomenon in an inviscid liquid partially filling a rigid vertical cylindrical tank. The liquid motion occurs due to a central cylindrical pulsating source at the base of the tank. Almost all the literature reports on the occurrence of this phenomenon due to the motion of the liquid container itself. Nevertheless, few works (Cf. Refs.[1-3]) have appeared that treat the liquid slosh due to the existence of sources of pulsating character.

The dynamic behaviour of the free surface of water in a stationary vertical cylinder due to the outlet velocity fluctuations at the base of the cylinder has been investigated by Buhta and Yeh Ref.[1]. Their analysis was based on linear free surface boundary conditions. Here, in the present study we reexamine the same problem of Buhta & Yeh in the light of exact nonlinear free surface conditions. In Sec.2 we formulate the boundary-value problem and in Secs.3&4 we present its solution followed by a discussion in Sec.5 .

2. THE PROBLEM

Let us consider the time-periodic irrotational three-dimensional motion of an inviscid incompressible liquid bounded partially by a stationary rigid vertical cylindrical tank. The liquid motion is due to a low-frequency pulsating source in a form of a cylinder that connects the tank with a pump. Let a be the radius of the cylindrical tank, ad the mean depth of the liquid, aβ the radius of the pulsating source $(\beta < 1)$, ar and az the distances along the radial and vertical axes, respectively, g the acceleration of the gravitational field, $\sqrt{a/g}\,t$ the time and $\sqrt{g/a}\,\Omega$ the circular frequency of the source. In addition let H be a measure of the wave amplitude. Then we define the parameter ε by $\varepsilon = a^{-1}H$ and let $\varepsilon a \eta(r,\theta;t)$ denote the elevation of the liquid free surface above the mean level given by the plane z=0 as shown in Fig.1. Finally, we introduce the

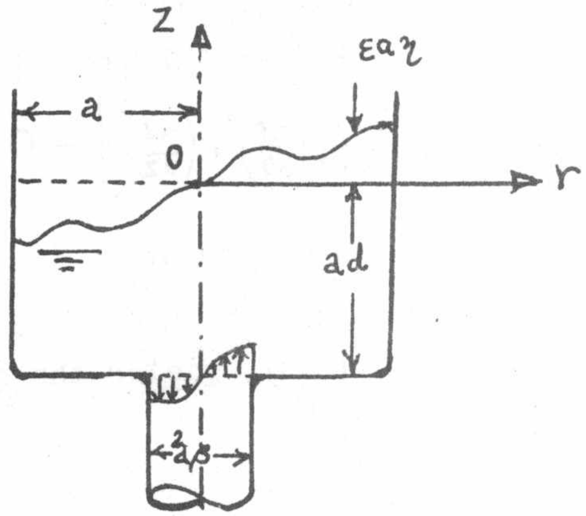

Fig. 1. Problem Geometry.

velocity-potential function in the form $\varepsilon a \sqrt{ag}\, \phi(r,\theta,z;t)$

At any instant t the liquid motion must satisfy the continuity condition, i.e., the velocity-potential function satisfies Laplace's Equation.

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

for $0 \leqslant r \leqslant 1 \ , -\pi < \theta \leqslant \pi \ , \ -d \leqslant z \leqslant \varepsilon\eta$ (2.1)

The requirement that the velocity normal to the tank curved boundary must vanish is expressed by the condition

$$\frac{\partial \phi}{\partial r} = 0 \qquad \text{on } r = 1 \qquad (2.2)$$

As for the tank base, the velocity takes the form

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z = -d \text{ and } \beta \leqslant r \leqslant 1 \qquad (2.3a)$$

$$\frac{\partial \phi}{\partial z} = A \cos\theta \ \sin \Omega t \qquad \text{on } z = -d \text{ and } 0 \leqslant r < \beta \qquad (2.3b)$$

On the liquid free surface, the kinematic condition which states that a particle on the surface remains on it, is given by

$$\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t} = \varepsilon\left(\frac{\partial \phi}{\partial r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}\frac{\partial \eta}{\partial \theta}\right) \qquad \text{on } z = \varepsilon\eta(r,\theta;t) \qquad (2.4)$$

The pressure $p(r,\theta,z;t)$ inside the liquid is given by Bernoulli's Equation (Cf.Wehausen & Laitone [4])

$$\frac{p-p_0}{a\rho g}+z+\varepsilon\frac{\partial\phi}{\partial t}+\frac{\varepsilon^2}{2}\left[\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]=0 \qquad (2.5)$$

in which p_o denotes the ullage pressure.

Setting the free surface pressure which must be constant equal to the ullage pressure in Eq.(2.5), yields the dynamic condition;

$$\frac{\partial\phi}{\partial t}+\eta = -\frac{\varepsilon}{2}\left[\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right] \qquad \text{on} \quad z=\varepsilon\eta(r,\theta;t) \quad (2.6)$$

Thus,the problem under study is that of determining the dimensionless free surface elevation $\eta(r,\theta;t)$ and the dimensionless velocity-potential function $\phi(r,\theta,z;t)$. It is obvious that the solution depends on the dimensionless depth d , the dimensionless source radius β and frequency Ω and lastly the dimensionless parameter ε . We shall solve the problem by determining the first two terms in the expansion of the solution in powersof the parameter ε . Naturally, additional terms can be obtained by continuing the procedure.

3. FIRST ORDER SOLUTION

We assume that the free surface elevation η and the velocity-potential function ϕ have the limits η_o and ϕ_o as the constant parameter ε tends to zero. Setting $\varepsilon = 0$ in Eqs. (2.1 - 2.6) , we find that all but Eqs.(2.4 - 2.6) are unchanged in form. Regarding Eqs.(2.4 & 2.6),they now take the forms:

$$\frac{\partial\phi_o}{\partial z} = \frac{\partial\eta_o}{\partial t} \qquad \text{on} \quad z = 0 \qquad (3.1a)$$

$$\frac{\partial\phi_o}{\partial t}+\eta_o = 0 \qquad \text{on} \quad z = 0 \qquad (3.1b)$$

Each of the normal modes of liquid oscillations for the circular tank requires four indices for its complete identification. These indices are : -

i for the azimuthal wave number

j for the radial wave number

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C or S	to distinguish between cosine-hyperbolic or sine-hyperbolic depth variations.
c or s	to distinguish between cosine and sine azimuthal variations.

Thus, each mode can be written as

$$ {}^{C,S}\Phi^{c,s}_{ij} = N^{-1}_{ij} \; J_i(\mu_{ij} r) \left(\text{Cosh}\,\mu_{ij} z \;,\; \text{Sinh}\,\mu_{ij} z\right) \left(\cos i\theta \;,\; \sin i\theta\right) \tag{3.2} $$

in which $i = 0,1,2 \ldots$ & $j = 1,2, \ldots$, N_{ij} is a normalization constant, J_i is a Bessel function of the first kind of the ith order and μ_{ij} is one of the infinite discrete set of eigenvalues. Out of these normal modes only the first, lowest frequency, antisymmetric mode $(i = j = 1)$ is the dominant mode.

Thus, based on these considerations we take ϕ_0 in the form

$$ \phi_0 = \left(\dot{q}_S(t) \;\text{Sinh}\; kz + \dot{q}_C(t)\;\text{Cosh}\; kz\right) \times J_1(kr) \times \cos\theta \tag{3.3} $$

in which q_S , q_C are generalised coordinates that are functions of time to be determined.

To find the value of the constant k we substitute Eq.(3.3) into Eq.(2.2). This gives

$$ J_1'(\mu_{11}) = 0 \tag{3.4} $$

in which $k = |\mu_{11}| = 1.841$.

To satisfy the conditions (2.3a & b) , we expand the constant A in a Dini series (Cf. Watson [5], Chap. 18) in the interval $[0,1]$. This series may take the form

$$ A = k\, C\, J_1(kr) + \sum_{i=2}^{\infty} D_i\, J_1(\mu_i r) \tag{3.5} $$

in which the constants μ_i are the positive roots of $J_1'(\mu_{1i})=0$ with k the smallest one.

Substituting Eq.(3.5) into Eqs.(2.3 a & b) yields

$$\dot{q}_S \;\text{Cosh}\; kd \;-\; \dot{q}_C \;\text{Sinh}\; kd \;=\; C \sin \Omega t \tag{3.6}$$

and

$$D_i = 0 \tag{3.7}$$

The constant C is given by

$$C = \frac{2A}{(k^2-1)\; J_1^2(k)}\; I \tag{3.8}$$

I is a Bessel correlation integral defined by (Cf. Mclachlan [6] , p.194)

$$\begin{aligned} I &= \int_0^{\beta} kr\; J_1(kr)\; dr \\ &= \pi\beta/2 \left[J_2(k\beta)\; H_1(k\beta) - J_1(k\beta)\; H_2(k\beta) \right] \\ &\quad + \frac{k}{3}\beta^2 \quad J_1(k\beta) \end{aligned} \tag{3.9}$$

in which H_1 and H_2 are Struve Functions of order one and order two respectively.

Integrating Eq.(3.6) with respect to time and taking the integration constant to be zero, yield

$$q_C(t) = q_S(t)\;\text{Coth}\; kd + \frac{C}{\Omega}\;\text{Csch}\; kd \cos \Omega t \tag{3.10}$$

Substituting Eq.(3.3) into Eq.(3.1 a) ,integrating with respect to time and also putting the integration constant equal to zero, give

$$\eta_o = k\, J_1(kr) \cdot \cos\theta \cdot q_S(t) \tag{3.11}$$

Substituting further Eq.(3.11) into Eq.(3.1 b) and eliminating $q_C(t)$ via Eq.(3.10) , we get

$$\ddot{q}_S + k\;\text{Tanh}\; kd \cdot q_S = \Omega C\;\text{Sech}\; kd \;\cos \Omega t \tag{3.12}$$

Writing $\omega^2 = k\;\text{Tanh}\; kd$, the steady state solution of Eq.(3.12) is

$$q_S(t) = \frac{\Omega C}{\omega^2 - \Omega^2} \text{ Sech } kd \cdot \cos \Omega t \tag{3.13}$$

Thus, the velocity-potential function and the free surface elevation in the linear regime are

$$\phi_0 = \frac{C}{\Omega^2 - \omega^2} \sin \Omega t \left[\Omega^2 \text{ Sech } kd \cdot \text{Sinh } kz + \omega^2 \text{ Csch } kd \times \cosh kz \right] J_1(kr) \cdot \cos\theta \tag{3.14}$$

and

$$\eta_0 = \frac{k C \Omega}{\omega^2 - \Omega^2} \text{ Sech } kd \cdot \cos \Omega t \cdot J_1(kr) \cdot \cos\theta \tag{3.15}$$

Eqs.(3.14 & 15) agree exactly with the results of Buhta and Yeh [1] .

4. SECOND ORDER SOLUTION

We now assume that ϕ and η possess first derivatives with respect to the parameter ε at $\varepsilon = 0$, and we denote them by ϕ_1 and η_1 . Then we differentiate Eqs.(2.1-6) with respect to ε and let ε tend to zero. Regarding the free surface equations namely, Eqs.(2.4 & 6) we utilise the relation

$$\frac{d}{d\varepsilon} \phi(r,\theta,\varepsilon\eta,t,\varepsilon) = \left[\frac{\partial}{\partial \varepsilon} + \left(\eta + \varepsilon\frac{\partial \eta}{\partial \varepsilon}\right)\frac{\partial}{\partial z}\right]\phi \tag{4.1}$$

Now, the kinematic condition Eq.(2.4) becomes

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial z \partial \varepsilon} + \left(\eta + \varepsilon\frac{\partial \eta}{\partial \varepsilon}\right)\frac{\partial^2 \phi}{\partial z^2} = {} & \frac{\partial^2 \eta}{\partial t \partial \varepsilon} + \frac{\partial \phi}{\partial r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}\frac{\partial \eta}{\partial \theta} + \\
& + \varepsilon \left\{ \frac{\partial^2 \eta}{\partial r \partial \varepsilon}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}\frac{\partial^2 \eta}{\partial \theta \partial \varepsilon} + \right. \\
& + \frac{\partial \eta}{\partial r}\left[\frac{\partial^2 \phi}{\partial r \partial \varepsilon} + \left(\eta + \varepsilon\frac{\partial \eta}{\partial \varepsilon}\right)\frac{\partial^2 \phi}{\partial r \partial z}\right] + \\
& \left. + \frac{1}{r^2}\frac{\partial \eta}{\partial \theta}\left[\frac{\partial^2 \phi}{\partial \theta \partial \varepsilon} + \left(\eta + \varepsilon\frac{\partial \eta}{\partial \varepsilon}\right)\frac{\partial^2 \phi}{\partial \theta \partial z}\right]\right\}
\end{aligned}$$

$$\text{on } z = \varepsilon\eta(r,\theta,t) \tag{4.2}$$

and also, the dynamic condition is

$$\frac{\partial \eta}{\partial \varepsilon} + \frac{\partial^2 \phi}{\partial t\, \partial \varepsilon} + \left(\eta + \varepsilon \frac{\partial \eta}{\partial \varepsilon}\right) \frac{\partial^2 \phi}{\partial t\, \partial z} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \phi}{\partial \theta}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right]$$
$$+ \varepsilon \frac{\partial \phi}{\partial r}\left[\frac{\partial^2 \phi}{\partial r\, \partial \varepsilon} + \left(\eta + \varepsilon \frac{\partial \eta}{\partial \varepsilon}\right)\frac{\partial^2 \phi}{\partial r\, \partial z}\right] + \frac{\varepsilon}{r^2}\frac{\partial \phi}{\partial \theta}\left[\frac{\partial^2 \phi}{\partial \theta\, \partial \varepsilon} + \left(\eta + \varepsilon \frac{\partial \eta}{\partial \varepsilon}\right)\frac{\partial^2 \phi}{\partial \theta\, \partial z}\right]$$
$$+ \varepsilon \frac{\partial \phi}{\partial z}\left[\frac{\partial^2 \phi}{\partial z\, \partial \varepsilon} + \left(\eta + \varepsilon \frac{\partial \eta}{\partial \varepsilon}\right)\frac{\partial^2 \phi}{\partial z^2}\right] = 0 \quad \text{on } z = \varepsilon \eta(r, \theta; t) \qquad (4.3)$$

By setting $\varepsilon = 0$ in Eqs.(4.2&3) , we obtain

$$\frac{\partial \phi_1}{\partial z} - \frac{\partial \eta_1}{\partial t} = \frac{\partial \phi_0}{\partial r}\frac{\partial \eta_0}{\partial r} + \frac{1}{r^2}\frac{\partial \phi_0}{\partial \theta}\frac{\partial \eta_0}{\partial \theta} - \eta_0 \frac{\partial^2 \phi_0}{\partial z^2} \quad \text{on } z=0 \qquad (4.4)$$

and

$$\frac{\partial \phi_1}{\partial t} + \eta_1 = -\eta_0 \frac{\partial^2 \phi_0}{\partial t\, \partial z} - \frac{1}{2}\left[\left(\frac{\partial \phi_0}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \phi_0}{\partial \theta}\right)^2 + \left(\frac{\partial \phi_0}{\partial z}\right)^2\right] \quad \text{on } z=0 \qquad (4.5)$$

To solve the differential Eqs.(4.4&5) , we first insert into R.H.S. of these equations the first order quantities given by Eqs.(3.14&15). Upon doing this and simplifying the results , we obtain

$$\frac{\partial \phi_1}{\partial z} - \frac{\partial \eta_1}{\partial t} = A_1(t)\left[J_0^2(kr) + 2\, J_1^2(kr) + J_2^2(kr)\right] + $$
$$+ A_2(t)\left[J_1^2(kr) - J_0(kr)\, J_2(kr)\right] \cos 2\theta$$
$$\text{on } z = 0 \qquad (4.6)$$

and

$$\frac{\partial \phi_1}{\partial t} + \eta_1 = A_3(t)\left[J_0^2(kr) + J_2^2(kr)\right] + A_4(t)\, J_1^2(kr) +$$
$$+ \left[A_5(t)\, J_0(kr)\, J_2(kr) + A_6(t)\, J_1^2(kr)\right] \cos 2\theta$$
$$\text{on } z = 0 \qquad (4.7)$$

where

$$A_1(t) = -\frac{k^3 c^2 \omega^2 \Omega}{4\,(\omega^2 - \Omega^2)^2} \,\text{Csch}\, 2kd \,.\, \sin 2\Omega t \qquad (4.8a)$$

$$A_2(t) = 2\, A_1(t) \qquad (4.8b)$$

$$A_3(t) = -\frac{k^2 c^2 \omega^4 \,\text{Csch}^2\, kd}{8\,(\omega^2 - \Omega^2)^2}\left(1 - \cos 2\Omega t\right) \qquad (4.8c)$$

$$A_4(t) = \frac{k^2 c^2 \Omega^4}{2\,(\omega^2 - \Omega^2)^2} \,\text{Sech}^2\, kd \,.\, \cos 2\Omega t \qquad (4.8d)$$

$$A_5(t) = -2\, A_3(t) \qquad (4.8e)$$

$$A_6(t) = -\frac{k^2C^2\Omega^4}{4(\omega^2-\Omega^2)^2}\ \text{Sech}^2\ kd\ (1-\cos 2\Omega t) \tag{4.8f}$$

It can be easily observed that Eqs.(4.6 & 7) are dominated by the two successive symmetric normal modes ${}^{C}\Phi_{01}$ and ${}^{C}\Phi^{c}_{21}$. Thus, the plausible form of ϕ_1 may be written as

$$\begin{aligned}\phi_1 &= \dot{P}_{01}(t)\cdot \text{Cosh}[\mu_{01}(z+d)]\cdot J_o(\mu_{01}\ r) \\ &+ \dot{P}_{21}(t)\cdot \text{Cosh}[\mu_{21}(z+d)]\cdot J_2(\mu_{21}\ r)\cdot \cos 2\theta \end{aligned} \tag{4.9}$$

Here, also, P_{01} and P_{21} are functions of time to be determined. Now, we expand the terms of R.H.S. of Eq.(4.6) that are independent of θ into a Dini series of the form $J_o(\mu_{0j}\ r)$ and retaining only the first term of the series since all the other terms are very small because they are not dominant modes. Repeating the same procedure for the terms of Eq.(4.6) that vary with cos 2θ and substituting for ϕ_1 into Eq. 4.6 ,lead to

$$\begin{aligned}\frac{\partial \eta_1}{\partial t} &= \mu_{01}\left[\dot{P}_{01}\ \text{Sinh}(\mu_{01}d) - B_1\ A_1(t)\right] J_o(\mu_{01}r) + \\ &+ \mu_{21}\left[\dot{P}_{21}\ \text{Sinh}(\mu_{21}d) - B_2\ A_2(t)\right] J_2(\mu_{21}r)\cdot\cos 2\theta \end{aligned} \tag{4.10}$$

The constants B_1 and B_2 are

$$B_1 = 2\ I_1 + I_2 \tag{4.11a}$$

$$B_2 = I_3 - I_4 \tag{4.11b}$$

in which I_1 , I_2 , I_3 and I_4 are correlation integrals (see the appendix) .

Also, for Eq.(4.7) we expand every term of its R.H.S. into the appropriate Dini series, integrating Eq.(4.10) with respect to time, substituting for ϕ_1 and η_1 and simplifying the results, we get

$$\begin{aligned}\ddot{P}_{01} + \omega^2_{01}\ P_{01} &= \mu_{01}\ \text{Sech}(\mu_{01}d)\left[B_1\ \overset{*}{A}_1(t) + B_3\ A_3(t) + \right. \\ &\left. + B_4\ A_4(t)\right] \end{aligned} \tag{4.12}$$

and

$$\ddot{P}_{21} + \omega_{21}^{2} P_{21} = \mu_{21} \operatorname{Sech}(\mu_{21} d) \left[B_2 \overset{*}{A}_2(t) + B_5 A_5(t) + \right.$$
$$\left. + B_6 A_6(t) \right] \quad (4.13)$$

where

$$\omega_{01}^{2} = \mu_{01} \operatorname{Tanh}(\mu_{01} d) \quad (4.14a)$$

$$\omega_{21}^{2} = \mu_{21} \operatorname{Tanh}(\mu_{21} d) \quad (4.14b)$$

$$\overset{*}{A}_{1,2}(t) = \int A_{1,2}(t)\, dt \quad (4.14c)$$

and the constants

$B_3 = I_2$, $B_4 = I_I$, $B_5 = I_4$ and $B_6 = I_3$ are defined in terms of the correlation integrals (see the appendix) .

Solving for the steady solution of Eqs.(4.12&13) and substituting into Eqs.(4.9) and (4.10), we obtain

$$\phi_1 = \frac{\mu_{01} k^2 C^2 \operatorname{Sech}(\mu_{01} d)}{4(\omega^2 - \Omega^2)^2 (4\Omega^2 - \omega_{01}^2)} \left[B_1 k \omega^2 \operatorname{Csch}(2kd) + \right.$$
$$\left. + B_3 \omega^4 \operatorname{Csch}^2 kd + 4 B_4 \Omega^4 \operatorname{Sech}^2 kd \right] \times \sin 2\Omega t$$
$$\times \operatorname{Cosh}\left[\mu_{01}(z+d)\right] \cdot J_o(\mu_{01} r) + \frac{\mu_{21} k^2 C^2 \operatorname{Sech}(\mu_{21} d)}{2(\omega^2 - \Omega^2)^2 (4\Omega^2 - \omega_{21}^2)} \times$$
$$\times \left[B_2 k \omega^2 \operatorname{Csch} 2kd - B_5 \omega^4 \operatorname{Csch}^2 kd + B_6 \operatorname{Sech}^2 kd \right]$$
$$\times \sin 2\Omega t \cdot \operatorname{Cosh}\left[\mu_{21}(z+d)\right] \cdot J_2(\mu_{21} r) \cdot \cos 2\theta \quad (4.15)$$

and

$$\eta_1 = \frac{\mu_{01}^2 k^2 C^2 \operatorname{Tanh}(\mu_{01} d)}{8(\omega^2 - \Omega^2)^2 (\omega_{01}^2 - 4\Omega^2)} \left[B_3 \omega^4 \operatorname{Csch}^2 kd + 4 B_4 \Omega^4 \right.$$
$$\left. \operatorname{Sech}^2 kd \right] \cos 2\Omega t \cdot J_o(\mu_{01} r) - \frac{\mu_{21}^2 k^2 C^2 \operatorname{Tanh}(\mu_{21} d)}{4(\omega^2 - \Omega^2)^2 (\omega_{01}^2 - 4\Omega^2)}$$
$$\left[B_5 \omega^4 \operatorname{Csch}^2 kd - B_6 \Omega^4 \operatorname{Sech}^2 kd \right] \cos 2\Omega t$$
$$\times J_2(\mu_{21} r) \cdot \cos 2\theta - \frac{\mu_{01}^2 k^2 C^2 \omega^4 B_3 \operatorname{Tanh}(\mu_{01} d)}{8(\omega^2 - \Omega^2)^2 \quad \omega_{01}^2}$$

$$\mathrm{Csch}^2\, kd \quad J_o(\mu_{01} r) \;+\; \frac{\mu_{21}^2 k^2 C^2\, \mathrm{Tanh}(\mu_{21} d)}{4\,(\omega^2 - \Omega^2)^2\, \omega_{21}^2}\; x$$
$$x \left[B_5\, \omega^4\, \mathrm{Csch}^2\, kd \;-\; B_6\, \Omega^4\, \mathrm{Sech}^2\, kd \right] J_2(\mu_{21} r) . \cos 2\theta \quad (4.16)$$

It may be of interest to observe that Eqs.(4.15 & 16) reveal that the motion is stable provided that $\Omega \neq \omega \neq \omega_{21}/2$.

5. DISCUSSION AND CONCLUSIONS

Based on the hypothesis of potential flow,a theoretical study has been done on the slosh of a liquid in a stationary circular tank due to a central pulsating source of a circular shape. Through the application of Dini expansions,the exact nonlinear governing equations of the phenomenon are solved by a perturbation procedure which has been carried out through the second approximation. No a priori limitations have been made on the depth of the liquid in the tank.

In the present study,the application of the exact nonlinear free surface boundary conditions to the slosh problem due to a pulsating source reveals certain observations illustrating how the characteristics of liquid slosh in the nonlinear regime differ from those of a linear one. For example, Eq.(3.15) indicates that the free surface elevation η_o in the linear regime vanishes twice during each period (i.e., the free surface is horizontal) at $\Omega t = \frac{\pi}{2}$ and $\Omega t = \frac{3\pi}{2}$. Also,Eq.(3.15) shows the existence of a one nodal diameter at the azimuth $\theta = \frac{\pi}{2}$ and a nodal circle at the radius r = 0.719. The crests and troughs are identical in shape; that is $\eta_o(r,0;\Omega t) = -\eta_o(r,\pi;\Omega t)$.

In contrast to the linear slosh waves,the finite waves do not possesss these characteristics. The finite Taylor expansions of $\varepsilon\eta$ and $\varepsilon\phi$ are

$$\varepsilon\eta \;=\; \varepsilon\,\eta_o(r,\theta;t) + \varepsilon^2\,\eta_1(r,\theta;t) + O(\varepsilon^3) \qquad (5.1)$$

$$\varepsilon\phi \;=\; \varepsilon\,\phi_o(r,\theta,z;t) + \varepsilon^2\,\phi_1(r,\theta,z;t) + O(\varepsilon^3) \qquad (5.2)$$

Eq.(4.16) shows that η_1 contains a non-constant term which is independent of time. Thus, from Eqs.(3.15),(4.16) and (5.1), we can easily see that the free surface of the liquid is never horizontal during the slosh motion. Also, these equations provide the nonexistence of any nodal diameter or circle. Regarding velocity distributions through the liquid, we can clearly notice from Eqs.(3.14),(4.15) and (5.2) that during each period, theliquid is everywhere twice momentarily at rest at the times $\Omega t = 0$ and $\Omega t = \pi$. Thus, at these times each part of the free surface is either at its highest or lowest position.

Finally, the present study is directly applicable to the design of many hydraulic systems subjected to a pump fluctuating pressure. By properly assigning the parameters of the pulsating source and integrating the slosh pressure on the tank walls, the total force exerted on the system due to the liquid motion is known.

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APPENDIX - Correlation Integrals

The following integrals appear in the text (Eqs. 4.11):

$$I_1 = \frac{2}{\mu_{01}\, J_0^2(\mu_{01})} \int_0^1 r\, J_1^2(kr)\, J_0(\mu_{01} r)\, dr \tag{A.1}$$

$$I_2 = \frac{2}{\mu_{01}\, J_0^2(\mu_{01})} \int_0^1 r \left[J_0^2(kr) + J_2^2(kr)\right] J_0(\mu_{01} r)\, dr \tag{A.2}$$

$$I_3 = \frac{2\mu_{21}}{(\mu_{21}^2 - 4)\, J_2^2(\mu_{21})} \int_0^1 r\, J_1^2(kr)\, J_2(\mu_{21} r)\, dr \tag{A.3}$$

$$I_4 = \frac{2\mu_{21}}{(\mu_{21}^2 - 4)\, J_2^2(\mu_{21})} \int_0^1 r\, J_0(kr)\, J_2(kr)\, J_2(\mu_{21} r)\, dr \tag{A.4}$$

Due to space limitations, we show a procedure for calculating the Correlation Integral "I_1". The other correlation integrals can be evaluated in a similar manner.

The data of the problem indicates that $k=1.841$, $\mu_{01}=3.832$, $J_0(3.832) = -0.4024$. From Ref. [7], we approximate $J_0(x)$ and $J_1(x)$ as :

$$J_0(x) = 1 - 2.25\,(x/3)^2 + 1.266\,(x/3)^4 - 0.316\,(x/3)^6 + 0.444\,(x/3)^8 + o(x^9) \tag{A.5}$$

$$x^{-1} J_1(x) = 1/2 - 0.562\,(x/3)^2 + 0.211\,(x/3)^4 - 0.04\,(x/3)^6 + 0.004\,(x/3)^8 + o\,(x^9) \qquad (A.6)$$

Substituting for $J_o(\mu_{01} r)$ from (A.5) and for $J_1(kr)$ from (A.6) , carrying out the integration term by term and simplifying , we get $I_1 = -0.064$.