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A COMPUTATIONAL PROCEDURE FOR MULTIVARIABLE STATE FEEDBACK ROBUST CONTROLLER DESIGN

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ABSTRACT

This paper describes a procedure for multivariable state feedback robust controller design. The plant in state space is given by the operational point description or in terms of a vector of slow varying physical parameters. Through solution of the Sylvester matrix equation, a nonunique static feedback controller, which assigns the prespecified closed-loop spectrum, is calculated. In addition, all the remaining feedback degrees of freedom are utilized to optimize a multiobjective function that reflects further design properties. The robust feedback gains is calculated through a three-phase computational algorithm. Numerical examples show that under the robust state feedback control, the closed-loop systems can both achieve satisfied transient characteristics and greatly reduce state trajectory sensitivity to small or large parameter variations in the plant. The proposed procedure is still applied to a VTOL aircraft model.

1. INTRODUCTION

The plant in state space is given by the operational point description corresponding to (r+1) parameters

$$\Sigma_j : \underset{n\times 1}{\dot{X}_j} = \underset{n\times n}{A_j}\, X_j + \underset{n\times m}{B_j}\, \underset{m\times 1}{U_j}$$

$$\underset{p\times 1}{Y_j} = \underset{p\times n}{C_j}\, X_j \qquad j=0,1,\ldots,r. \qquad (1)$$

where all matrices are real and the triplet $\{A_0, B_0, C_0\}$ is defined as the nominal system and is denoted by Σ_0. An alternative form of the plant is

$$\underset{n\times 1}{\dot{X}} = \underset{n\times n}{A(\alpha)}\, X + \underset{n\times m}{B(\alpha)}\, \underset{m\times 1}{U}$$

$$\underset{p\times 1}{Y} = \underset{p\times n}{C(\alpha)}\, X \qquad (2)$$

where the vector $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_r]^T$ represents slow varying physical

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parameters, while α_0 is the nominal value of α and the triplet $\{\mathbf{A}(\alpha_0), \mathbf{B}(\alpha_0), \mathbf{C}(\alpha_0)\}$ is defined as the nominal system Σ_0.

Geometric linear multivariable control theory [1] shows that for a multi-input state feedback assignment of $(\mathbf{A}+\mathbf{BF})$, where $\mathbf{F}$ is the feedback gains to be determined, only partially meets typical requirements. An interesting problem is how to utilize remaining freedom for choice of $\mathbf{F}$ to satisfy further desired design properties. These relate -for instance- to response shaping or parameter sensitivity. In papers [2] and [3], a significant method to assign insensitive closed-loop eigenvalues was posed. But design lost many freedom degrees in choosing $\mathbf{F}$, and needed coordinate transformations, which led to a complex computation.

Using an approach different from papers [3],[4] and [5], this paper develops a particular design procedure for robust state feedback.

2. STATEMENT OF THE PROBLEM

In this section, we give a detailed statement of the problem including the eigenstructure assignment selection and the control problem statement.

2.1 EIGENSTRUCTURE ASSIGNMENT SELECTION

Assuming that the nominal system Σ_0 is completely controllable, consider the matrix equation

$$\mathbf{A}\,\mathbf{V}_0 - \mathbf{V}\tilde{\mathbf{A}} = -\,\mathbf{B}_0\mathbf{U} \tag{3}$$

where $\mathbf{V}$ is the n-by-n solution matrix, $\mathbf{U}$ is the m-by-n control matrix and $\tilde{\mathbf{A}} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with λ_i's are a specified set of desired closed-loop eigenvalues that take different real values. We say that equation(3) has a unique solution if $\sigma(\mathbf{A}_0) \neq \sigma(\tilde{\mathbf{A}})$ where $\sigma(.)$ denotes the spectrum of the matrix. In addition, for almost any control matrices, the corresponding solution matrices $\mathbf{V} = \mathbf{V}(\mathbf{U})$ are nonsingular [6]. Substituting $\mathbf{U} = \mathbf{FV}$, where $\mathbf{F}$ is the feedback gains to be determined, into equation(3), we get

$$\mathbf{V}^{-1}(\mathbf{A}_0 + \mathbf{B}_0\mathbf{F})\mathbf{V} = \tilde{\mathbf{A}} \tag{4}$$

If $\mathbf{V}^{-1}$ exists, hence $\mathbf{v}_i$ - the ith column of $\mathbf{V}$ - is a right-hand closed-loop eigenvector corresponding to λ_i. The feedback gains $\mathbf{F}$ is calculated from

$$\mathbf{F} = \mathbf{U}\,\mathbf{V}^{-1} \tag{5}$$

2.2 CONTROL PROBLEM STATEMENT

As equation(5) is valid for almost any control matrix $\mathbf{U}$, we expect to choose a special $\tilde{\mathbf{U}}$ from the field of m-by-n real matrices to determine $\mathbf{F} = \tilde{\mathbf{U}}\mathbf{V}^{-1}$. When such a feedback control law

$$\mathbf{U}_j = \mathbf{F}\,\mathbf{x}_j(t) \quad , \; j=0,1,\ldots,r. \tag{6}$$

is applied to the plant(1), its nominal system Σ_0 has the desired closed-loop spectrum and its nominal state $x_0(t)$ will have satisfied transient characteristics.

Once plant(1) works on other operational points with subscript $j\neq 0$, we still require all the state trajectories $X_j(t)$ deviate from $x_0(t)$ as small as possible.

3. THE MULTIOBJECTIVE COST FUNCTION

For a closed loop system (A_0+B_0F, B_0, C_0), its state vector $X_0(t)$ can be represented -using the spectrum expansion- as

$$X_0(t) = \sum_{i=1}^{n} v_i(w_i^T x_0(0)) \exp(\lambda_i t) \quad , \quad w_i^T v_j = 1 \quad i=j; \quad = 0 \quad i\neq j \tag{7}$$

where $\lambda_i \in \sigma(A_0 + B_0F)$, w_i^T, v_i are the left and right-hand eigenvectors of $(A_0 + B_0F)$ and $x_0(0)$ is an initial state. From (7), the sensitivity of $X_0(t)$ to parameter variations is estimated by the sensitivity of $\{\lambda_i, w_i^T, v_i\}$. When the operational point in plant(1) changes from a subscript j=0 to j=1, the new triplet (A_1, B_1, C_1) has the following representation:

$$A_1 = A_0 + \delta A_0 \quad , \quad B_1 = B_0 + \delta B_0 \quad , \quad C_1 = C_0 + \delta C_0$$

where

$$\delta A_0 = A_1 - A_0 \quad , \quad \delta B_0 = B_1 - B_0 \quad , \quad \delta C_0 = C_1 - C_0 .$$

Let us define the perturbation matrix P:

$$P = W\,\delta\hat{A}\,V = \begin{bmatrix} \beta_{11} & & \beta_{1n} \\ \beta_{n1} & & \beta_{nn} \end{bmatrix} \quad \text{with} \quad \beta_{ij} = w_i^T\,\delta\hat{A}\,v_j \tag{8}$$

where $W = V^{-1}$ and $\delta\hat{A} = \delta A_0 - \delta B_0 F$

In [7], it was shown that the closed-loop eigenvalue sensitivity to parameter perturbation $\delta\hat{A}$ is

$$\delta\lambda_i = \beta_{ii} = w_i^T\,\delta\hat{A}\,v_i \qquad i\leqslant n \tag{9}$$

and eigenvector sensitivity is

$$\delta w_i^T = \sum_{i\neq j}^{n}\{\beta_{ij}/(\lambda_j - \lambda_i)\}w_j^T \quad \text{and} \quad \delta v_i = \sum_{i\neq j}^{n}\{\beta_{ji}/(\lambda_j - \lambda_i)\}v_j \ , \ i\leqslant n \tag{10}$$

3.1 MAIN RESULTS

Based on the sensitivity measures (9) and (10), a useful approximate relation between parameter sensitivity and response shaping is now derived. Defining

$$S_p = \sum_{i=1}^{n} \| w_i \| . \| v_i \| \tag{11}$$

under the assumption of $|\lambda_j - \lambda_i| \geqslant 1$, for $j\neq i$ and $\| w_i \| = \| v_i \|$ which also implies $\| w_i \| \geqslant 1$ and $\| v_i \| \geqslant 1$ since

$$1 = w_i^T v_i = \| w_i \| \| v_i \| \cos\theta_i$$

Applying the norm inequality property to (9) and (10) and doing some little calculations, we get an upper bound to eigenvalue and eigenvector sensitivity as follows:

$$\frac{|\delta\lambda_i| + \|\delta \mathbf{w}_i\|}{\|\delta\hat{\mathbf{A}}\|} \leqslant S_p \|\mathbf{w}_i\| \; ; \; \frac{|\delta\lambda_i| + \|\delta \mathbf{v}_i\|}{\|\delta\hat{\mathbf{A}}\|} \leqslant S_p \|\mathbf{v}_i\| \qquad (12)$$

even for some $j \neq i$ and $|\lambda_j - \lambda_i| < 1$, the upper bounds (12) remain valid if these bounds are multiplied by a scalar quantity "c" where

$$c = 1/\min|\lambda_j - \lambda_i| \quad , \; j \neq i. \qquad (13)$$

With equation(7), another norm inequality shows that

$$\frac{\|x_0(t)\|}{\|x_0(0)\|} \leqslant S_p \max_i e^{\lambda_i t} \qquad (14)$$

We notice that the inequalities in (12) and (14) include an important property that minimizing S_p. By this property, we can both reduce state trajectory sensitivity to parameter perturbations (whether they are known or not) and limit state response magnitude or overshoot in a norm sense. On the other hand, from (9) and (10) with well-separated closed-loop eigenvalues, minimizing $|\beta_{ij}|$'s also decreases trajectory sensitivity. Especially when the ith row and column of the perturbation matrix $\mathbf{P}$ tends to zero.i.e.,

$$[\beta_{i1}, \beta_{i2}, \ldots, \beta_{in}] = \mathbf{O} \quad \text{and} \quad [\beta_{1i}, \beta_{2i}, \ldots, \beta_{ni}] = \mathbf{O} \qquad (15)$$

there follows

$$\delta\lambda_i = 0 \quad , \quad \delta\mathbf{w}_i^T = \mathbf{O} \quad \text{and} \quad \delta\mathbf{v}_i = \mathbf{O}.$$

This result can be even extended to the case of larger parameter variations of $\lambda\hat{\mathbf{A}}$ by the following two theorems:

Theorem(1) For any finite or larger parameter variations of $\delta\mathbf{A}$, each $\tilde{\lambda} \in \sigma(\mathbf{A}_0+\mathbf{B}_0\mathbf{F}+\delta\mathbf{A})$ lies in the least one of the circular discs centered at λ_i (see[7]), i.e.,

$$|\tilde{\lambda} - \lambda_i| \leqslant \sum_{j=1}^{n}|\beta_{ji}| \quad \text{or} \quad |\tilde{\lambda} - \lambda_i| \leqslant \sum_{j=1}^{n}|\beta_{ij}| \quad , \; i \leqslant n \qquad (16)$$

Theorem(2) For any finite or larger $\delta\hat{\mathbf{A}}$, let $\lambda_i \in \sigma(\mathbf{A}_0+\mathbf{B}_0\mathbf{F}+\delta\mathbf{A})$, $\tilde{\mathbf{w}}_i^T$ and $\tilde{\mathbf{v}}_i$ be the ith eigenvectors of $(\mathbf{A}_0+\mathbf{B}_0\mathbf{F}+\delta\hat{\mathbf{A}})$, if the condition (15) is satisfied, then

$$\tilde{\lambda}_i = \lambda_i \quad , \quad \tilde{\mathbf{w}}_i^T = \mathbf{w}_i^T \quad \text{and} \quad \tilde{\mathbf{v}}_i = \mathbf{v}_i$$

In other words, the ith mode in expansion(7) is invariant under variation $\delta\hat{\mathbf{A}}$. Since $[\beta_{i1}, \beta_{i2}, \ldots, \beta_{in}] = \mathbf{w}_i^T\delta\hat{\mathbf{A}}\,\mathbf{V} = \mathbf{O}$ iff $\mathbf{w}_i^T\,\delta\hat{\mathbf{A}} = \mathbf{O}$, hence the condition(15) holds and consequently,

$$\lambda_i w_i^T = w_i^T(A_0 + B_0F) = w_i^T(A_0 + B_0F) + w_i^T \delta\hat{A} = w_i^T(A_0 + B_0F + \delta\hat{A})$$

The same reason is for $\tilde{V}_i = V_i$, as another result of (15), the radius of the ith circular disc in (16) also becomes zero.

3.2 A MULTIOBJECTIVE FUNCTION

In order to solve the control problem posed in section(2), we set the multi-objective cost function

$$J = J_1 + \tau J_2 \tag{17}$$

where τ is a weighting factor; J_1 and J_2 are defined as

$$J_1 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij}^2 = \frac{1}{2}\,tr(P^TP) \tag{18}$$

$$J_2 = \frac{1}{2}\sum_{i=1}^{n}\{\| w_i \| + \| v_i \|\} = \frac{1}{2}\,tr(V^TV + WW^T) \tag{19}$$

Another general form of J_1 is

$$J_1 = \frac{1}{2}\,tr(G_1P^TG_2P) \tag{20}$$

where $G_1 = diag(f_i)$, $G_2 = diag(g_i)$ are positive weighting matrices and f_i, g_i are weighting factors which force the values of the ith column and row of P to become very small during minimization of J_1. Minimization of J_2 produces a small S_p which, as stated before, limits the state magnitude and reduces the trajectory sensitivity. Furthermore, since $F = UW$, a small value for the norm of W ($\|W\|$) in J_2 usually means small feedback gains, hence J_2 plays a role which is similar to that the weighting matrices R and Q plays in Linear Optimal Control Problem [8]

$$J_0 = \int_0^{\infty}(x^TQ\,x + u^TR\,u)\,dt \tag{21}$$

where increasing Q can limit state magnitude and reduce trajectory sensitivity and the matrix R usually results in small feedback gains.
The value of J depends on how to select a closed-loop eigenmatrix of V, or directly, how to choose a control matrix of U. So far the problem is simplified into minimizing $J = J(U)$ by optimal choice of a control matrix from the set of all m-by-n real matrices.

4. THE PROPOSED COMPUTATIONAL PROCEDURE

In this section, all the results of the former section are extended to the general case of the plant(1) with $r \geqslant 1$.
Defining the ith perturbation matrix as

$$P_i = W\,\delta\hat{A}_i\,V \quad , \quad R_i = W\,\delta A_{0i}\,V \quad , \quad i \leqslant r. \tag{22}$$

with

$$\delta\hat{A}_i = \delta A_{0i} + \delta B_{0i}F \ , \quad \delta A_{0i} = A_i - A_0 \ , \quad \delta B_{0i} = B_i - B_0 \tag{23}$$

Taking the expanded multiobjective function:

$$J = \sum_{i=1}^{r} J_{1i} + r J_2 \qquad \text{with} \qquad J_{1i} = \frac{1}{2}\,tr(P_i^T P_i) \quad , \; i \leqslant r \qquad (24)$$

we compute the robust feedback gains in the following main three steps:

Step(1): Choose the desired closed-loop eigenvalues λ_i to meet some typical requirements, let $\tilde{A} = diag(\lambda_1 , \lambda_2 , \ldots, \lambda_n)$.

Step(2): Minimize the objective function $J(U)$ and make $\tilde{J} = J(\tilde{U}) < \varepsilon$, ε is a properly specified positive number. This problem is solved using the gradient search procedure [9]. Let

$$Q = \sum_{i=1}^{r} (P_i^T R_i - P_i P_i^T) + V^T V - W W^T \qquad (25)$$

The gradient of J with respect to U is computed this way:(assume $r = 1$)

$$\frac{\partial J}{\partial U} = B_0^T X^T + \sum_{i=1}^{r} (B_{0i}^T W^T P_i \quad , \quad \tilde{A}X - XA_0 = QW \qquad (26)$$

Step(3): When $J(\tilde{U}) < \varepsilon$ is satisfied, solve

$$A_0 V - V\tilde{A} = -B_0 \tilde{U} \qquad (27)$$

Step(4): Compute $F = \tilde{U}\, V^{-1}$ (28)

REMARKS AND DISCUSSION

1. In all the proposed steps, the design computational procedure is mainly based on the solution of Sylvester equation [10].

2. If a pair of complex eigenvalues $\lambda_{1,2} = a \pm ib$ is required, the computation procedure still keeps unchanged except for step(1), where we let

$$\tilde{A} = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ -b & a & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

3. When the plant is described by the alternative form (2), we can apply the same procedure. This is available when, in equations (22)-(26) A_0 and B_0 are replaced by $A(\alpha_0)$ and $B(\alpha_0)$ and the increments δA_{0i} , δB_{0i} by derivatives $\partial A/\partial\alpha_{i0}$, $\partial B/\partial\alpha_{i0}$ for $i = 1$ to r.

4. On choosing the closed-loop eigenvalues, it is possible to fix some leading eigenvalues, say $\lambda_1, \lambda_2, \ldots, \lambda_q$; $q < n$. The remainings may be allowed in the region $a_i < \lambda_i < b_i$ where $i = q+1, q+2, \ldots, n$ and $b_i \ll min(Re\lambda_j , j \leqslant q)$. Now optimizing $\tilde{J} = J(\tilde{U}, \tilde{\lambda}_{q+1}, \ldots, \tilde{\lambda}_n)$ with eigenvalue inequality constraints, we get much more freedom degrees to improve system properties, and the gradient of J with respect to λ_i, $i=q+1,\ldots,n$ are $\partial J/\partial\lambda_i = -x_i^T v_i$ where x_i^T is the ith row of the matrix X in (26).

5. NUMERICAL EXAMPLES

Example (1): Consider the unstable system

$$\dot{X}_j = A_j X_j + B_j U_j \quad , \; j = 0 \; , \; 1 \; , \; 2. \qquad \text{with}$$

$$A_j = \begin{bmatrix} 0 & 0 & a & 0 & -1 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & a & 0 & a & -1.0130 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix}$$

$$B_j = \begin{bmatrix} 0 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 0 \\ 4.419 & 0 & -1.6650 \\ 1.575 & 0 & -0.0732 \end{bmatrix}$$

This system works on three operational points (j=0,1,2.) on which elements a_{13}, a_{42}, a_{44} and b_{21} have different values , which are

	j=0	j=1	j=2
a_{13}	1.3200	0.5000	4.0000
a_{42}	0.0485	-0.5000	1.0000
a_{44}	-0.8556	-1.0000	0.2300
b_{21}	-0.1200	-0.3000	1.0000

while leaving the other elements unchanged. The desired closed-loop eigenvalues are

$$\lambda_{1,2} = -2 \pm i2 \quad , \quad \lambda_3 = -3 \quad , \quad \lambda_4 = -4 \quad , \quad \lambda_5 = -5$$

Following our proposed procedure, the robust controller $F = \tilde{U} V^{-1}$ is

$$F = \begin{bmatrix} 5.4325 & 0.2738 & 3.2012 & 0.3120 & 2.3313 \\ 2.4389 & -3.8878 & -1.3248 & -0.6497 & -0.2735 \\ 16.1894 & 0.8826 & 20.3753 & 4.1702 & -7.0554 \end{bmatrix}$$

Below is the table of eigenvalue sensitivity under this robust controller

	percent change in elements				percent change in locations of the eigenvalues of the closed-loop system				
	a_{13}	a_{42}	a_{44}	b_{21}	$\Delta\lambda_1$	$\Delta\lambda_2$	$\Delta\lambda_3$	$\Delta\lambda_4$	$\Delta\lambda_5$
j=1	56	1131	16.9	150	4.6	4.6	62.3	47.8	22.8
j=2	253	1962	127	933	2.7	2.7	12.2	6.7	6.7

Example (2): Consider the system

$$\dot{X} = A(\alpha)\ X + B(\alpha)\ U \qquad \text{where}$$

$$A(\alpha) = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2/\alpha & 1/\alpha \\ 1 & 1 & -2 \end{bmatrix} , \quad B(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 1/\alpha \\ 0 & 0 \end{bmatrix}$$

with $\alpha_0 = 1$ and the desired closed-loop eigenvalues are

$$\lambda_1 = -1 \quad , \quad \lambda_2 = -1.2 \quad , \quad \lambda_3 = -3$$

The leading eigenvalue λ_1 is expected to have zero sensitivity, therefore the weighting matrices G_1 and G_2 are $G_1 = G_2 = \mathrm{diag}(100,1,1)$.
An initial objective function value $J(U_0) = 10445$ with an arbitrary chosen U_0.
The corresponding controller is

$$F_0 = U_0 V^{-1} = \begin{bmatrix} 2.96 & 1.44 & -2.44 \\ -4.94 & -2.16 & 2.16 \end{bmatrix}$$

After minimization of J, then $J(\tilde{U}) = 4.5$ and the robust controller $F = \tilde{U}V^{-1}$ is

$$F = \begin{bmatrix} 0.0737 & -0.9255 & -0.0736 \\ -0.0680 & 0.7263 & -0.0357 \end{bmatrix}$$

The related perturbation matrix follows

$$P = \begin{bmatrix} 0.00 & 0.00 & 0.00 \\ 0.01 & 1.15 & 0.19 \\ 0.00 & 0.72 & 0.12 \end{bmatrix}$$

Since the first row and column of P nearly become zero, the mode

$$v_1 (w_1^T x_0(0)) e^{\lambda_1 t}$$

in expansion (7) has zero sensitivity to parameter α_0 when perturbed +20%, the closed-loop eigenvalues are

	λ_1	λ_2	λ_3
$\alpha_0 = 1$	-1	-1.2000	-3.0000
$\Delta\alpha_0 = +20\%$	-1	-1.3267	-3.0147
$\Delta\alpha_0 = -20\%$	-1	-1.0948	-2.9894

It is found that the closed-loop eigenvalues are insensitive to perturbed parameter α. Another advantage is the feedback gains of F are smaller than those of the arbitrary controller F_0.

6. CONCLUSION

An optimal design procedure for control systems with good transient characteristics and robustness is described. In comparison with the robust suboptimal LQR design, it wins an advantage over [5] in aspect of free assignment of closed-loop spectrum while keeping the other advantages that developed in [5] like limitation of overshoots, control of energy and reduction of sensitivity. Instead of solution of the Reccati equation, here solves the Sylvester equation.

The procedure is still applied to a VTOL aircraft model, which was ever designed in [2], but the design results are more satisfied than those given in [2]. Still another suggestion for partially assignment of closed-loop spectrum is posed in the paper. Besides, based on sensitivity measures, a useful approximate relation between parameter sensitivity and response shaping was derived.

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