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ON THE STABILITY OF LARGE-SCALE INTERCONNECTED SYSTEMS

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ABSTRACT

In this paper we study the stability of large-scale systems. We consider those systems which may be viewed as an interconnection of lower order subsystems. Lyapunov stability results were considered using norm type function, which led to the use of the measure of a matrix to analyze the stability. The resultant test matrix is an M-matrix with entries that are expressed in terms of the stability properties of the subsystems and in terms of the qualitative properties of the system interconnecting structure.

I. INTRODUCTION

In this paper we study the stability of large-scale interconnected systems using norm type Lyapunov function. This leads to the use of the measure of a matrix, as studied by Desoer (1972) and Vidyasagar (1978), to analyze the stability. The results obtained ,make it possible to utilize the measure of matrix as a tool to analyze the stability of interconnected dynamical systems. The resultant test matrix is an M-matrix with entires that are expressed in terms of the qualitative properties of the free subsystems and the interconnections of such subsystems. In section II we discuss the stability conditions of the free subsystems first, then we extend this to the stability of the large-scale interconnected systems(III). Numerical examples are given in section IV to illustrate our results.

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II. STABILITY OF THE FREE SUBSYSTEMS

In this section we will study the stability of the free subsystems, and establish conditions that will be used in the next sections.

Consider a linear large-scale interconnected system, as given in Jamshidi (1983), Michel (1977), Michel (1983), and Siljak (1978), of the form

$$\dot{x}_i = A_i x_i + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} N_{ij} x_j \quad , \quad i=1,\dots,\ell \tag{1}$$

where $x_i \in R^{n_i}$, $A_i \in R^{n_i \times n_i}$ and $N_{ij} \in R^{n_i \times n_j}$.

Let $x \in R^n$ with $x^T = (x_1^T,\dots,x_\ell^T)$ and $n = \sum_{i=1}^{\ell} n_i$.

Equation (1) may be viewed as the interconnection of ℓ free subsystems or isolated subsystems described by the equation of the form

$$\dot{x}_i = A_i x_i \quad , \quad i=1,\dots,\ell \tag{2}$$

Definition 1

Let $v_i(x_i)$ be continuous, convex and positive definite function, the right-hand derivative at each t is given by Vidyasagar (1978)

$$\dot{v}_i^+(x_i) = \lim_{\varepsilon \to 0^+} \frac{v_i(x_i + \varepsilon \dot{x}_i) - v_i(x_i)}{\varepsilon} \tag{3}$$

The modified matrix measure $\mu_{v_i}(A_i)$ as studied by Vidyasagar (1978), is defined by

$$\mu_{v_i}(A_i) = \sup_{x_i \in R^{n_i}/\{0\}} \lim_{\varepsilon \to 0^+} \frac{v_i(x_i + \varepsilon A_i x_i) - v_i(x_i)}{\varepsilon v_i(x_i)} \tag{4}$$

Definition 2

If A_i is an $n_i \times n_i$ real matrix, the measure of A_i, denoted by $\mu(A_i)$, is the scalar defined by Desoer (1972)

$$\mu(A_i) = \lim_{\varepsilon \to 0^+} \frac{\|I + \varepsilon A_i\|_i - 1}{\varepsilon} \tag{5}$$

where $\|\cdot\|_i$ denote the induced matrix norm corresponding to the vector norm $\|\cdot\|$.

It is known from the properties of matrix measure in Desoer (1972) that if $\mu(A_i) < 0$ then Re $\lambda_j(A_i) < 0$, where Re $\lambda_j(A_i)$ denotes the real part of the eigenvalue $\lambda_j(A_i)$ of A_i, $j=1,\ldots,n_i$, and consequently the system (2) is globally asymptotically stable. However if the system (2) is stable this does not imply that $\mu(A_i) < 0$. It is also known that

$$\mu_2(A_i) = \lambda_{max}\left(\frac{A_i + A_i^T}{2}\right) \tag{6}$$

where $\mu_2(A_i)$ denotes the matrix measure of A_i corresponding to ℓ^2 - norm and λ_{max} is the maximum eigenvalue of the indicated matrix.

Let $v_i(x_i) = (x_i^T\ x_i)^{\frac{1}{2}}$ be Lyapunov function of the system (2), then the Lyapunov matrix equation will be

$$\left[-(A_i + A_i^T)\right] = Q_i \tag{7}$$

where Q_i is positive definite. From equation (7) it is clear that $v_i(x_i)$ is a suitable Lyapunov function of the system (2) if $\left[-(A_i + A_i^T)\right]$ is Positive definite, and hence from equation (6) $\mu_2(A_i) < 0$. The last result will be generalized by the following two lemmas.

Lemma 1

Let $x_i = 0$ be the equilibrium of the free subsystem described by (2) and let $Q_i = \left[-(A_i + A_i^T)\right]$ be a positive definite. The system (2) is globally asymptotically stable if and only if the scalar $\sigma_i < 0$, where $\sigma_i = \mu_2(A_i)$.

Lemma 2

Let $x_i = 0$ be the equilibrium of the free subsystem described by (2) and let $Q_i = \left[-(A_i + A_i^T)\right]$ is not positive definite. The system (2) is globally asymptotically stable if and only if the scalar $\sigma_i < 0$, where $\sigma_i = \mu_2(\hat{A}_i)$, and $\hat{A}_i = T_i A_i T_i^{-1}$ such that $\hat{Q}_i = \left[-(\hat{A}_i + \hat{A}_i^T)\right]$ is positive definite, and T_i is a nonsingular transformation.

In the following section we are going to study the stability of large-scale interconnected systems using the previous results.

III. STABILITY OF LINEAR TIME-INVARIANT LARGE-SCALE INTERCONNECTED SYSTEMS

We are now in the position to state our main result.

Theorem 1

The equilibrium x = 0 of (1) is globally asymptotically stable if the following assumptions are satisfied:

1. For each free subsystem given by (2), $\sigma_i < 0$, where σ_i is defined by Lemma 1 and Lemma 2.
2. The $\ell \times \ell$ matrix $D = \left[d_{ij}\right]$ is an M - matrix, where

$$d_{ij} = \begin{cases} -\sigma_i & , \quad i = j \\ -\beta_{ij} & , \quad i \neq j \end{cases}$$

and β_{ij} is defined as follows :

case a: if $\left[-(A_i + A_i^T)\right]$ is positive definite then $\beta_{ij} = \| N_{ij} \|_{i2}$

case b: If $\left[-(A_i + A_i^T)\right]$ is not positive definite then $\beta_{ij} = \| T_i \; N_{ij} T_j^{-1} \|_{i2}$

where T_i and T_j are the nonsingular transformations discussed in lemma 2.

Proof :

We are going to prove the general case, b, since case a is considered a special form of case b.

Suppose there exist nonsingular transformations T_i and T_j such that, $z_i = T_i \, x_i$, $\hat{A}_i = T_i \, A_i \, T_i^{-1}$, $\hat{A}_j = T_j A_j T_j^{-1}$ and $\nu_{ij} = T_i N_{ij} T_j^{-1}$. Then from equation (1) we have

$$\dot{z}_i = \hat{A}_i \, z_i + \sum_{\substack{j=1 \\ j \neq 1}}^{\ell} \nu_{ij} \, z_j \, , \qquad i=1,\ldots,\ell \tag{8}$$

Let $v_i(z_i)$ be continuous, convex and positive definite function for $i=1,\ldots,\ell$. For any scalar $\alpha_i > 0$, we define $\alpha^T = (\alpha_1, \ldots, \alpha_\ell)$. Choose as a Lyapunov function for (8)

$$v(z) = \sum_{i=1}^{\ell} \alpha_i v_i (z_i) \qquad (9)$$

where $z = Tx$ and $T = \mathrm{diag}\,(T_1, \ldots, T_\ell)$.

Clearly, $v(z)$ is positive definite and radially unbounded. The right-hand derivative $\dot{v}^+(z)$ is given by

$$\dot{v}^+(z) = \sum_{i=1}^{\ell} \alpha_i \dot{v}_i^+ (z_i) \qquad (10)$$

From equations (3) and (10) we have

$$\dot{v}^+(z) = \lim_{\varepsilon \to 0^+} \sum_{i=1}^{\ell} \alpha_i \left\{ \frac{v_i(z_i + \varepsilon \dot{z}_i) - v_i(z_i)}{\varepsilon} \right\} \qquad (11)$$

From equations (8) and (11) we have

$$\dot{v}^+(z) = \lim_{\varepsilon \to 0^+} \sum_{i=1}^{\ell} \alpha_i \left\{ \frac{v_i(z_i + \varepsilon \hat{A}_i z_i + \varepsilon \sum_{\substack{j=1 \\ j\neq 1}}^{\ell} \nu_{ij} z_j) - v_i(z_i)}{\varepsilon} \right\}$$

$$\dot{v}^+(z) = \lim_{\varepsilon \to 0^+} \sum_{i=1}^{\ell} \alpha_i \left[\left\{ \frac{v_i(z_i + \varepsilon \hat{A}_i z_i) - v_i(z_i)}{\varepsilon} \right\} + \left\{ \frac{v_i(z_i + \varepsilon \hat{A}_i z_i + \varepsilon \sum_{\substack{j=1 \\ j\neq 1}}^{\ell} \nu_{ij} z_j) - v_i(z_i + \varepsilon \hat{A}_i z_i)}{\varepsilon} \right\} \right]$$

$$= \sum_{i=1}^{\ell} \alpha_i \left[\dot{v}_i^{*}(z_i) + \dot{\omega}^+(z) \right] \qquad (12)$$

where

$$\dot{\omega}^+(z) = \lim_{\varepsilon \to 0^+} \frac{v_i(z_i + \varepsilon \hat{A}_i z_i + \varepsilon \sum_{\substack{j=1 \\ j\neq 1}}^{\ell} \nu_{ij} z_j) - v_i(z_i + \varepsilon \hat{A}_i z_i)}{\varepsilon} \qquad (13)$$

and

$$\dot{v}_i^{*}(z_i) \leq \mu_{v_i}(\hat{A}_i)\, v_i(z_i) \qquad (14)$$

From the convexity property of $v_i(.)$, equation (13) can be written as

$$\dot{\omega}^{+}(z) \leqslant \sum_{\substack{j=1 \\ j\neq i}}^{\ell} v_i\,(\nu_{ij} z_j) \qquad (15)$$

From equations (12), (14) and (15) we have

$$\dot{v}^{+}(z) \leqslant \sum_{i=1}^{\ell} \alpha_i \left\{ \mu_{v_i}(\hat{A}_i) v_i(z_i) + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} v_i(\nu_{ij} z_j) \right\} \qquad (16)$$

For the case that $v_i(z_i) = (z_i^T z_i)^{\frac{1}{2}}$ equation (16) will have the form

$$\dot{v}^{+}(z) \leqslant \sum_{i=1}^{\ell} \alpha_i \left\{ \mu_2(\hat{A}_i) \| z_i \|_2 + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \| \nu_{ij} z_j \|_2 \right\} \qquad (17)$$

Equation (17) can be rewritten as

$$\dot{v}^{+}(z) \leqslant \sum_{i=1}^{\ell} \alpha_i \left\{ \mu_2(\hat{A}_i) \| z_i \|_2 + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \| T_i N_{ij} T_j^{-1} \|_{i2} \| z_j \|_2 \right\} \qquad (18)$$

Defining

$$\beta_{ij} = \| T_i N_{ij} T_j^{-1} \|_{i2} \qquad (19)$$

From equations (18) and (19) and lemma 2 we have

$$\dot{v}^{+}(z) \leqslant \sum_{i=1}^{\ell} \alpha_i \left\{ \sigma_i \| z_i \|_2 + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \beta_{ij} \| z_j \|_2 \right\} \qquad (20)$$

Equation (20) can be rewritten as

$$\dot{v}^{+}(z) \leqslant -\alpha^T D\, y(z) = -\,\delta^T y(z) \qquad (21)$$

where

$$D = \left[d_{ij} \right] \text{ with } d_{ij} = \begin{cases} -\sigma_i, & i=j \\ -\beta_{ij}, & i\neq j \end{cases},$$

$$y^T(z) = \left[\| z_1 \|_2, \ldots\ldots, \| z_\ell \|_2 \right] \text{ and } \alpha^T D = \delta^T.$$

Since D is an M-matrix, it follows that D^{-1} exists and $D^{-1} \geq 0$ (i.e., every element of D^{-1} is nonnegative). Thus, given $\delta > 0$, we can choose $\alpha = (D^{-1})^T \delta > 0$. Hence for this choice of α it follows from equation (21) that $\dot{v}^+(z)$ is negative definite for all $x \in R^n$.

Consequently, for the equilibrium x = 0 of (1) to be globally asymptotically stable, the matrix D must be an M-matrix.

Now, the application of Theorem 1 to study the stability of a large-scale interconnected system that is described by equation (1) is formulated by the following algorithm that provides the computation procedure.

Algorithm 1. Application of Theorem 1

Step 1: Assuming the local subsystems A_k, $k = 1,\ldots,\ell$ be globally asymptotically stable, calculate $Q_k = \left[-(A_k + A_k^T)\right]$.

Step 2: If both Q_i and Q_j for $i \neq j$ are positive definite, then $T_i = I_{n_i}$, $T_j = I_{n_j}$, $\hat{A}_i = A_i$, $\hat{A}_j = A_j$ and $\nu_{ij} = N_{ij}$, and go to step 5 (where I_{n_i} denotes identity matrix with dimension n_i).

If Q_i is positive definite while Q_j is not positive definite, then $T_i = I_{n_i}$, $\hat{A}_i = A_i$, and go to step 3 to evaluate T_j and $\hat{A}_j$.

If Q_i is not positive definite while Q_j is positive definite, then $T_j = I_{n_j}$, $\hat{A}_j = A_j$ and go to step 3 to evaluate T_i and $\hat{A}_i$.

If both Q_i and Q_j are not positive definite, then go to step 3 to evaluate T_i, $\hat{A}_i$, T_j and $\hat{A}_j$.

Step 3: If A_i has real distinct eigenvalues $\lambda_{1k},\ldots,\lambda_{n_k k}$, then $\hat{A}_k = \text{diag}(\lambda_{1k},\ldots,\lambda_{n_k k})$ and T_k is the transformation such that $\hat{A}_k = T_k A_k T_k^{-1}$. Go to step 5.

If A_k has complex eigenvalues, then $\hat{A}_k$ will be in the complex conjugate form and T_k is the transformation such that $\hat{A}_k = T_k A_k T_k^{-1}$. Go to step 5.

If A_k has not distinct eigenvalues, find the Jordan form $\hat{A}_k$ and the transformation T_k such that $\hat{A}_k = T_k A_k T_k^{-1}$. If

$\hat{Q}_k = \left[-(\hat{A}_k + \hat{A}_k^T)\right]$ is not positive definite go to step 4.
If $\hat{Q}_k$ is positive definite go to step 5.

Step 4: For given F_k solve the Lyapunov matrix equation

$F_k = \left[-(P_k A_k + A_k^T P_k)\right]$ and find P_k. Then calculate T_k from

the equation $T_k^T T_k = P_k$ and consequently $\hat{A}_k = T_k A_k T_k^{-1}$.

Step 5: Calculate $\sigma_i = \mu_2(\hat{A}_i)$ and $\beta_{ij} = \| \nu_{ij} \|_{i2}$.

Form the $\ell \times \ell$ D matrix. If D is an M-matrix, then the system (1) is globally asymptotically stable, Stop.

IV. EXAMPLES

Example 1.

Consider a fifith-order linear system decomposed into a third-order and Second-order subsystems, and it is represented by

$$\begin{bmatrix} \dot{x}_1 \\ \hline \dot{x}_2 \end{bmatrix} = \left[\begin{array}{ccc|cc} -1 & .1 & .2 & .1 & .2 \\ .2 & -2 & .5 & .1 & .1 \\ .1 & -1 & -3 & .5 & .4 \\ \hline 1 & 0 & 1 & -4 & .2 \\ .2 & .5 & 0 & 1 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} \qquad (22)$$

Now, following Algorithm 1 to study the stability of system (22), we have

Step 1: A_1 is asymptotically stable, with eigenvalues -.9900, -2.5050 $\pm$ j.4745

A_2 is asymptotically stable, with eigenvalues -3.8292, -5.1708

$$Q_1 = \begin{bmatrix} 2 & -.3 & -.3 \\ -.3 & 4 & .5 \\ -.3 & .5 & 6 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} 8 & -1.2 \\ -1.2 & 10 \end{bmatrix}$$

Step 2: Both Q_1 and Q_2 are positive definite, then $T_1 = I_3$, $T_2 = I_2$, $\hat{A}_1 = A_1$ and $\hat{A}_2 = A_2$.

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Step 5: $\sigma_1 = \mu_2 (A_1) = -.9715$, $\sigma_2 = \mu_2(A_2) = -3.718975$

$$\beta_{12} = \| N_{12} \|_{i2} = .68698491, \quad \beta_{21} = \| N_{21} \|_{i2} = 1.4223502$$

Then, the test matrix D will be

$$D = \begin{bmatrix} .9715 & -.68698491 \\ -1.4223502 & 3.718975 \end{bmatrix}$$

It is clear that D is an M-matrix, consequently the equilibrium x=0, where $x^T = (x_1^T , x_2^T)$, of the system (22) is globally asymptotically stable, which agrees with the fact that the overall system has the eigenvalues at (-.9, -2.32 $\pm$ j. 54, -4.52 , -4.94).

Example 2.

Consider a third-order linear system decomposed into a second-order and a first-order subsystem, and it is represented by

$$\begin{bmatrix} \dot{x}_1 \\ \hline \dot{x}_2 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -9.425 & -5.182 & -3.396 \\ \hline -.192 & -.768 & -5.099 \end{array}\right] \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} \qquad (23)$$

Now, following Algorithm 1 to study the stability of system (23), we have

Step 1: A_1 is asymptotically stable, with eigenvalues -1.7315, -2.4505

A_2 is asymptotically stable, with eigenvalues -5.099

$$Q_1 = \begin{bmatrix} -2 & 8.425 \\ 8.425 & 10.364 \end{bmatrix}, \qquad Q_2 = 10.198$$

Step 2: Q_1 is not positive definite, while Q_2 is positive definite, thus $T_2 = I_1$, $\hat{A}_2 = A_2$.

Step 3: A_1 has distinct eigenvalues, and

$$\hat{A}_1 = \begin{bmatrix} -2.4506 & 0 \\ 0 & -1.7314 \end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix} -3.7983 & -1.3906 \\ -4.7983 & -1.3906 \end{bmatrix}$$

Step 5: $\sigma_1 = \mu_2(\hat{A}_1) = -1.7314$, $\sigma_2 = \mu_2(A_2) = -5.099$

$\beta_{12} = \| T_1 N_{12} \|_{i2} = .9272$ and $\beta_{21} = \| N_{21} T_1^{-1} \|_{i2} = 3.1103$

Then, the test matrix D will be

$$D = \begin{bmatrix} 1.7314 & -.9272 \\ -3.1103 & 5.099 \end{bmatrix}$$

It is clear that D is an M-matrix, consequently the equilibrium x=0 where $x^T = (x_1^T , x_2^T)$, of the system (23) is globally asymptotically stable, which agrees with the fact that the overall system has the eigenvalues -1.576, -1.895, -5.81.

V. CONCLUSIONS

The above results make it possible to utilize the measure of a matrix as a tool to analyze the stability of large-scale interconnected dynamical systems, which offer different approach than other existing stability results.

The resultant test matrix is an M-matrix which agrees with the results in Jamshidi (1983), Michel (1977), Michel (1983), and Siljak (1978).

We note the use of ℓ^2- norm in our work was essential, because both the ℓ^1 - norm and ℓ^∞-norm would result in weak coupling conditions on the subsystem level.

In this work the system is supposed to consist of interconnected subsystems. It is assumed that this decomposition or tearing has already been specified, and that a description of each subsystem and a description of the interconnection is available.

At the present time, extension of the given algorithm to the nonlinear large-scale dynamical systems as well as discrete time dynamical systems is under consideration.

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