Modified Quasilinearization For Optimal Flight Trajectory

I. Mansour * , M.abdel Hamid ** , A.Y. Bilal ***

ABSTRACT

The application of Pontryagin's maximum principle to optimal control problems results in a two-point boundary value problem. The quasilinearization method is used to solve the resulting problem and the method is very effective if the final time is fixed. In this paper the modified quasilinearization with time transform is used to find the minimum time flight path in a vertical plane. The method is based on Newton-Raphson combined with long's method in order to solve the problem of unknown final time. The advantage of the method is the fact that it does not need large memary size compared with other methods of trajectory optimization. The minimum time trajectory of climb in a vertical plane is obtained after a few iterations which proves that the method is very efficient in this case.

INTRODUCTION

The problem of flight path optimization has attracted many researchers to deal with it. In Ref.[1] the indirect method of the calculus of variations was used for the solution of optimal rocket flight in vaccum and in a resisting meduim. The method of gradient was introduced [2 &3] for the solution of the problem of flight path optimization. In Ref.[4] the conjugate gradient method was generalized in order to be applied for optimal control problems. The gradient method with penalty function [5] was used to find the solution of the fixed end point problem of optimum trajectory in a vertical plane. The three dimensional minimum time flight path was found using the indirect method of variational calculus [6] The conjugate gradient method [7] was used for the solution of optimum flight path of V/STOL airplanes. In the above mentioned works the quasilinearization method was not used inspite of the fact that it needs less memory space. This suggests the use of quasilinearization method for

---

* Aeronautical Department, Military Technical College, Cairo, Egypt. Member AIAA.

** Information Center of Armed Forces, Egypt.

*** Prof., Department of Electrical Engineering, Cairo, University, Giza, Egypt.

the solution of minimum time flight path in this paper. Since the method is not applicable to problems of unknown final time, the method must be modified to be used for minimum time problems. This will be the main task of this paper.

MATHEMATICAL FORMULATION

The problem of flight in a vertical plane (see fig.1) is represented by a system of four non-linear differential equations according to Ref. [8] they are

$$\dot{x} = V(t).\cos\gamma(t) \qquad (1)$$

$$\dot{z} = V(t).\sin\gamma(t) \qquad (2)$$

$$\dot{V} = [\, T(z,V).\cos\alpha - D(z,V,\alpha)\,]/m - g.\sin\gamma(t) \qquad (3)$$

$$\dot{\gamma} = [\, T(z,V).\sin\alpha + F(z,v,\alpha)\,]/m.V - g.\cos\gamma(t)/m.V \qquad (4)$$

If the cost function P is defined by

$$P = t_f ,$$

then the problem is to find the minimum of P subject to equations (1) to (4) with the four states prescribed at the initial time and z,V and γ given at the unknown final time while x is free.The fact that x_f is free leads to possible reduction of the model order. In dealing with the above fourth order system, the Hamiltonian of it will be independent upon x , this means that the corresponding adjoint variable λ_x is constant along the optimum flight path. Since the final value of x is free, then the final value of the corresponding adjoint variable λ_x will be zero. So λ_x is zero along the optimal flight path and it is possible to drop the first equation from the model. The mathematical mcdel is composed from three non-linear differential equations in z, V and γ . Since the problem is the minimum time climb in the vertical plane, the thrust is assumed to be maximum but function of z and V. This means that the system has only one control variable, the angle of attack .

The quasilinearization method is not directly applicable to problems of unknown final time. To overcome this difficulty, Long's method [9] is used. It is based on changing the time variable

$$t = a.\tau \quad ,\text{where } 0 \leqslant \tau \leqslant 1 \qquad (6)$$

a is a parameter giving the final time t_f . If y is a vector defined as

$$y = (z,V,\gamma,\lambda_z,\lambda_V,\lambda_\gamma,a)$$

then a suitable form will be

$$y = (\, y_1 , y_2 , y_3 , y_4 , y_5 , y_6 , y_7 \,) \qquad (7)$$

The state equations can be written in the following form

$$y_1' = y_7 . y_2 . \sin y_3 \qquad (8)$$

$$y_2' = y_7 . [[T(y_1,y_2) - D(y_1,y_2,\alpha)]/ m - g. \sin y_3] \qquad (9)$$

$$y_3' = y_7 . [[T(y_1,y_2) . \alpha + F(y_1,y_2,\alpha)]/m.y_2 - g.\cos y_3/m.y_2] \qquad (10)$$

According to Pontryagin's maximum principle, the Hamiltonian of the above given state equations is of the following form:

$$H = 1 + y_7 . y_4 . y_2 \sin y_3 + y_5 . y_7 . [[T(y_1,y_2) - D(y_1,y_2,\alpha)]/m - g. \sin y_3] + y_6 . y_7 . [[T(y_1,y_2) . \alpha + F(y_1,y_2,\alpha)]/m.y_2 - g. \cos y_3 / y_2] \qquad (11)$$

The costate equations are obtained as the negative derivative of the Hamiltonian by the corresponding state variable,they are

$$y_4' = - y_5 . y_7 . \frac{\partial T(y_1,y_2)}{\partial y_1} /m - y_6 . y_7 . \frac{\partial T(y_1,y_2)}{\partial y_1} /m.y_2 + y_5 . y_7 . \frac{\partial D(y_1,y_2)}{\partial y_1} /m - y_6 . y_7 . \frac{\partial F(y_1,y_2,\alpha)}{\partial y_1} / m.y_2 \qquad (12)$$

$$y_5' = - y_4 . y_7 . \sin y_3 + y_5 . y_7 . [\frac{\partial D(y_1,y_2,\alpha)}{\partial y_2} - \frac{\partial T(y_1,y_2)}{\partial y_2}]/m + y_6 . y_7 . [T(y_1,y_2) . + F(y_1,y_2,\alpha)] / m.y_2^2 - [\frac{\partial T(y_1,y_2)}{\partial y_2} . + \frac{\partial F(y_1,y_2,\alpha)}{\partial y_2}]/m.y_2 - g.\cos y_3/y_2^2 \qquad (13)$$

$$y_6' = - y_4 . y_7 . y_2 . \cos y_3 + y_7 . y_5 . g.\cos y_3 - y_6 . y_7 . g.\sin y_3/y_2 \qquad (14)$$

$$y_7' = 0 \qquad (15)$$

The optimality condition is obtained by differentiating the Hamiltonian by the control variable and equating the derivative by zero.So the optimality condition is

$$\frac{\partial H}{\partial \alpha} = 0 \quad , \text{ which means that:}$$

$$- y_5 \cdot \frac{\partial D(y_1, y_2, \alpha)}{\partial \alpha} /m + y_6 \cdot [\ T(y_1, y_2) + \frac{\partial F(y_1, y_2, \alpha)}{\partial \alpha}]/m.y_2 = 0 \qquad (16)$$

The optimal control α is obtained by solving the above equation and taking into cosideration that α must be smaller than the stalling angle of attck α_{max}.

The transversality condition is

$$H_{max} = 0 \qquad ; \qquad t_o \leqslant a.\tau \leqslant t_f$$

At $t = 0$ the above equation gives y_6 as function of the other variables.

MODIFIED QUASILINEARIZATION METHOD

The problem is now transformed to a two point boundary value problem. The mathematical model can be written in the following form:

$$y' = h(y, \alpha) \qquad (17)$$

with the boundary conditions:

$$y_i(0) = y_{io} \qquad i = 1,2,3 \qquad (18)$$

$$y_i(1) = y_{if} \qquad i = 1,2,3 \qquad (19)$$

where h is a seven-dimensional vector function. The modified quasilinearization treats the non-linear differential equation (17) as an initial value problem at each iteration and quasilinearization is used to find an improved initial condition vector. Equation (17) can be written as follows:

$$y'^k = h(y^k) \qquad ; \ y^k(0) = q^{k-1}(0) \qquad (20)$$

The associated linear equation is

$$q'^k = h(y^k) + J(y^k) \ . \ (q^k - y^k) \qquad (21)$$

with the boundary conditions

$$q_i^k (0) = y_{io} \qquad i = 1,2,3 \qquad (22)$$

$$q_i^k (1) = y_{io} \qquad i = 1,2,3 \qquad (23)$$

where $J(y^k)$ is the Jacobian matrix

6

It is possible to rewrite the equations (20) - (23) in the following form:

$$\dot{y}^k = h(y^k) \quad : \quad y^k(0) = y^{k-1}(0) + \eta\, z^{k-1} \qquad (24)$$

$$\dot{z}^k = J(y^k).\, z^k \qquad (25)$$

with the following boundary conditions

$$z_i^k(0) = y_{io} - y_i^k(0) \quad : \quad i = 1,2,3 \qquad (26)$$

$$z_i^k(0) = y_{if} - y_i^k(1) \quad : \quad i = 1,2,3 \qquad (27)$$

where $z = q - y$, η is an improving coefficient, the set of equations (24) - (27) is a set of homogenous differential equations which can be easily solved as a homogenous linear two-point boundary value problem Ref.[10]. It is possible to get the missing initial conditions using the known final conditions and the following equation

$$z^k(\tau) = C_1^k\, z_{h1}^k(\tau) + C_2^k\, z_{2h}^k(\tau) + \ldots . \qquad (28)$$

where $z^k(\tau), z^k(\tau), \ldots\ldots\ldots$ are the homogenous solutions of equation (25). According to Ref.[11] C_1 , C_2 ... can be determined by putting $\tau = 1$ the improved initial condition vector is:

$$y^{k+1}(0) = y^k(0) + \eta . C^k \qquad (29)$$

where $C^k = z^k(0)$.

COMPUTATIONAL ALGORITHM

The computational algorith for the modified quasilinearization based upon the above given analysis is as follows;

1. For $k = 0$; guess an initial condition vector to equation (24). They are y_4^0 , y_5^0 and y_7^0 ; y_6^0 is calculated using the transversality condition.
2. Integrate equation(24) forward with initial condition $y_1^0, y_2^0, \ldots, y_7^0$.
3. Integrate equation(25) forward with the initial condition vectors (0,0,0,1,0,0,0) ,(0,0,0,0,1,0,0) and (0,0,0,0,0,0,1). Note that the ones are corresponding to the missing initial conditions.
4. Calculate the constants C^k from equation(28).
5. Calculate the new initial conditions using equation(29).
6. Check if $| y^{k+1}(0) - y^k(0)| \leq \varepsilon$,where ε is the required accuracy. If the result is yes the problem is solved, otherwise go to step 2 and continue till the required accuracy is reached.

RESULTS AND DISCUSSIONS

The method is applied to calculate the minimum time climb of an airplane in a vertical plane. It is required to find the minimum time flight path and the corresponding angle of attack control necessary to change the aircraft state from horizontal flight at $z = 100$ m and $V = 140$ m/s to $z = 2000$ m and flight speed $V = 150$ m/s. The determination of the unknown parameter " a " plays an important role in the iteration process. If the initial guess of the parameter " a " is not close to the optimal value, the iteration process may not converge. It was found that if the initial guess of the final time (parameter " a ") is far from 150 seconds convergence was not obtained. If the final time was chosen to be 176 seconds, convergence was achieved after three iterations. In fig.(2) is given the parameter " a " as function of the iteration number k. It is clear that the iteration process has a good convergence. In fig. (3) is given the control variable as function of τ for $k = 3$. It is clear the angle of attack α is contineously increasing till the maximum permissible value $\alpha = 15$ at $\tau \approx 0.73$. The change of flight altitude z with τ is given in fig (4), It is clear that the obtained result is of physical significance. In fig.(5) is shown the flight velocity as function of τ . The velocity is changing very slightly. The flight path angle γ as function of τ is given in fig (6). In fig(7) is shown the velocity altitude curve which is physically acceptable in climb performance.

CONCLUSION

An optimal flight path is obtained using the modified quasilinearization method. For the possible application of quasilinearization for problems with unknown final time it is useful to combine long's method with quasilinearization. Special attention must be given to the choice of the parameter " a " and to the guess of the missing initial conditions. The proposed method is very effective and the convergence is very good.

REFERENCES

1. Miele A., " The calculus of variations in Applied Aerodynamics and Flight Mechanics ", In " Optimization Techniques, Leitman (ed), Academic Press, 1962 .
2. Kelly H., " Method of Gradients " in " Optimization Techniques ", G. Leitman (ed), Academic Press, 1962 .
3. Kelly H., " Gradient Theory of Flight Paths ", ARS Journal, Vol 30, October 1960 .
4. Ladson L.S., Mitter S.K. and waren A.D. " The conjugate gradient Method for Optimal control Problems ", IEEE, AC-12, 1967 .
5. Falco M. and Kelly H., " Aircraft Symmetric Flight Optimization, Grumman Research Department, Report, RE-42, 1972 .

6. Parson m.G. and Bryson A.E., " Three Dimensional Minimum Time Flight Paths to a Point and Onto a Line for a Supersonic Aircraft",Stanford University, 1972 .
7. Mehra R.K. and Bryson A.E.,"Conjugate Gradient Methods With an Application to V/STOL Flight Path Optimization", Journal of Aircraft , Vol 6 No 2,1969 .
8. Sage A.P., " Optimum System Control ",Prentice-Hall, INC,Englewood Cliffs, N.J.
9. Long R.S.," Newton-Raphson Operator,Problems with Undetermined End Points ",AIAA Journal ,3 ,1965.
10. Rothenberger B.F. and Lapidus L.,"The Control of Nonlinear Systems, Part IV ; Quasilinearizationa as a Numerical Method", Aiche J.13,1967
11. Aly G.M, and Chan W.C.,"Numerical Computation of Optimal Control with Unknown Final Time", Journal of Mathematical Analysis and Applications Vol,45,No.2, February,1974.

NOMENCLATURE

D	Drag force
F	Lift force
g	Gravity acceleration
H	Variational Hamiltonian
m	Mas of airplane
P	Cost function
T	Thrust force
t	Time
V	Flight speed
x	Horizontal distance
z	Flight altitude
α	Angle of attack
γ	Flight path angle
λ	Adjoint variable
τ	Normalized time

Used Indices:

$(\ )_o$	Initial value
$(\ )_f$	Final Value
$(\ ')$	Derivative w.r.t.
$(\ \dot{} \ )$	derivative w.r.t. t

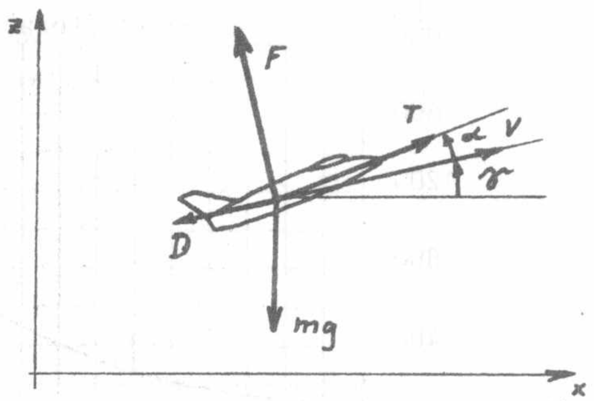

Fig.(1),Coordinate System Used in the Problem of Flight in a vertical Plane.

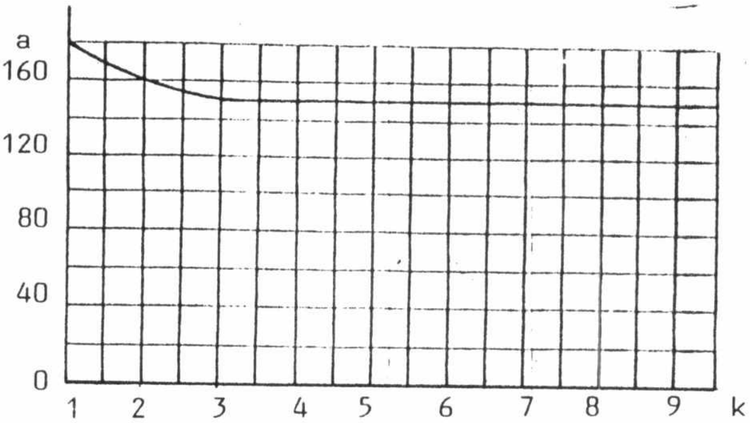

Fig.(2), The Parameter "a" as Function of the Number of Iterations

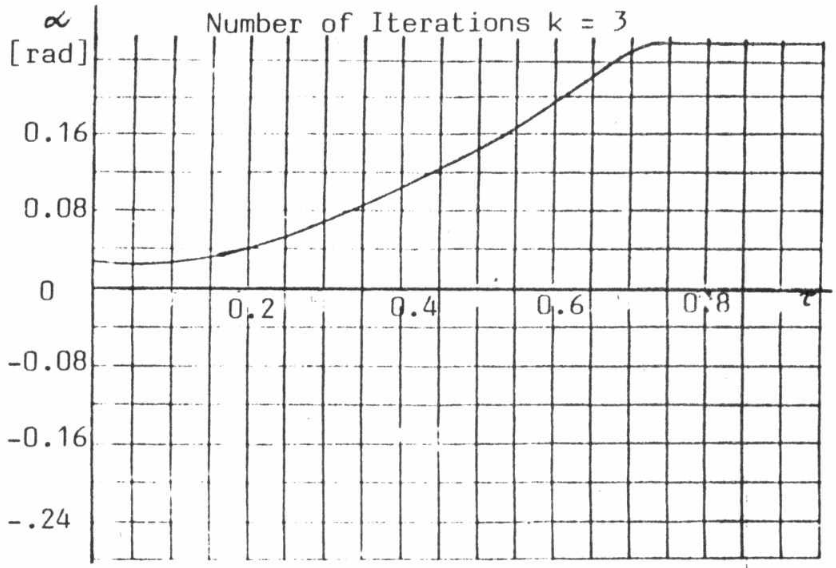

Fig.(3),The Control Variable as Function of the Normalized Time

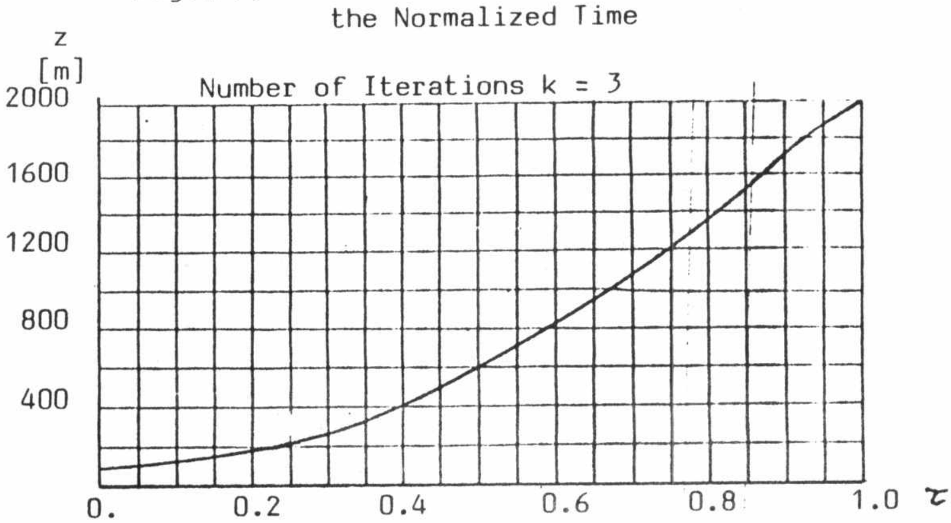

Fig.(4),Flight Altitude as function of the Normalized Time

6

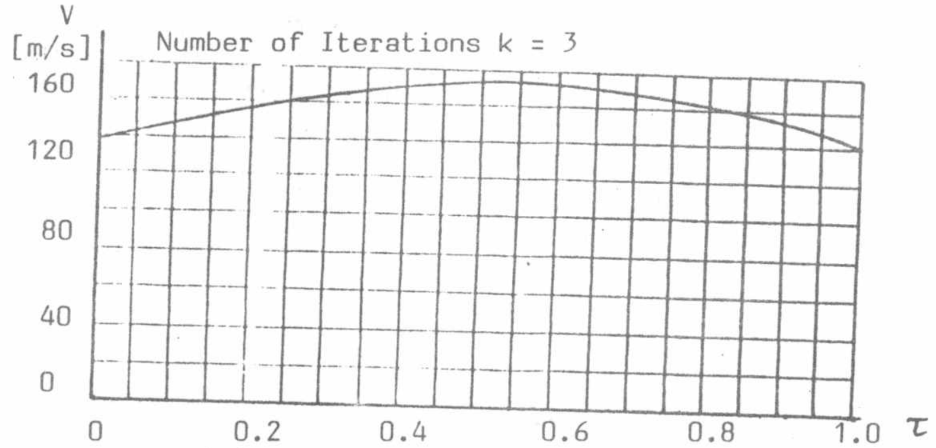

Fig.(5),The flight speed Versus the Normalized time

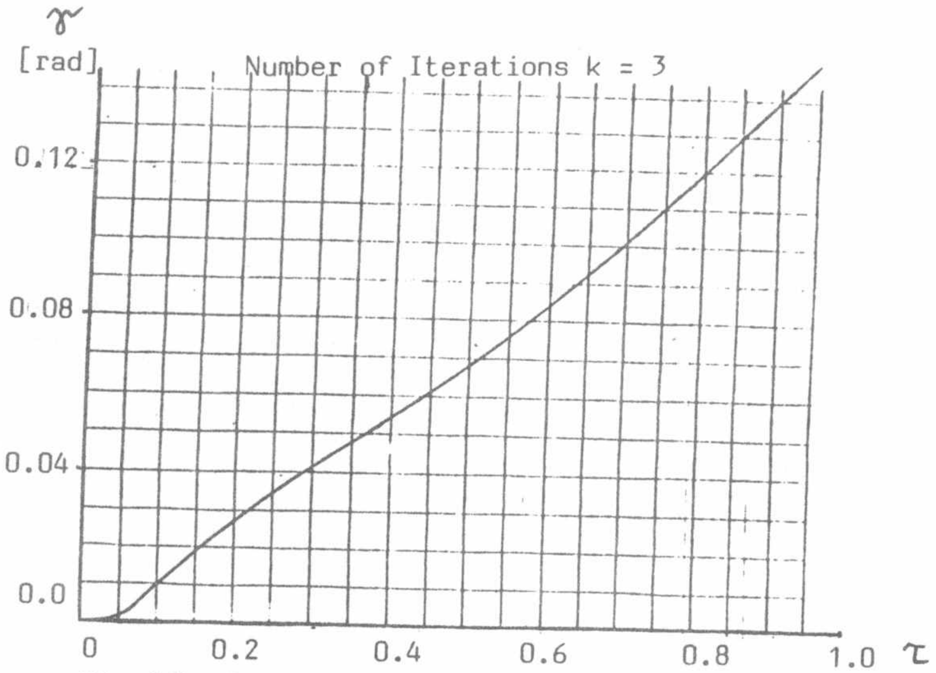

Fig.(6),Flight Path Angle vesus Normalized time

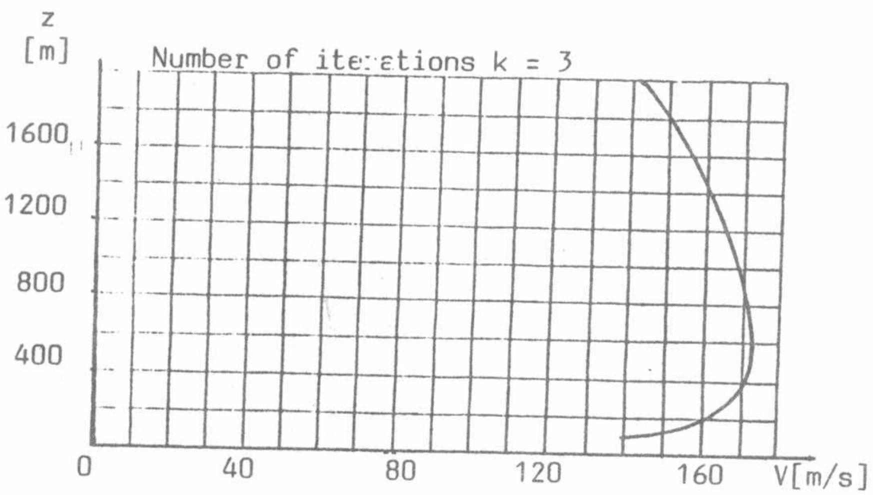

Fig.(7),Flight altitude Versus Flight Speed