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MILITARY TECHNICAL COLLEGE
CAIRO - EGYPT

ERROR ANALYSIS OF INERTIAL NAVIGATION SYSTEMS

M.Samy Abo El Soud*, E.M.Zakzouk*, A.M.Hamad*, and M.M.El-Dakiky**

ABSTRACT

The accuracy of modern multisensor integrated inertial navigation systems (INS) is affected by various noise sources, such as accelerometer and gyro instabilities. These error sources are often modeled as Gauss-Markov random processes and their effects on navigation system accuracy are determined by covariance simulation. Environmental error sources such as gravity uncertainties have been treated similarly. In this paper, the error equations for a local-level INS are derived. In order to apply covariance error analysis to the system, the linear dynamic error equations in their state-space representation are presented. The error sources considered are gyro drift, accelerometer error, and gravity uncertainties.
We have obtained finally a general case model representing error propagation equations including three dimensional free vehicle motion .

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* Prof.Dr.M.Samy Abo El Soud, Prof.Dr.E.M.Zakzouk, and Dr.A.M.Hamad, Military Technical College, Cairo, Egypt.
**Eng. M.M.El-Dakiky, Air Defence College, Alexandria, Egypt .

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I.INTRODUCTION

INS have found universal application both militarily and commercially. They are self-contained, nonradiating, nonjammable, and sufficiently accurate to meet the requirements of users in most satisfactory manner. The mechanization and error analysis of INS is not entirely new. Simplified analyses of such systems can be found, for example, in [1] - [3] .

INS utilize inertial elements (i.e., accelerometers and gyros) to sense vehicle acceleration in a known direction and then integrate this acceleration to determine velocity and position. These sensors are conventionally mounted on gimbaled platforms which isolate them from vehicle motion and physically locate them in the desired coordinate reference frame.

In analyzing INS, three coordinate frames are usually defined. The reference (or true), the platform (or instrumented), and the computer (or indicated) coordinate frames. In local-level north-pointing systems, the reference frame is the local geographic (north-east-down) frame ; it is desired that the platform should be forced to remain in as close coincidence as possible with the geographic frame. If this is achieved, physical indication will always be available of the direction of north and east, and of the vertical too. The platform coordinate frame has a gyro and an accelerometer input axes along each of its axes; three gyros to establish the rotation of the platform frame, and three accelerometers to sense acceleration in that frame .

The Kalman filter is used frequently for mixing data in modern, integrated, aided INS. For the design of such a filter, state-space (Gauss-Markov) models are needed for all error sources.

In this paper, the appropriate dynamics of the INS are presented in differential equations form. The error sources considered are gyro drift, accelerometer error, and gravity uncertainties. The linear dynamic error equations are presented in their state-space representation. This will be useful for designing optimal estimation software to infer instrument and environmental errors from INS outputs .

II. THE SPECIFIC-FORCE EQUATION AND ITS RESOLUTION IN THE LOCAL GEOGRAPHIC FRAME

The components of the angular space rate of a moving local geographic frame (fig.1) about its three axes(north, east,and down) are expressed in terms of longitude λ, geographic latitude L, the rate of change of these quantities, and the earth rate Ω as follows:

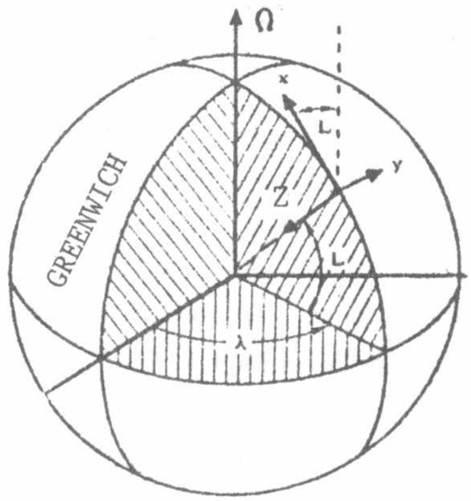

Fig.1 Locally level coordinate frame

$$\begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} (\Omega + \dot{\lambda}) \cos L \\ -\dot{L} \\ -(\Omega + \dot{\lambda}) \sin L \end{bmatrix} = \begin{bmatrix} \Omega_h + w_e \\ -w_n \\ -\Omega_v - w_e \tan L \end{bmatrix} \qquad (1)$$

where

$$\begin{aligned} \Omega_h &= \Omega \cos L \ , \\ \Omega_v &= \Omega \sin L \ , \\ w_n &= \dot{L} \ , \\ w_e &= \dot{\lambda} \cos L \ . \end{aligned} \qquad (2)$$

The components of the specific force, which is defined as the sum of kinematic acceleration plus gravity, for the moving local geographic frame along its axes are given by

$$\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} -\dot{V}_x \\ -\dot{V}_y \\ -\dot{V}z \end{bmatrix} + \begin{bmatrix} -(2\Omega_v + w_e \tan L)V_y + w_n V_z \\ (2\Omega_v + w_e \tan L)V_x + (2\Omega_h + w_e)V_z \\ -w_n V_x - (2\Omega_h + w_e)V_y + g_o \end{bmatrix} \qquad (3)$$

where V_x, V_y, and V_z are the ground velocities in the north, east, and down directions, respectively, and g_o is the nominal value of gravity. The ground velocities are related to the longitude and geographic latitude by

$$V_x = \dot{L}R \qquad = w_n R \;,$$
$$V_y = (\dot{\lambda}\cos L)R \qquad = w_e R \;, \qquad (4)$$
$$V_z = -\dot{R} \qquad .$$

where R is the radial distance to the earth's center. The computer of an INS implements the set of equations relating position and velocity informations to the sensed specific-force components. It is desired that the platform frame (ix, iy, iz) should be forced to remain in as close coincidence as possible with the local geographic frame(x,y,z). The axes of the platform frame are almost never aligned perfectly with the local geographic frame; the actual directions of the axes of the platform depend on the interaction of the navigation system with its error sources.

III. DERIVATION OF VELOCITY ERROR EQUATIONS

The uncertainty of gravity (or gravity disturbance vector) is that portion of the gravity field not accounted by the formula used to calculate gravity [4]. In other words, the gravity disturbance vector at a point is the difference between the true value of gravity and a reference value based on some model of the earth.(In an error analysis, it is generally assumed that the earth is a perfect sphere of radius R_e). In local-level geographic frame, this vector is normally expressed as

$$\underline{\delta g} = \begin{bmatrix} -g_o\xi \\ -g_o\zeta \\ -\Delta g \end{bmatrix} \qquad (5)$$

where g_o is the nominal value of gravity. The quantities ξ and ζ are called vertical deflections and Δg is called the gravity anomaly.So, the specific-force components given by (3) will be corrected for the gravity disturbance vector . Therefore ,

$$\begin{bmatrix} f'_x \\ f'_y \\ f'_z \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} + \begin{bmatrix} -g_o\xi \\ -g_o\zeta \\ \Delta g \end{bmatrix} \qquad (6)$$

Now, consider a platform frame misaligned from the local geographic frame by the vector $\bar{\Psi}$. The specific-force components associated with the moving local geographic frame, given by equation(6), along the misaligned (tilted) axes are, therefore, for small misalignment angles

$$\begin{bmatrix} f_{ix} \\ f_{iy} \\ f_{iz} \end{bmatrix} = \begin{bmatrix} 1 & \Psi_z & -\Psi_y \\ -\Psi_z & 1 & \Psi_x \\ \Psi_y & -\Psi_x & 1 \end{bmatrix} \begin{bmatrix} f'_x \\ f'_y \\ f'_z \end{bmatrix} \qquad (7)$$

The platform frame has an accelerometer input axis along each of its axes to sense specific-force components in this frame. The outputs of the accelerometers can be written as :

$$\begin{bmatrix} f'_{ix} \\ f'_{iy} \\ f'_{iz} \end{bmatrix} = \begin{bmatrix} f_{ix} \\ f_{iy} \\ f_{iz} \end{bmatrix} + \begin{bmatrix} e_{ax} \\ e_{ay} \\ e_{az} \end{bmatrix} \qquad (8)$$

where e_{ax} , e_{ay}, and e_{az} are the accelerometer errors .

The computer of an INS implements the accelerometer outputs (eq.8) to determine the vehicle position in the local geographic frame. The position and velocity information are computed from

$$\begin{bmatrix} f'_{ix} \\ f'_{iy} \\ f'_{iz} \end{bmatrix} = \begin{bmatrix} -\dot{V}'_x \\ -\dot{V}'_y \\ -\dot{V}'_z \end{bmatrix} + \begin{bmatrix} -(2\Omega'_v + w'_e \tan L')V'_y + w'_n V'_z \\ (2\Omega'_v + w'_e \tan L')V'_x + (2\Omega'_h + w'_e)V'_z \\ -w'_n V'_x - (2\Omega'_h + w'_e)V'_y + g_o \end{bmatrix} \qquad (9)$$

and

$$\begin{aligned} V'_x &= w'_n R' = w'_n R \quad , \\ V'_y &= w'_e R' = w'_e R \quad , \\ V'_z &= -\dot{R}' = -\dot{h}' \quad . \end{aligned} \qquad (10)$$

where the 'prime' denotes the computed (indicated) quantities and R is the radial distance to the earth's center,($R=R_e+h$, h is the altitude).

Using Taylor's series expansion of the computed (primed) quantities about their true values, they may be expressed as follows :

$$\begin{aligned}
\dot V_x' &= \dot V_x + \delta\dot V_x \\
\dot V_y' &= \dot V_y + \delta\dot V_y \\
\dot V_z' &= \dot V_z + \delta\dot V_z \\
V_x' &= V_x + \delta V_x \\
V_y' &= V_y + \delta V_y \\
V_z' &= V_z + \delta V_z \\
\lambda' &= \lambda + \delta\lambda \\
L' &= L + \delta L \quad .
\end{aligned} \qquad (11)$$

Therefore

$$\begin{aligned}
\sin L' &= \sin L + \delta L \cos L \quad , \\
\cos L' &= \cos L - \delta L \sin L \quad , \\
\tan L' &= \tan L + \delta L \sec^2 L \quad .
\end{aligned} \qquad (12)$$

For $\cos\delta L \simeq 1$, $\sin\delta L \simeq \delta L$, and neglecting products of errors .

Replacing the corresponding terms in equation(9) by notations and approximations given in equations (11) and (12), and neglecting products and powers of δ's higher than the first ; substituting for f_{ix}, f_{iy} , and f_{iz} using equations (8), (7), and (6); and making use of equation (3) gives :

$$\begin{aligned}
\delta\dot V_x = {} & \delta V_x\left(\frac{V_z}{R}\right) - \delta V_y\left(2\Omega_v + 2\frac{V_y}{R}\tan L\right) + \delta V_z\left(\frac{V_x}{R}\right) - \delta L\left(2\Omega_h V_y + \frac{V_y^2}{R}\sec^2 L\right) \\
& - \Psi_y\left(\dot V_z + \frac{V_x^2}{R} + \frac{V_y^2}{R} + 2\Omega_h V_y - g_o\right) - \Psi_z\left(-\dot V_y + 2\Omega_v V_x + \frac{V_x V_y}{R}\tan L + \right. \\
& \left. 2\Omega_h V_z + \frac{V_y V_z}{R}\right) + e_{ax} + g_o\xi \quad .
\end{aligned}$$

$$\delta\dot{V}_y = \delta V_x\left(2\Omega_v + \frac{V_y}{R}\tan L\right) + \delta V_y\left(\frac{V_x}{R}\tan L + \frac{V_z}{R}\right) + \delta V_z\left(2\Omega_h + \frac{V_y}{R}\right) +$$
$$\delta L\left(2\Omega_h V_x + \frac{V_x V_y}{R}\sec^2 L - 2\Omega_v V_z\right) + \Psi_x\left(\dot{V}_z + \frac{V_x^2}{R} + \frac{V_y^2}{R} + 2\Omega_h V_y - g_o\right) \tag{13}$$
$$-\Psi_z\left(\dot{V}_x + 2\Omega_v V_y + \frac{V_y^2}{R}\tan L - \frac{V_x V_z}{R}\right) + e_{ay} + g_o\xi \quad .$$

$$\delta\dot{V}_z = -\delta V_x\left(2\frac{V_x}{R}\right) - \delta V_y\left(2\Omega_h + 2\frac{V_y}{R}\right) + \delta L(2\Omega_v V_y) + \Psi_x\left(-\dot{V}_y + 2\Omega_v V_x + \frac{V_x V_y}{R}\tan L\right.$$
$$\left. + 2\Omega_h V_z + \frac{V_y V_z}{R}\right) + \Psi_y\left(\dot{V}_x + 2\Omega_v V_y + \frac{V_y^2}{R}\tan L - \frac{V_x V_z}{R}\right) + e_{az} - \Delta g \ .$$

IV. DERIVATION OF LATITUDE AND LONGITUDE ERROR EQUATIONS

The latitude error is given by

$$\delta\dot{L} = \dot{L}' - \dot{L} = \frac{V_x'}{R} - \frac{V_x}{R} \quad ,$$

$$\delta\dot{L} = \frac{\delta V_x}{R} \tag{14}$$

The longitude error is given by

$$\delta\dot{\lambda} = \dot{\lambda}' - \dot{\lambda} = \frac{V_y'}{R\cos L'} - \frac{V_y}{R\cos L}$$

$$= \frac{V_y + \delta V_y}{R(\cos L - \delta L \sin L)} - \frac{V_y}{R\cos L}$$

$$\delta\dot{\lambda} = \delta V_y\left(\frac{1}{R\cos L}\right) + \delta L\left(\frac{V_y \tan L}{R\cos L}\right) \tag{15}$$

V. DERIVATION OF TILT RATE EQUATIONS

Considering the instrumented rotating platform frame misaligned from the geographic rotating frame by the vector $\bar{\Psi}$, the components of the geographic rotating platform frame along the misaligned axes are, therefore, for small misalignment angles

$$\begin{bmatrix} \tilde{w}_{ix} \\ \tilde{w}_{iy} \\ \tilde{w}_{iz} \end{bmatrix} = \begin{bmatrix} 1 & \Psi_z & -\Psi_y \\ -\Psi_z & 1 & \Psi_x \\ \Psi_y & -\Psi_x & 1 \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \tag{16}$$

but $\dot{\Psi}_x = w_{ix} - \tilde{w}_{ix}$, $\dot{\Psi}_y = w_{iy} - \tilde{w}_{iy}$, and $\dot{\Psi}_z = w_{iz} - \tilde{w}_{iz}$ (17)

where w_{ix}, w_{iy}, and w_{iz} are the instrumented space rate components of the platform about the instrumented platform. These components are obtained by torquing the gyros with the rates required to maintain the platform local level ,

$$
\begin{aligned}
w_{ix} &= \Omega' \cos L' + w_e' + e_{gx} \quad , \\
w_{iy} &= \quad - w_n' \quad + e_{gy} \quad , \\
w_{iz} &= -\Omega' \sin L' - w_e' \tan L' + e_{gz} \quad .
\end{aligned}
\tag{18}
$$

where e_{gx}, e_{gy}, and e_{gz} are the gyro drift rates .

Using equations (1), (16), (17), and (18) , the set (17) gives

$$
\begin{aligned}
\dot{\Psi}_x &= \frac{\delta V_y}{R} - \delta L(\Omega_v) - \Psi_y(\Omega_v + \frac{V_y}{R}\tan L) + \Psi_z(\frac{V_x}{R}) + e_{gx} \\
\dot{\Psi}_y &= - \frac{\delta V_x}{R} + \Psi_x(\Omega_v + \frac{V_y}{R}\tan L) + \Psi_z(\Omega_h + \frac{V_y}{R}) + e_{gy} \\
\dot{\Psi}_z &= -\delta V_y(\frac{\tan L}{R}) - \delta L(\Omega_h + \frac{V_y}{R}\sec^2 L) - \Psi_x(\frac{V_x}{R}) \\
&\quad - \Psi_y(\Omega_h + \frac{V_y}{R}) + e_{gz} \quad .
\end{aligned}
\tag{19}
$$

Finally, the sets (13), (14), (15), and (19) reprsent eight differential equations containing 8 variables, which can be written directly in matrix form.

VI. CONCLUSION

We have obtained an 8th order linear dynamical system model with the gyro errors, accelerometer errors, and gravity uncertainties as the driving functions. Depending upon vehicle manoeuvre, users have a wide variety of simplifications of the time varying matrix depending on their particular interest .

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