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ESTIMATION OF THE AVERAGE RATE OF PHOTOELECTRONS USING THE TIME SEQUENTIAL TECHNIQUES

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ABSTRACT

The estimation of the average rate of photoelectrons λa based on the time sequential techniques, is examined for two modes of the optical field, the ideal laser light (Poisson), and the Gaussian light, (Bose Einstein, BE). For the second model, the effect of the used statistics is examined by applying the waiting time statistics and the life time statistics. The numerical values of the instants of occurrence of photoevents are obtained by simulation of the process.

The results are compared with what are obtained using the fixed sample size techniques. The comparison assured that the two techniques are equivalent for Poisson model, but for thermal model, estimation based on the sequential techniques are superior to those based on the fixed sample size techniques.

INTRODUCTION

The determination of the average rate of photoelectrons per second λa of an optical field as one of its basic parameters using the statistical data received from a photodetector is considered as a very important requirement in many applications.

Two modes of statistical techniques may be used to estimate λa. Fixed sample size techniques involve the following procedures: The number of output pulses, $n i$, from a photodetector is observed for a counting subinterval time T and recorded. A sequence of similar measurements is made. Then the sequence, $\{ni\}$, is used to determine the maximum likelihood estimate of λa. The number of fixed subintervals is determined by the required statistical accuracy of the estimate.

Recently, sequential detection techniques have been used to determine λa from statistical data [1]. These techniques involve the measurement of the arrival time t(m) required to observe m successive events. Again a sequence of $\{t(m)\}$ values is made and the results are used to determine a maximum likelihood estimate.

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The objective of both techniques is to obtain a fast estimate, since the communication system needs to maintain a particular value of λa, for the accurate estimation of $\hat{\lambda}$.

The task of this work is to study the features of the clock receiver when it is used to estimate the basic parameters of the optical field. This receiver is based on recording the arrival times of the mth photoelectron [2].

To attain this objective, the effect of the mode of the incident field on the accuracy of the estimate is examined, and then the role of the order of the received event is investigated. Two models of optical field have been chosen, the Poisson model and the thermal model.

POISSON MODEL

For coherent light, the intensity is nonfluctuating and therefore, the point process of detected photoelectrons must be Poisson distributed. The maximum likelihood estimate (MLE), $\hat{\lambda}$, based on the fixed sample size techniques is given by [3].

$$\hat{\lambda} = \frac{1}{TN} \sum_{i=1}^{N} n_i \tag{1}$$

where N is the number of measured samples, T is the subinterval counting time (sample size), and i is the number of photoelectrons measured in the subinterval time T for the sample i.

Generally, the bias measured by $(\lambda_a - \hat{\lambda})$ and the variance may be introduced as two separate error measures of the estimate.

For this model, the estimate is an unbiased estimate, where the average $\langle\hat{\lambda}\rangle$ equals to λa and the normalized mean square error e_λ^2 is simply given by:

$$e_\lambda^2 \triangleq \operatorname{var}(\hat{\lambda}) / \lambda_a^2 = \frac{1}{N\bar{n}} \tag{2}$$

where $\bar{n} = \langle n \rangle = \lambda a \, T$

Another practical measurement of the accuracy of the estimator is statistical confidence level νn [1] which is defined by:

$$\nu_n = \int_{\hat{n}-\alpha}^{\hat{n}+\beta} p(n|\bar{n})\, p(\bar{n})\, d\bar{n} \Big/ \int_0^{\infty} p(n|\bar{n})\, p(\bar{n})\, d\bar{n} \tag{3}$$

where $p(\bar{n})$ is the probability density of the means of the photoelectrons distribution. The interval $(\hat{n}+\beta, \hat{n}-\alpha)$ is defined as the confidence interval and must be chosen on the basis of desired accuracy of the estimate, i.e., by requiring the estimate to have a $(1 - \nu n)$ % chance of being in error by more than a fixed amount (confidence interval). For Poisson model, νn is given by:

$$P_n = \int_{N(\hat{n}-\alpha)}^{N(\hat{n}+\beta)} (\bar{X})^n / n! \; . \; \exp(-\bar{X})\, d\bar{X} \qquad (4)$$

where $\bar{X} = N.\bar{n}$, and $n = \sum_{i=1}^{N} n_i$.

Now for the estimation based on sequential techniques, the probability density function (p.d.f.) for the occurance time tm of the mth event for the Poisson process is given by [4]:

$$p(tm \mid \lambda_a) = \frac{\lambda_a^m\, (tm)^{m-1}}{(m-1)!} \exp(-\lambda_a tm) \qquad (5)$$

The joint probability density that a particular sequence of arrival times t_m^1 , t_m^2 , t_m^3 , ... , t_m^N will be observed after N independent measurments is given by:

$$P(t_m^1, t_m^2, \ldots\ldots t_m^N) = \prod_{i=1}^{N} P(t_m^i \mid \lambda a) \qquad (6)$$

The MLE of λa is obtained by finding the value of λa that renders Eq.(6) the maximum. This value is given by:

$$\hat{\lambda} = \frac{Nm}{\sum_{i=1}^{N} t_m^i} \qquad (7)$$

The sequence t_m^i is found by simulation. The simulation of the first event is done using the systematic sampling method. Then the first, second and third moments of the distributions are computed numerically and compared with those based on the actual distributions. The comparison verified the simulatiom is reasonably accurate.

For this model, the three separate error measures of the estimate have been checked. Using Eqs.(5) and (7), and by induction we get the average of the estimate:

$$\langle\hat{\lambda}\rangle = \lambda_a \cdot mN / (mN-1) \qquad (8)$$

provided that m.N > 1.

This equation states that the mean value of the estimate approaches the real value λ a when the observed photoelectrons are as much as possible, which is equivalent to observe as much of the optical energy as possible.

Following a similar procedure, an analytical expression for the mean square error e_λ^2 is derived:

$$e_\lambda^2 = (mN)^2 / (mN-2)(mN-1)^2 \qquad (9)$$

provided that m.N > 2.

For very high value of m.N, this equation will be:

$$e_{\lambda}^{2} = 1/mN \tag{10}$$

This gives that the estimation error is equal to the inverse of the total number of detected photoelectrons in the overall observation time,

$$T_0 = \sum_{i=1}^{N} t_m^{(i)} \tag{11}$$

Comparing Eqs. (2) and (10), we come to the conclusion that both techniques are equivalent.

The confidence level, ν_t , is found by using the definition given by Eq. (3), where the corresponding distribution and parameters are used:

$$\nu_t = \int_{(\hat{\lambda}-\alpha)T}^{(\hat{\lambda}+\beta)T} x^{K} / K! \cdot \exp(-x)\,dx \tag{12}$$

where $X = \lambda a\ T_o$, and $K = m.N$.

Recalling Eq. (4), and comparing it with Eq. (12), we get that for the same observation time T_o , νn and νt, based on the first and second techniques respectively, are identical for getting an equal $\hat{\lambda}$ given the same confidence intervals. This assures once more that the two techniques are equivalent.

THERMAL MODEL

When the counting time interval T is short compared to the coherence time of the field (τc), and the detector area A is small compared to the coherence area Ac, the counting statistics of n is given by the Bose-Einstien distribution (Geometric distribution).

The estimation based on the arrival time of the mth photoelectron, either measured from an arbitrary chosen origin of the time axis (waiting time statistics, W.T.S.) or provided an event received at the origin (life time statistics, L.T.S.) is investigated. For Poisson model, both statistics are equivalent.

For W.T.S., the probability that a particular sequence of occurance times t_m^1 , t_m^2 , . . . , t_m^N will be observed after N independent measurements of the arrival time of the mth event is given by [4]:

$$P(t_m^1, t_m^2, \ldots\ t_m^N) = \prod_{i=1}^{N} \frac{m\lambda_a^m\ (t_m^{(i)})^{m-1}}{(1+\lambda_a t_m^{(i)})^{m-1}} \tag{13}$$

The maximum likelihood estimate $\hat{\lambda}$ is obtained simply by numerical computation of the value of λa which maximizes the R.H.S. of Eq. (13). For this case $\hat{\lambda}$ has no explicit expression, and therefore it is difficult to find an expression for the mean square error e_{λ}^{2} or even the mean of the estimate.

The values { t_m^i } are simulated, and the histograms for the corresponding distributions when m = 1,2 are shown in Fig.1 and Fig.2, respectively. The confidence level γ_t is computed for different number of samples N.

For the same optical field, the probability density of the arrival time of the mth photoelectron, based on L.T.S., is given by[4]:

$$P(0|t_m) = m\,(m+1)\,\lambda_a \frac{(\lambda_a t_m)^{m-1}}{(1+\lambda_a t_m)^{m+2}} \qquad (14)$$

By the same procedures used for waiting time statistics, the maximum likelihood estimate $\hat{\lambda}$ can be determined for this case. The simulation of the arrival time of the first photoelectron (m = 1) is done for three different values of λ_a. The accuracy of the simulated distribution has been checked by comparing the first moment < t_1 > based on the analytical distribution given by Eq. (14) with the one computed for the simulated distribution. Satisfactory accuracy is obtained.

RESULTS AND DISCUSSION

The estimation based on the occurance times of photoevents was examined for two models. The numerical values of those events were evaluated by the simulation of the process. The accuracy of the simulation was checked, and a satisfactory result was obtained.

For Poisson model, two analytical expressions for the mean value of the estimate and the mean square error were derived. For large number of received events, the estimate will be unbiased and the expression of e^2_{λ} is equivalent to that given by the fixed sample size technique. On the other hand by comparing the confidence levels γ_n and γ_t for fixed sample size and sequential techniques, respectively, we find that they are identical when finding the same estimate, throughout the same observation time T_o . These comparisons assure that the two techniques are equivalent for Poisson model, and no gain in accuracy is obtained by the sequential technique.

For the sequential technique, by comparing the two optical field models, we find that the total number of the counts mN is sufficient for estimating λ in the case of Poisson, whereas the order i of the samples t_m^i is involved for estimating $\hat{\lambda}$ in the thermal field (BE).

The computations showed that for BE distribution, the required number of samples to estimate λ_a within a certain accuracy, is higher that required for Poisson model. This may be concluded from the results shown in Fig.3. Moreover, the total observation time T_o for N samples of Poisson model is less than the corresponding one of BE (W.T.S.) model. The previous results come from the fact that BE distribution is characterized by a variance larger than that of Poisson distribution. This difference makes the detection and estimation in the case of Poisson model easier than in the BE one.

For the BE distribution, the results demonstrated in Fig.3 shows thst the accuracy of the estimation based on life time statistics (L.T.S.) is better than that based on waiting time statistics (W.T.S.). This conclusion arises from the fact that the L.T.S. carries more information about the process.

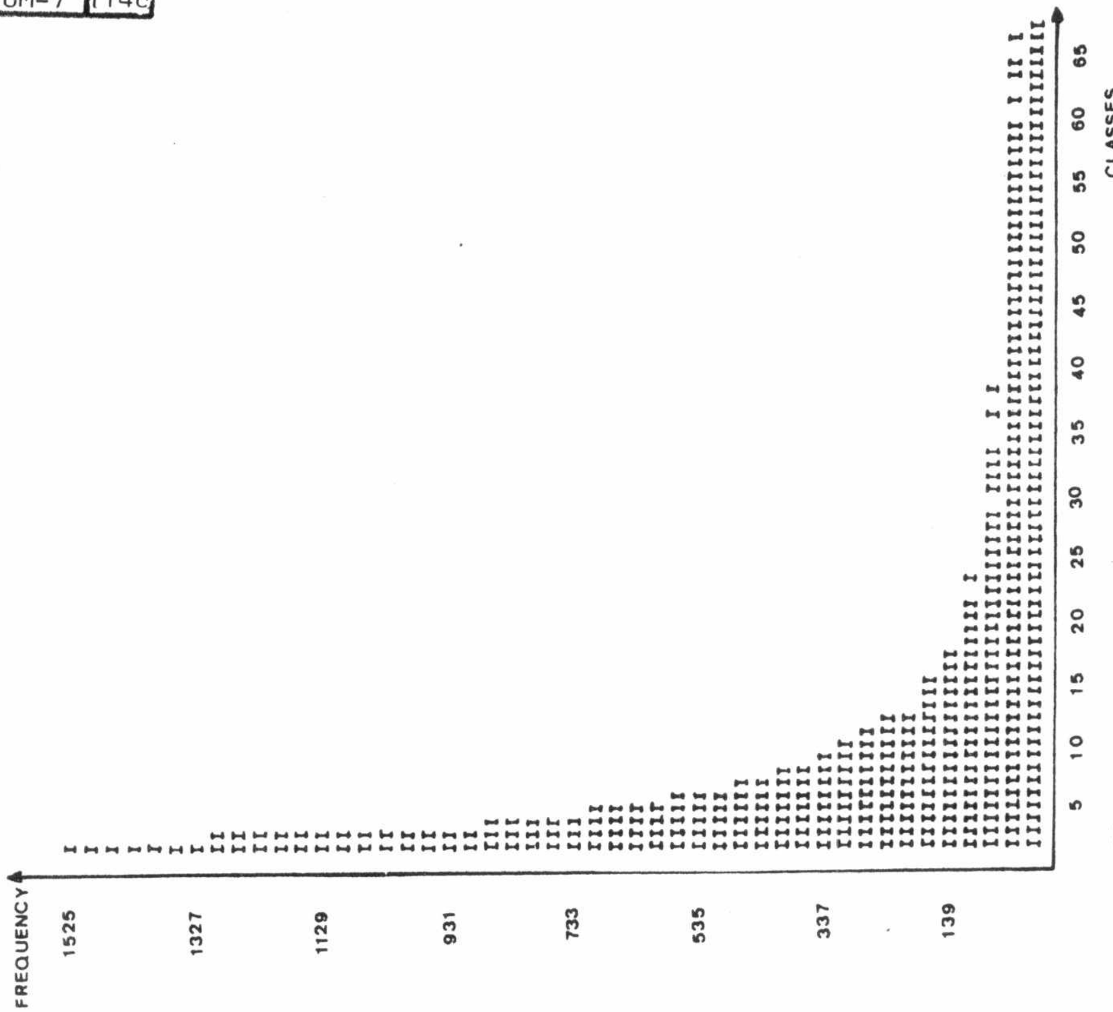

Fig.1: Histogram for the simulated BE distribution with m= 1, $\lambda_a = 1.x10^{+6}$ sec.$^{-1}$.

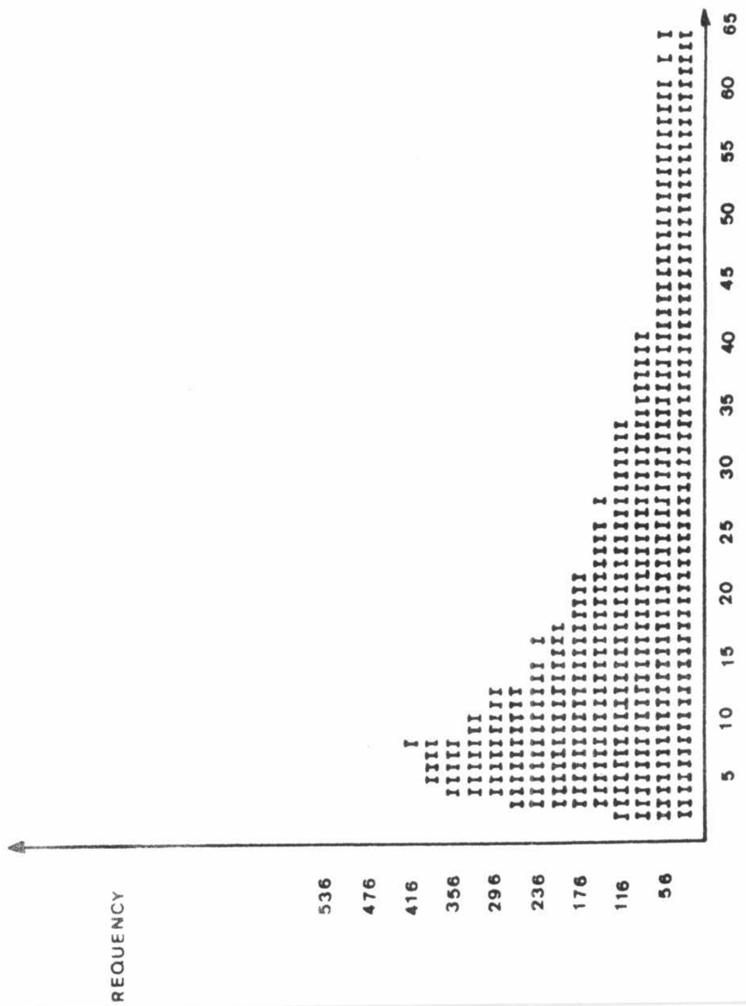

Fig. 2: Histogram for the simulated BE distribution with m = 2, $\lambda_a = 0.25 \times 10^{+6}$ sec.$^{-1}$.

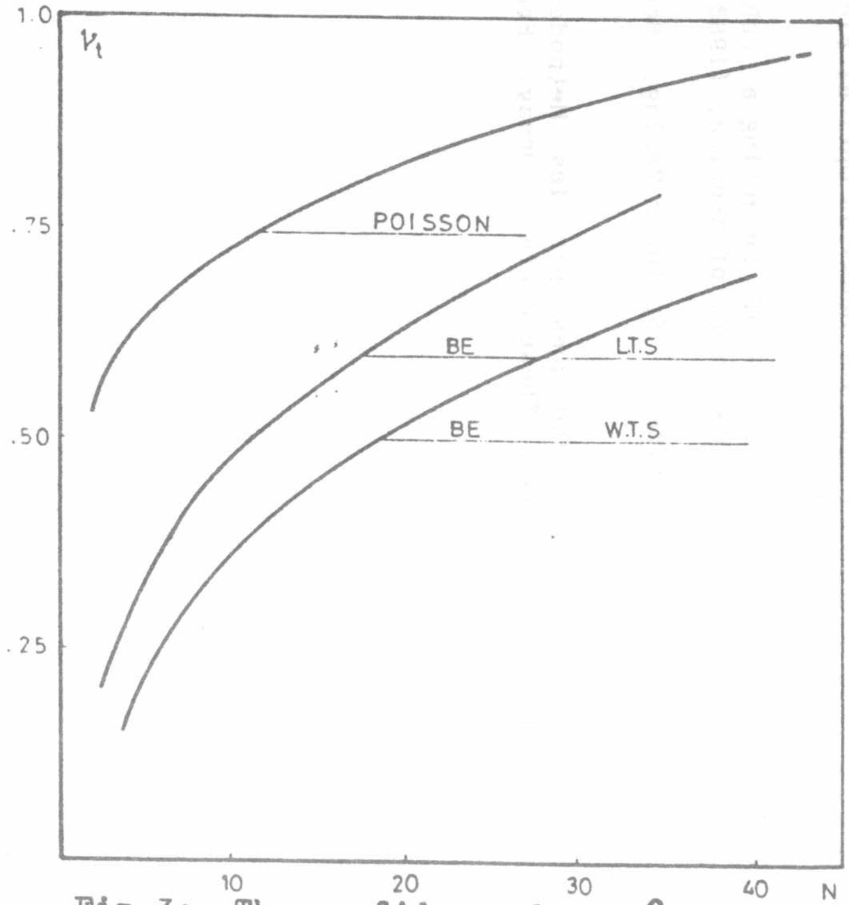

Fig. 3: The confidence level ν_t versus the used number of samples N for poisson, BE models. For BE, waiting time and life time statistics are used .

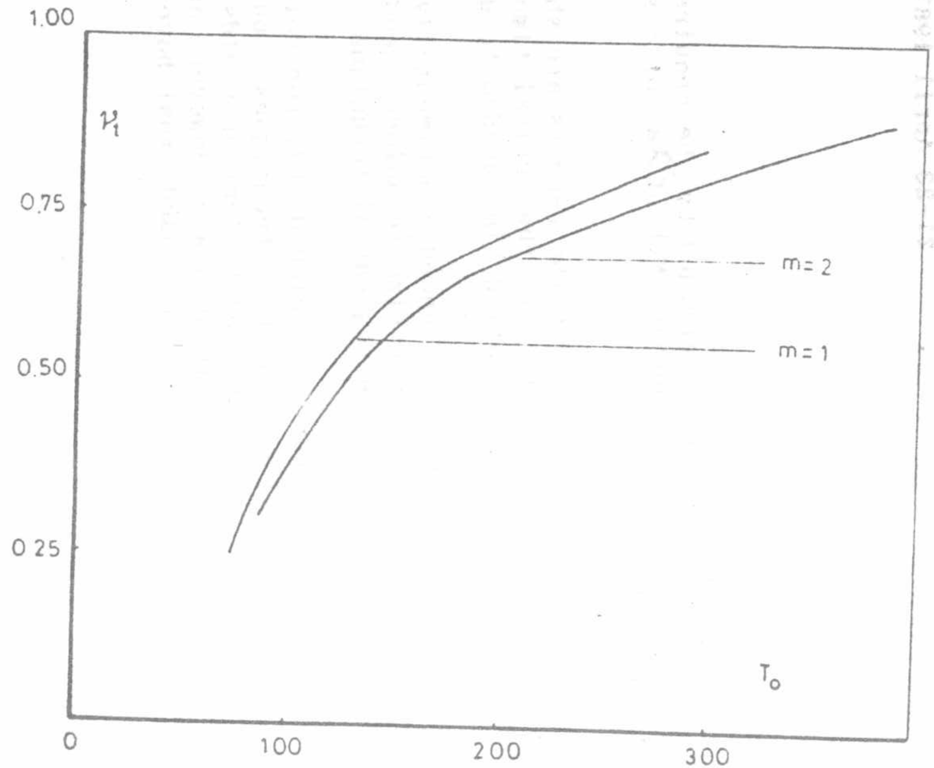

Fig. 4: The confidence level ν_t versus the total observation time T_o with m=1,2; for BE.

The optimum choice of the order m is determined by the requirement that the total observation time T_o needed to obtain MLE of λa at a specific confidence level has to be a minimum.

The calculations were done for m = 1,2, and the results are shown in Fig.4. The results indicate that the simulation of the arrival time of the first photoelectron yields the shortest observation time needed to determine $\hat{\lambda}$.

For this model, Davidson and Amoss [1] showed experimentally that for the same estimation accuracy the fixed sample size techniques need more time for estimating λa than that required by the sequential technique.

From the previous analysis, it may be concluded that for Poisson model, no gain in accuracy is obtained by the sequential techniques, but for thermal model, the estimation based on the first photoelectron yields the shortest time of observation while giving the same accuracy. However, as the observation time is decreased, the electronic devices used must have a more rapid response.

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