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A METHOD FOR SOLVING SUPERSONIC FLOW OVER A FLAT PLATE

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ABSTRACT

A new method for solving the compressible, laminar, boundary layer equations over a flat plate is developed in the present study. The method accounts for the variations of all boundary layer parameters which control the flow, and it is able to solve the equations for any gas. In this method, the Illingworth transformation is used to transform the equations of motion to a set of ordinary differential equations (ODE). The fourth-order Runge-Kutta method in conjunction with the shooting method are used to solve the set of ODE. A very efficient procedure is developed to find out the unknown boundary conditions at the wall.

The method is applied to investigate the characteristics of supersonic flow of air over cooled, adiabatic, and heated walls, where wide ranges of free stream Mach numbers and wall-free stream temperature ratios are used. Comparsons of the present results with previous ones show that the method is very efficient. In addition to the verification of the success of the method, investigation of the characteristics of the flow leads to very useful results.

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INTRODUCTION

Many methods have been developed to solve the laminar, compressible, boundary layer equations over a flat plate [1-3]. Most of these methods are restricted by some approximations which lead to simplified equations. However the resulting soution may not be consistant with that of the real flow due to the nonrealistic assumptions. For example, if one considers Pr = 1 for air (the actual value is 0.72), the simplified solution [3] is greatly deviated from the experimental one [4]. Thus in order to efficiently solve the equations, one must not put any restriction on any controling parameter and must account for the variations of the flow properties in a realistic manner. Of cours the equations will be more complicated and many simplified methods fail to solve them.

The present study provides a new method which takes into considerations all the above precautions. As will be shown, the method can solve the boundary layer equations for any arbitrary value of Pr and for any correlations of the viscosity with temperature, that is, the method is able to solve the equations for any gas by appropraitely selecting the above two parameters.

GOVERNING EQUATIONS

The two dimensional compressible, laminar, boundary layer equations are [6]

$$(\rho u)_x + (\rho v)_y = 0 \qquad (1)$$

$$\rho u u_x + \rho v u_y = (\mu u_y)_y \qquad (2)$$

$$\rho u\, h_x + \rho v\, h_y = \left(\frac{\mu}{pr}\, h_y \right)_y + \mu\, u_y^2 \qquad (3)$$

The energy equation has been derived upon the assumption of perfect gas relations with constant specific heats:

$$h = Cp\, T \qquad (4)$$

$$P = \rho R\, T \qquad (5)$$

The viscosity is related to the temperature by the power law [7]

$$\frac{\mu}{\mu_o} = \left(\frac{T}{T_o}\right)^n \qquad (6)$$

Where μ_o and T_o are reference values and n is an exponent. Empirical values of μ_o , T_o, and n are tabulated in Ref.[7] for various gases.

METHOD OF SOLUTION

The method of solution consists of the transformation of the equations of motion to a set of ordinary differential equations (ODE), numerical method for solving the ODE, and a procedure for finding out the unknown boundary conditions (B.C.) at the wall.

Equations (1) to (3) are transformed into a set of ODE by using the Illingworth transformation [5]. In this situation a similarity solution is obtained. The transformations used are given by

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$$\eta = (Re/2)^{\frac{1}{2}} \int_o^y \frac{\rho}{\rho_e} \frac{dy}{x} \qquad (7)$$

$$\psi = \sqrt{2 \rho_e U_e \mu_e x} \; f(\eta) \qquad (8)$$

$$\frac{U}{U_e} = f'(\eta) \qquad (9)$$

$$\frac{h}{h_e} = \frac{T}{Te} = g(\eta) \qquad (10)$$

$$\frac{\rho \mu}{\rho_e \mu_e} = C = g^{n-1} \qquad (11)$$

Where C is the Cahpman-Rubesin parameter [6]. The continuity equation is satisfied identically. Transformations of the other equations yield:

$$f''' + (n-1)\, g^{-1} g' f'' + g^{1-n} f f'' = 0 \qquad (12)$$

$$g'' + (n-1)\, g^{-1} g'^2 + g^{1-n} Pr\, f g' + Pr\, (\gamma-1) M_e^2 f''^2 = 0 \qquad (13)$$

The boundary conditions are

No slip $\quad f_w = f'_w = 0$ (14)

Free streem $f'_e = g_e = 1$ (15)

In addition, either one of the following B.C. must be satisfied

Adiabatic wall $\quad g'_w = 0$ (16)

Heat transfer $\quad g_w = T_w/T_e$ (17)

In order to solve equations (12) and (13) subjected to the B.C. [equations (14) to (17)], the fourth order Runge-Kutta method [8] is used where solution starts at the wall and extends throughout the boundary layer until the free stream is reached. The shooting method is used to predict the unknown B.C. at the wall [g'_w (or g_w) and f''_w] where guessed values are assumed and the solution in the whole boundary layer is obtained. The computed values of f'_e and g_e are then compared with their exact values and the procedure is repeated until convergence is achieved.

To obtain the exact values of the unknown boundary conditions at the wall [g'_w (or g_w) and f''_w], the following procedure is performed: As seen there are two unknowns. For simplicity, let these two unknowns be denoted by the letters a and b. The procedure is based upon the assumption that the computed free stream conditions are functions of a and b. When the exact values of a and b are obtained, the following equations must be statisfied:

$$F = f'_e - 1 = F(a,b) = 0 \qquad (18)$$

$$G = g_e - 1 = G(a,b) = 0 \qquad (19)$$

Expanding the above equations into Taylor series about the point (a_i, b_i) yields

$$F(a_i + \Delta a, b_i + \Delta b) = F_i + \Delta a\, F_a(a_i, b_i) + \Delta b\, F_b(a_i, b_i) = 0 \qquad (20)$$

$$G(a_i + \Delta a, b_i + \Delta b) = G_i + \Delta a\, G_a(a_i, b_i) + \Delta b\, G_b(a_i, b_i) = 0 \qquad (21)$$

Where subscripts a and b are used to denote partial differentiation with respect to the subscript at point (a_i, b_i). Equations (20) and (21) can be solved for Δa and Δb. The values of a_i and b_i are then corrected by adding Δa to a_i and Δb to b_i and the procedure is repeated until equations (18) and (19) are satisfied.

The partial derivatives with respect to a are computed as follows: The value of a_i is increased by a small amount $\varepsilon = 0.001$ such that its value becomes $a_1 = a_i + \varepsilon$, and the value of b_i is kept unchanged. Equations (12) and (13) are then solved with B.C. a_1 and b_i. Let the corresponding free streem values denoted by F_1 and G_1. The partial derivatives are computed from the equations

$$F_a\,(a_i, b_i) = (F_1 - F_i)/(a_1 - a_i) = (F_1 - F_i)/\varepsilon \tag{22}$$

$$G_a\,(a_i, b_i) = (G_1 - G_i)/(a_1 - a_i) = (G_1 - G_i)/\varepsilon \tag{23}$$

Similar procedure is performed such that the partial derivatives with respect to b can be obtained.

RESULTS AND DISCUSSIONS

The flow of air has been choosen in our application. The values of Pr, n, and γ are 0.72, 2/3, and 1.4, respectively. Keeping these values unchanged, it is clear that the remainder parameters which control the solution are M_e and g_w. The effect of these two parameters is investigated in the present work. The solutions are presented in the nondimensional physical coordinate $\bar{y}$, given by:

$$\bar{y} = \frac{y}{x}\sqrt{Re} = \sqrt{2}\int_0^{\eta} g\, d\eta \tag{24}$$

Figure 1 illustrates the profiles of the adiabatic flow. The results of Ref. [9] are also shown. It is seen that as Me is increased, there is a considerable thickness of the boundary layer $\bar{y}_e$. The profiles of f' are approximately linear for values of Me equal to or higher than 4. The adiabatic wall temperature g_{wa} is seen to increase by increasing Me. A very important conclusion is obtained from the figure by noticing the similarity of the profiles of g and f" (notice that both g and f" have maximum values with zero slopes at the wall). This similarity of the profiles may suggest the existance of a solution in the form g = g (f"). No attempt has been done in the present work to investigate such solution. The present results are in excellent agreement with that of Ref. [9].

Variation of the adiabatic wall temperature g_{wa} with Mach number sequared is shown in Figure 2. As seen, the profile can be approximated to a very high accuracy by a straight line which indicates that g_{wa} is proportional to M_e^2. Variation of r with Me, is illustrated in Figure 3. The results of Ref. [10] are also presented. The agreement is shown to be excellent. The figure shows that r is slightly decreased by increasing Me.

The flow with heat transfer is illustrated in the remainder figures of this work. Figure 4 illustrates the velocity profiles $f' = u/U_e$. As seen the effect of increasing g_w, keeping Me unchanged, results in an increase of the boundary layer thickness $\bar{y}_e$. Linearity of the profiles

6 is seen to be achieved for any value of Me when g_w reaches a minimum value g_{wm}. The ratio of g_{wm} to the corresponding adiabatic value g_{wa} is of high values at low Me and it decreases by increasing Me.

Figure 5 illustrates the temperature profiles $g = T/Te$. Comparison of the thermal and velocity boundary layers shows that the first is slightly larger than the later. The shape of the profiles is greatly affected by the values of g_w and Me. For cooled wall, the profile is increased from g_w to a maximum value then it decreases gradually. For insulated and heated walls the profile is decreased gradually from its maximum value at the wall. A very important conclusion can be deduced from the profiles over heated walls at low Me: As shown from figure 5-c, the profiles are approximately linear at high values of g_w (4 and 8).

Figure 6 illustrates the g'-profiles. Again the profiles are greatly affect by the values of Me and g_w. The values of g'_w are positive for cooled walls and negative for heated walls. The profiles where Me = 2 and $g_w = 4$ and 8, are shown to have constant values over a wide range of $\bar{y}$. This supports the linearity of the corresponding profiles of g. Figure 7 shows the f"-profiles. As seen the slopes of f" are positive for cooled walls, zero for adiabatic walls, and negative for heated walls. The effect of increasing g_w, keeping Me unchanged, is to increase f''_w. Inspection of the profiles of f" and g shows similarity between them for values of Me of 6 and 12. However this similarity is shown to be lost for low values of Me. The skin firction coefficient C_f and the stanton number C_h [3] are shown in figure 8. The results of Ref. [9] are also illustrated. Excellent agreement of both results are clearly seen.

CONCLUSIONS

A new method for solving the supersonic boundary layer equations over a flat plate is presented in this study. The method is able for solving the equations as applied to any gas by selecting the appropriate values of Pr, n, and γ. The method is applied to the flow of air and it has proven to give excellent results.

In addition to the verification of the success of the method, investigation of the characteristics of the flow leads to the following useful results: (1) The adiabatic wall temperature is proportional to M_e^2. (2) The profiles of f' are approximately linear for any value of Me when g_w reaches a minimum value g_{wm}. The ratio g_{wm}/g_{wa} is of order of multiples of unity at low values of Me and it decreases by increasing Me until it reaches a fraction of unity at high values of Me. (3) The profiles of g are approximately linear for low values of Me and high values of g_w. (4) For values of Me which are greater than 4, the profiles of f" and g are similar. This may suggest the existance of a solution in the form $g = g(f'')$.

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NOMENCLATURES

C	Chapman-Rubesin parameter
C_f	Skin friction coefficient
C_p	Specific heat at constant pressure
f	Transformed stream function
g	Temperature ratio
h	Static enthalpy
M	Mach number
n	Exponent
P	Static pressure
Pr	Prandtl number
R	Gas constant
Re	Reynold's number
U_e	Free stream velocity
u,v	Velocity components in the x and y directions.
x,y	Cartesian coordinates
η	Transformed coordinate
ρ	Density
μ	Viscosity
γ	Specific heat ratio
ψ	Stream function.

Subscripts

e	Free stream condition
w	Wall

Prime denotes differentiation with respect to η

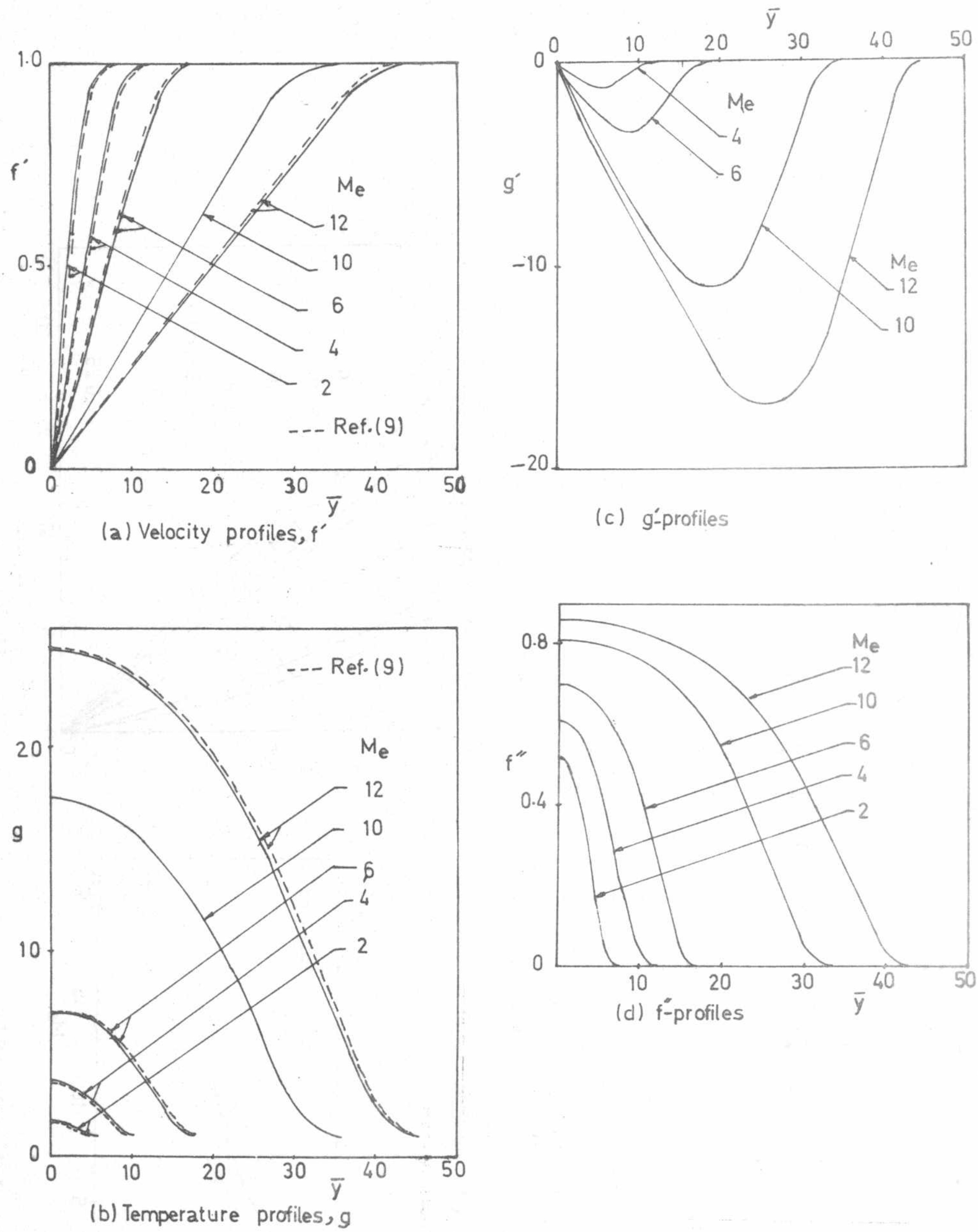

Figure 1. Adiabatic flow

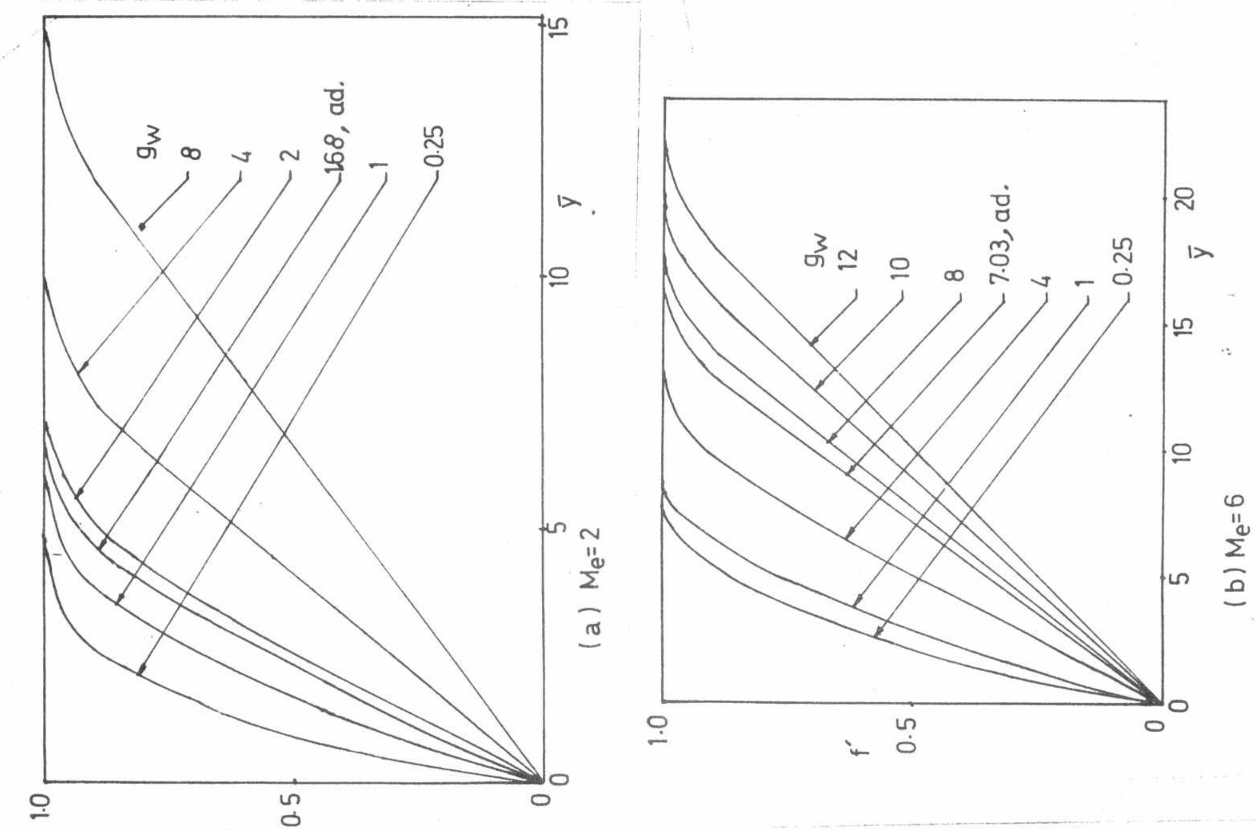

Figure 4. Velocity profiles, f'

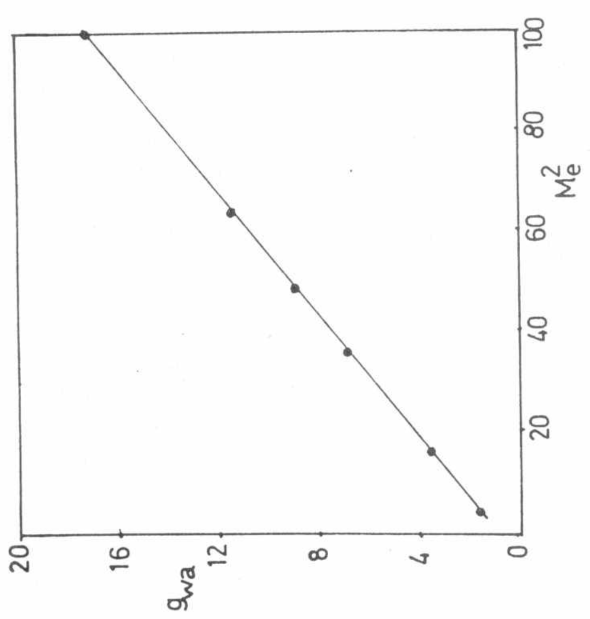

Figure 2. Adiabatic wall temperature

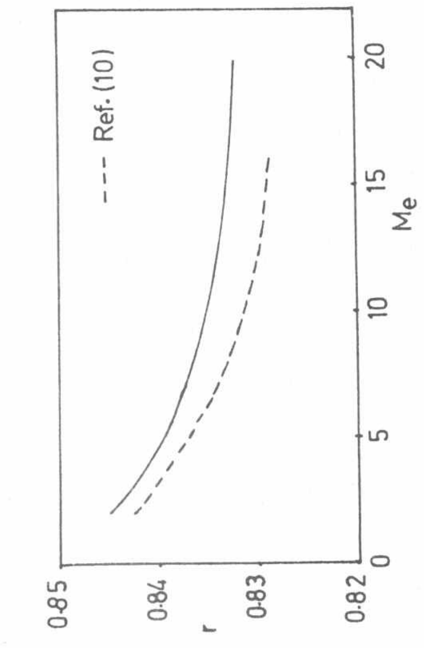

Figure 3. Recovery factor

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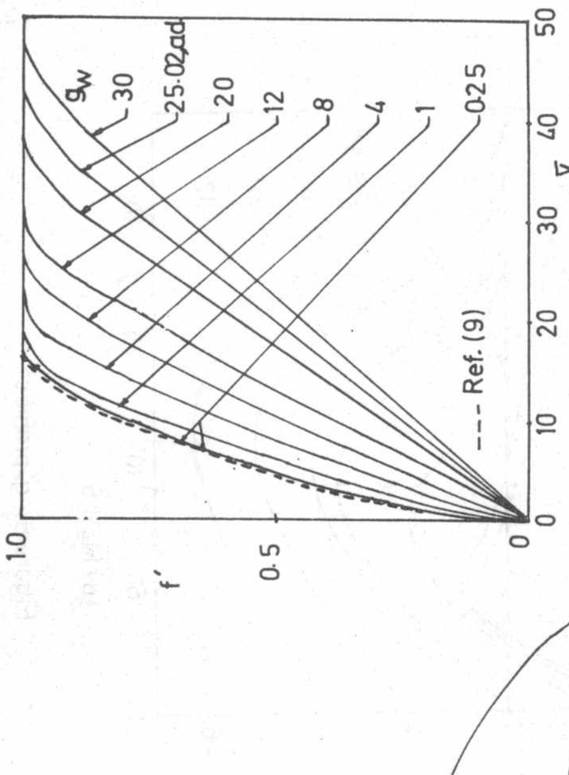
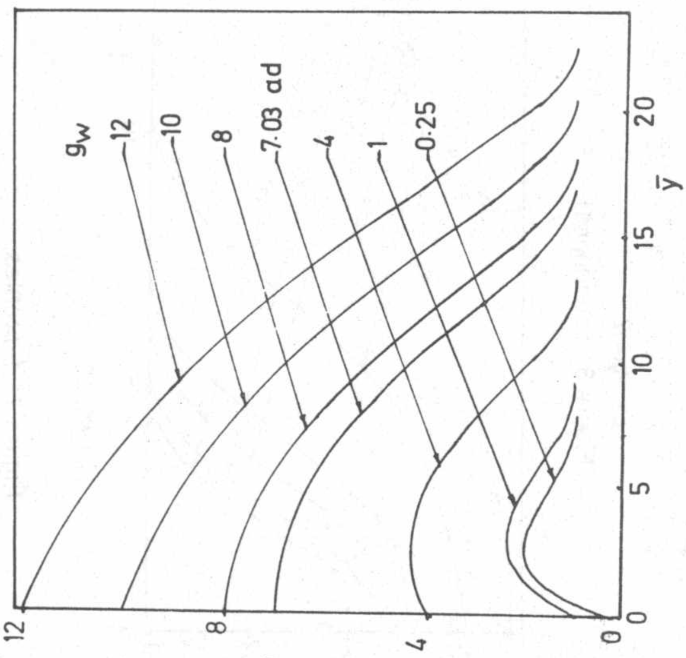
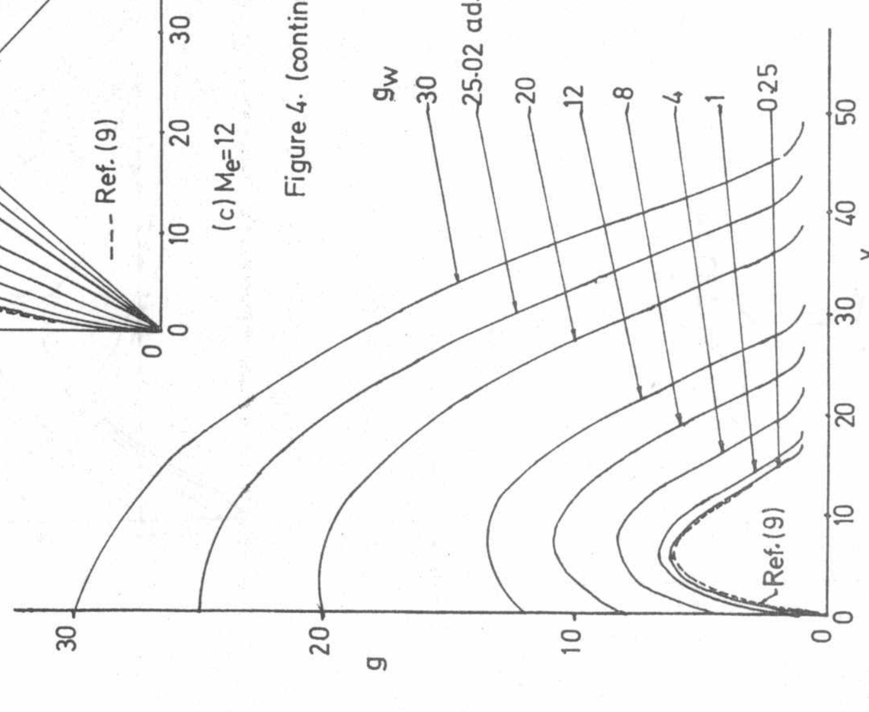


(c) M_e=12

Figure 4. (continued)

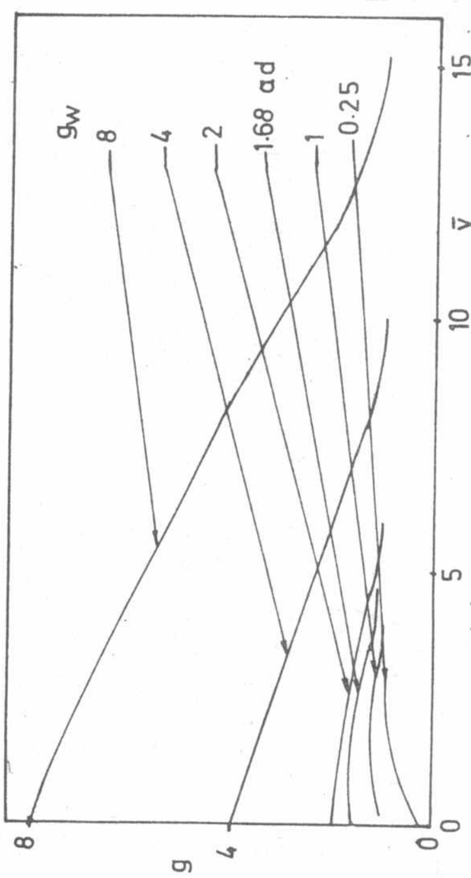


(b) M_e = 6



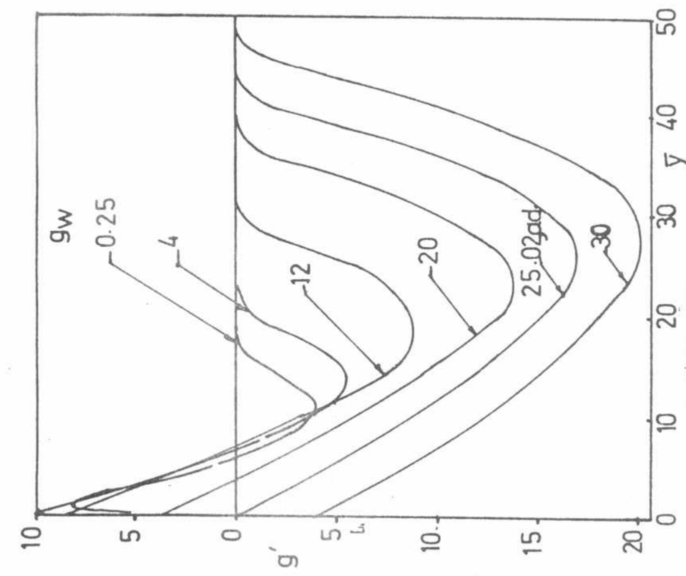
(a) M_e = 12

Figure 5. Temperature profiles, g

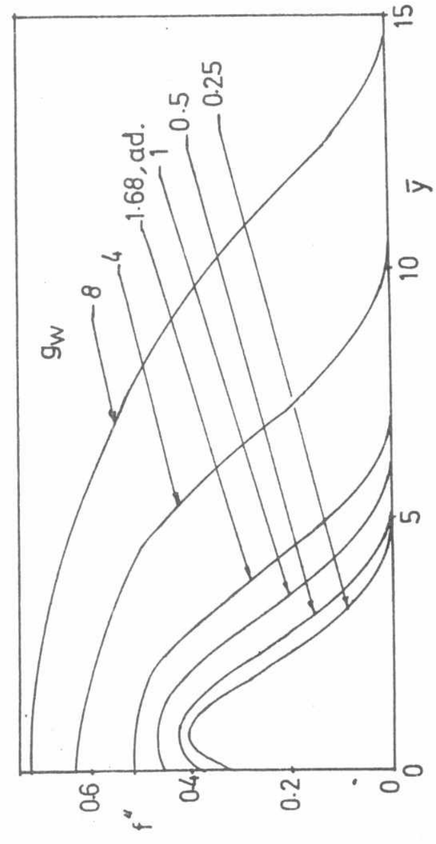


(c) M_e = 2

Figure 5. (continued)

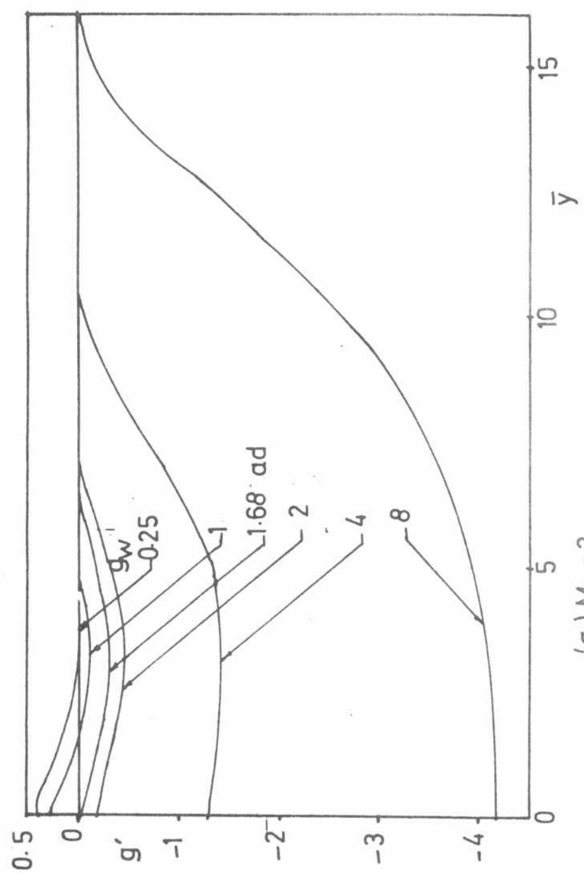


(c) M_e=12
Figure 6. (continued)

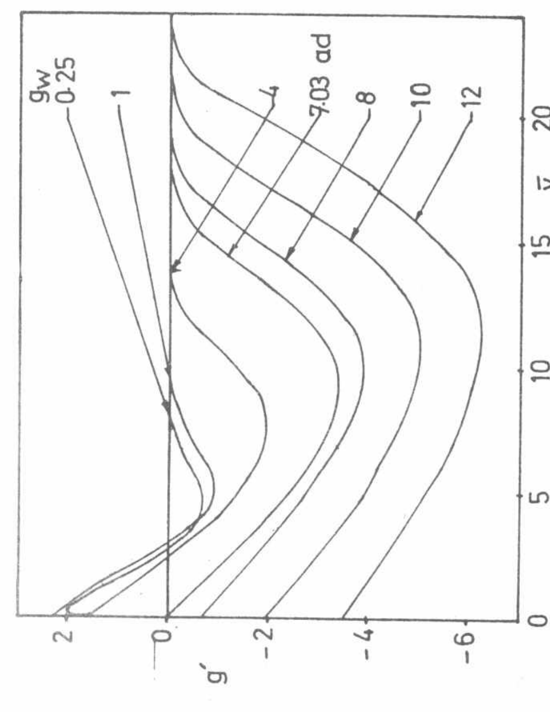


(a) M_e = 2

Figure 7. f''-profiles



(a) M_e = 2



(b) M_e = 6

Figure 6. g'-profiles

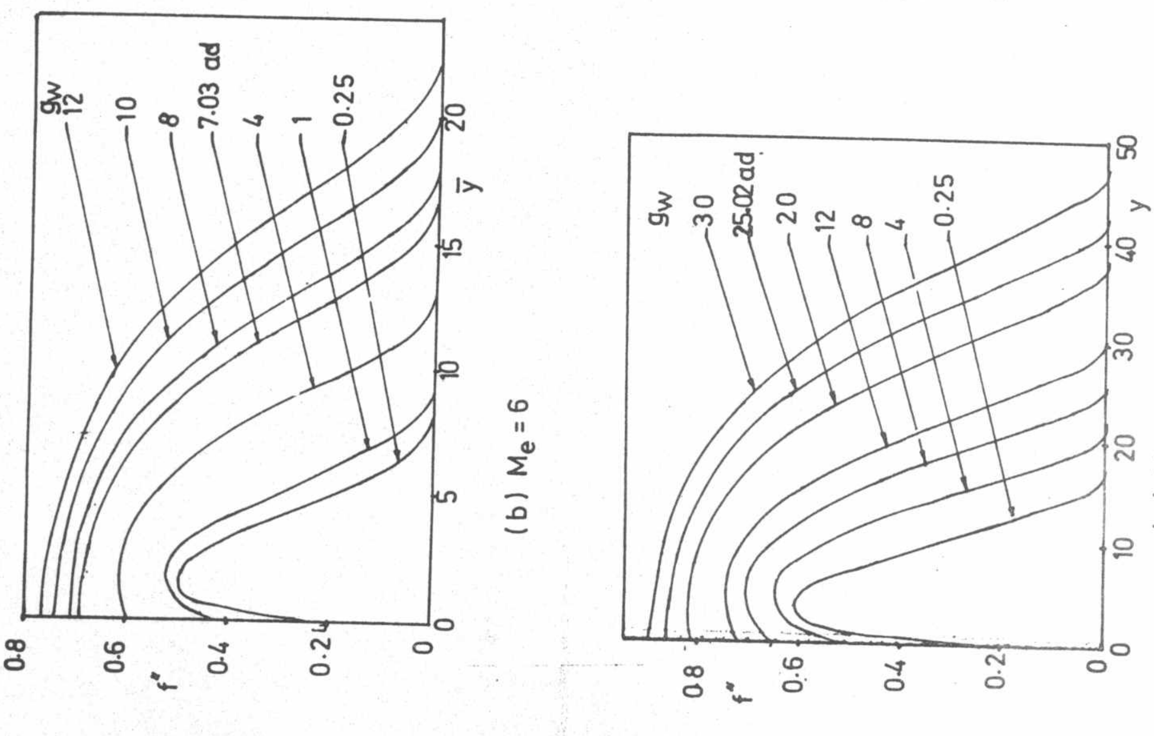


Figure 7. (continued)

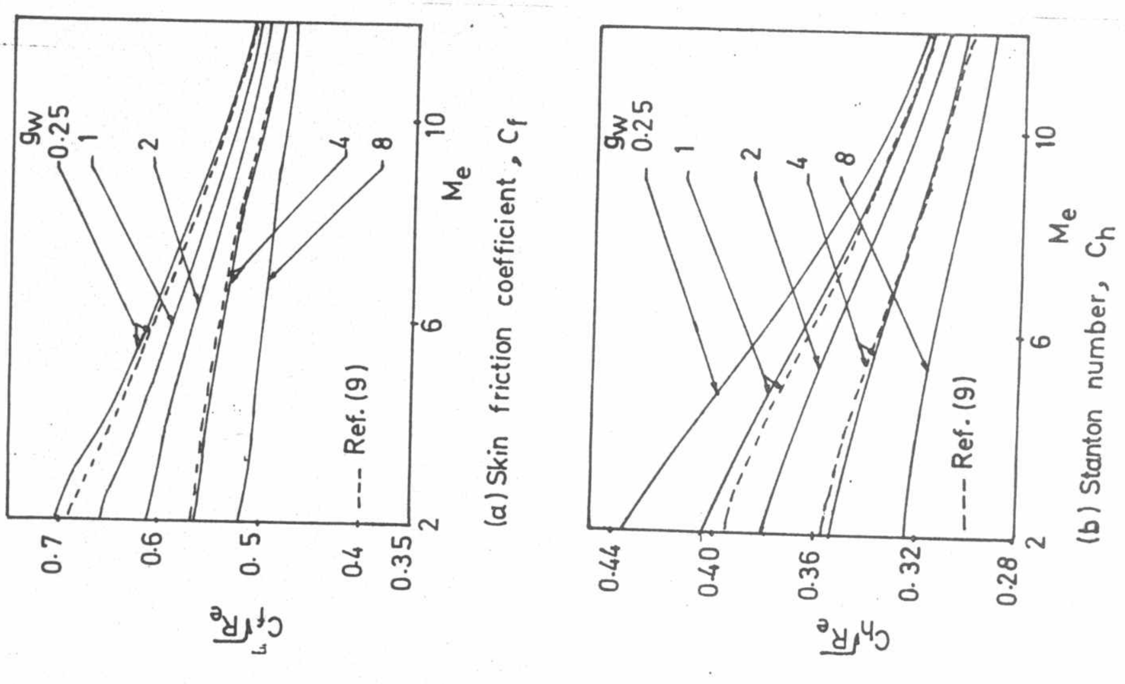


Figure 8. Wall friction and heat transfer