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" 3-D STANDING WAVES IN A RECTANGULAR BASIN DUE TO A LINEAR ELASTIC RESTORING FORCE "

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ABSTRACT Three-dimensional standing gravity waves on the surface of an inviscid liquid in a moving rectangular basin are considered. The basin horizontal plane motion is due to a linear elastic restoring force. The initial-boundary value problem (I.Bv.P) has been formulated and solved. The surface-wave profile, velocity-potential and pressure distribution are determined. The three-dimensional results reduce to the two-dimensional ones when one of the wavelengths becomes infinite. Reduced results agree exactly with previous studies .

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1. INTRODUCTION

STANDING gravity waves problems are more difficult to analyse than those of progressive wave motions. The reason is that the progressive wave motions can be easily reduced to steady flows of known solutions by an appropriate choice of the frame of reference while standing wave motions possess complications introduced by their time dependence. Nevertheless, treatments have been done to many cases of interest such as: two- and three-dimensional standing waves on a fluid of finite and infinite depth ; interfacial standing waves in multi-layered fluids and effects due to surface tension. A survey of these and other standing wave problems may be found in the review article of Wehausen & Laitone [1] . In particular , standing wave problems in fluids of limited mass in two- and three-dimensions are given in Abramson [2] and Moiseev & Rumyantsev [3] . Further progress has been reported in Refs. [4 - 9] .

Usually, researchers ,in their treatment for standing waves in liquids contained in moving basins, do not consider certain factors such as basin inertia. They also prescribe , á priori , the basin motion which is always in a straight line either parallel or perpendicular to the gravitational field direction. This may be attributed, in the present writer's opinion, to that the horizontal or vertical straight line translation of the basin is the simplest method of excitation in the laboratory. Besides, it is the simplest case amenable mathematically. In the present paper, we consider the three-dimensional standing gravity waves on the surface of an inviscid liquid bounded by a rectangular basin of specified mass. The basin describes a two-dimensional motion in a horizontal plane due to an elastic restoring force, linear in magnitude but varying in direction. In sec.2 we formulate the initial-boundary value problem that governs the phenomenon under consideration. In sec.3 we present its solution and in sec.4 we give a discussion and limiting cases.

2. HYPOTHESES AND FIELD EQUATIONS

We consider a rectangular basin of mass m_b whose horizontal base dimensions are π and πL in which L is width to length ratio. The basin is filled with liquid of mass m_ℓ, forming a finite domain D in R^3, up to a height h from its base. If the liquid-containing basin, whose mass now is $m = m_b + m_\ell$, is displaced from its equilibrium position to

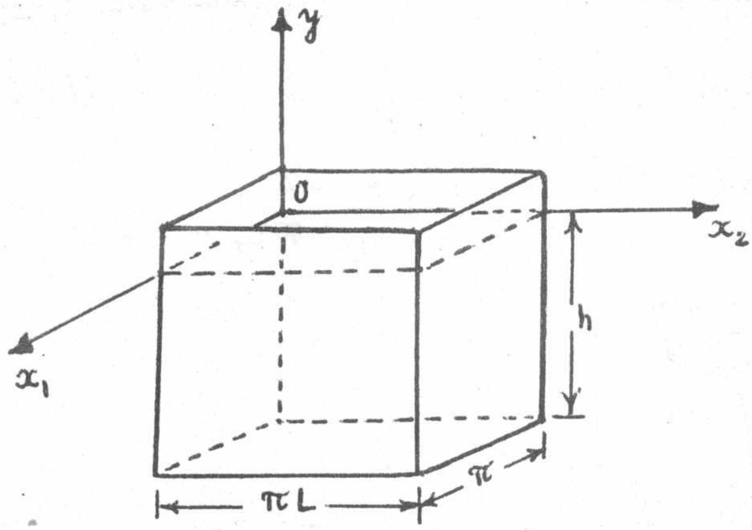

Fig.1. Diagram of partially filled basin showing co-ordinate system.

describe a translational motion in the horizontal plane due to an elastic linear restoring force with modulus $m k^2$ then a motion in the liquid is set in. To describe this generated motion of the liquid by following the eulerian representation, we take the origin O at one corner, the axes x_1, x_2 along two of the sides of the basin and the y axis points vertically upward as shown in Fig. 1 . The triad x_1, x_2, y is fixed in the basin and the coordinates ξ and ζ denote the position of the basin at any instant. We adopt the following hypotheses:

1. The basin walls and base are rigid, impermeable and free from geometric irregularities.
2. The basin displacement in the vertical direction as well as its rotations about all axes are fully constrained.

3. The liquid is homogeneous, incompressible and nonviscous ; and the capillary contact effects between the liquid and the basin walls are negligible.
4. The deflection and slope of the free surface of the liquid are everywhere small during the motion.
5. No spraying or tumbling over occurs from the liquid during the motion.

Thus, the equations governing the motion are

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \forall P \in D \tag{2.1}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad x_1 = 0 \,,\ x_1 = \pi \,,\ x_2 = 0 \,,\ x_2 = \pi L,\ y = -h \tag{2.2}$$

$$\frac{\partial \eta}{\partial t} = - \frac{\partial \phi}{\partial y} \quad \text{on} \quad y = 0 \tag{2.3}$$

$$\int_0^{\pi L}\int_0^{\pi} \eta (x_1, x_2; t)\, dx_1\, dx_2 = 0 \tag{2.4}$$

$$p = \rho \left(\frac{\partial \phi}{\partial t} - g\, y - \ddot{\xi}\, x_1 - \ddot{\zeta}\, x_2 \right) \tag{2.5}$$

$$m_b \ddot{\xi} = \iint_{\partial D_S} p \cos(n, x_1)\, dA - m\, k^2 \xi \tag{2.6}$$

$$m_b \ddot{\zeta} = \iint_{\partial D_S} p \cos(n, x_2)\, dA - m\, k^2 \zeta \tag{2.7}$$

in which $\phi(x_1, x_2, y; t)$ is the velocity-potential whose negative gradient is the liquid velocity vector, η is the free surface elevation, p is the pressure inside the liquid, $\vec{n}$ is the outward normal vector, g is the acceleration of the gravitational field, ρ is the liquid density, ∂D_S is the liquid boundary in contact with the basin walls and base, ∂D_F is the free surface boundary and, finally, the dot denotes as usual the time differentiation.

Equations (2.1 - 2.3) are the usual equations of the theory of surface waves (Cf. Coulson [10] , Chap.5). The first expresses the condition of continuity ; the second follows from rigidity of basin walls and base ; the third expresses the condition that a particle on the surface remains on the surface. Eq.(2.4) indicates the condition of constant volume of the liquid. Eq.(2.5) is Bernoulli's law taking into effect the base accelerations . Lastly,Eqs.(2.6 & 7) imply the principle of conservation of linear momentum of the whole dynamical system in the ξ and ζ directions respectively.

3. I.Bv.P. SOLUTION

Proceeding for the solution of Eqs.(2.1 - 7), we take the free surface elevation and velocity-potential in the forms

$$\eta(x_1,x_2;t) = \cos x_1 \cdot q_{10} + \cos L^{-1}x_2 \cdot q_{01} + \cos x_1 \cdot \cos L^{-1}x_2 \cdot q_{11} \tag{3.1}$$

and

$$\begin{aligned}\phi(x_1,x_2,y;t) = &- \mathrm{Csch}(h) \cdot \cos x_1 \cdot \mathrm{Cosh}(y+h) \cdot \dot{q}_{10} \\ &- L\, \mathrm{Csch}(L^{-1}h) \cdot \cos L^{-1}x_2 \; \mathrm{Cosh}\left[L^{-1}(y+h)\right] \cdot \dot{q}_{01} \\ &- (1+L^{-2})^{-1/2}\, \mathrm{Csch}\left[h(1+L^{-2})^{1/2}\right] \cdot \cos x_1 \\ &\cos L^{-1}x_2 \; \mathrm{Cosh}\left[(1+L^{-2})^{1/2}(y+h)\right] \cdot \dot{q}_{11}\end{aligned} \tag{3.2}$$

where q_{10}, q_{01} and q_{11} are generalised coordinates that are functions of time with $\max\left(|q_{10}|, |q_{01}|, |q_{11}|\right) = O(\varepsilon h)$ in which $\varepsilon \ll 1$.

Here,consideration is given only to the lowest mode pairs,since for the higher modes $\max |q_{ij}|$, $(i,j > 1)$ are of higher order of smallness,i.e. $o(\varepsilon h)$,(Cf. Abramson[2], p. 280).

It is obvious that Eq.(3.2) satisfies Eqs.(2.1 & 2), Eq.(3.1) satisfies Eq.(2.4) and both Eqs.(3.1 & 2) satisfy Eq.(2.3) . Instead of further manipulating directly Eqs.(2.5 - 7) , we proceed with a Hamiltonian formalism as follows.

The kinetic energy of the dynamic system is

$$T = \frac{m_b}{2}(\dot{\xi}^2+\dot{\zeta}^2)+\frac{\rho}{2}\iiint_D \left[(\dot{\xi}-\frac{\partial \phi}{\partial x_1})^2+(\dot{\zeta}-\frac{\partial \phi}{\partial x_2})^2+(\frac{\partial \phi}{\partial y})^2\right] dV \qquad (3.3)$$

Expanding,we get

$$T = \frac{m}{2}(\dot{\xi}^2+\dot{\zeta}^2)+\frac{\rho}{2}\iiint_D [\nabla\phi\cdot\nabla\phi]\, dV - \rho\dot{\xi}\iiint_D \frac{\partial \phi}{\partial x_1}\, dV - \rho\dot{\zeta}\iiint_D \frac{\partial \phi}{\partial x_2}\, dV \qquad (3.4)$$

Applying Green's identity to the term $\iiint_D [\nabla\phi\cdot\nabla\phi]\, dV$ results in

$$\iiint_D [\nabla\phi\cdot\nabla\phi]\, dV = \iint_{\partial D_S} \phi\frac{\partial \phi}{\partial n}\, dA - \iiint_D \phi\nabla^2\phi\, dV + \iint_{\partial D_F} \phi\frac{\partial \phi}{\partial n}\, dA \qquad (3.5)$$

By virtue of Eqs.(2.1 & 2), the first two integrals on the R.H.S. of Eq.(3.5) vanish and via Eq.(2.3) , Eq.(3.5) becomes

$$\iiint_D [\nabla\phi\cdot\nabla\phi]\, dV = -\int_0^{\pi L}\int_0^{\pi} \phi(x_1, x_2, 0; t)\frac{\partial \eta}{\partial t}\, dx_1\, dx_2 \qquad (3.6)$$

The next two integrals on the R.H.S. of Eq.(3.4) can also be simplified as follows :

$$\text{Since} \quad \frac{\partial \phi}{\partial x_1} = \nabla\cdot(x_1\nabla\phi) - x_1\nabla^2\phi \qquad (3.7)$$

then by integrating Eq.(3.7) and applying Gauss divergence theorem to the first term on the R.H.S., we get

$$\iiint_D \nabla\cdot(x_1\nabla\phi)\, dV = \iint_{\partial D_S} x_1\frac{\partial \phi}{\partial n}\, dA + \iint_{\partial D_F} x_1\frac{\partial \phi}{\partial n}\, dA \qquad (3.8)$$

The first integral on the R.H.S. of Eq.(3.8) vanishes by Eq.(2.2)and , also , via Eq.(2.3), Eq.(3.8)can be written as

$$\iiint_D \nabla\cdot(x_1\nabla\phi)\, dV = -\iint_{\partial D_F} x_1\frac{\partial \eta}{\partial t}\, dA \qquad (3.9)$$

Noting that the last term on the R.H.S. of Eq.(3.7) vanishes by Eq.(2.1) , we finally get

$$\iiint_D \frac{\partial \phi}{\partial x_1}\, dV = -\int_0^{\pi L}\int_0^{\pi} x_1\frac{\partial \eta}{\partial t}\, dx_1\, dx_2 \qquad (3.10)$$

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and similarly

$$\iiint_D \frac{\partial \phi}{\partial x_2}\, dV = -\int_0^{\pi L}\int_0^{\pi} x_2 \frac{\partial \eta}{\partial t}\, dx_1\, dx_2 \qquad (3.11)$$

Thus, from Eqs.(3.1 - 3 , 6 , 10 , 11) , the final expression of the kinetic energy of the system is

$$T = \frac{\pi^2}{4} L\rho \left\{ \mathrm{Coth}(h)\,\dot{q}_{10}^2 + L\,\mathrm{Coth}(L^{-1}h)\,\dot{q}_{01}^2 + \frac{1}{2}(1+L^{-2})^{-1/2} \mathrm{Coth}\left[h(1+L^{-2})^{\frac{1}{2}}\right]\dot{q}_{11}^2 \right\} - 2\pi L\rho(\dot{q}_{10}\dot{\xi} + L\dot{q}_{01}\dot{\zeta}) + \frac{m}{2}(\dot{\xi}^2 + \dot{\zeta}^2) \qquad (3.12)$$

Also, the potential energy of the system is

$$\Pi = \frac{\rho g}{2}\iint_{\partial D_F} \eta^2\, dA + \frac{mk^2}{2}(\xi^2 + \eta^2) \qquad (3.13)$$

Substituting for η from Eq.(3.1) into Eq.(3.13) and evaluating the resulting integrals, we obtain

$$\Pi = \frac{\pi^2}{4} L\rho g\left(q_{10}^2 + q_{01}^2 + \frac{1}{2}q_{11}^2\right) + \frac{mk^2}{2}(\xi^2 + \eta^2) \qquad (3.14)$$

We introduce the generalised momenta p_{10} , p_{01} , p_{11} , p_ξ , p_ζ that are defined by

$$p_{10} = \frac{\partial T}{\partial \dot{q}_{10}} = \frac{\pi^2}{2} L\rho\left(\pi\,\mathrm{Coth}(h)\,\dot{q}_{10} - 4\dot{\xi}\right) \qquad (3.15a)$$

$$p_{01} = \frac{\partial T}{\partial \dot{q}_{01}} = \frac{\pi^2}{2} L^2\rho\left(\pi\,\mathrm{Coth}(L^{-1}h)\,\dot{q}_{01} - 4\dot{\zeta}\right) \qquad (3.15b)$$

$$p_{11} = \frac{\partial T}{\partial \dot{q}_{11}} = \frac{\pi^2}{4} L\rho(1+L^{-2})^{-1/2}\,\mathrm{Coth}\left[h(1+L^{-2})^{1/2}\right]\dot{q}_{11} \qquad (3.15c)$$

$$p_\xi = \frac{\partial T}{\partial \dot{\xi}} = m\dot{\xi} - 2\pi L\rho\,\dot{q}_{10} \qquad (3.15d)$$

$$p_\zeta = \frac{\partial T}{\partial \dot{\zeta}} = m\dot{\zeta} - 2\pi L^2\rho\,\dot{q}_{01} \qquad (3.15e)$$

Thus, the corresponding Hamiltonian H is

$$H = T + \Pi = \frac{m}{\pi^2 L\rho}\left[m\,\mathrm{Coth}(h) - 8L\rho\right]^{-1} p_{10}^2 + \frac{m}{\pi^2 L^2\rho}\left[m\,\mathrm{Coth}(L^{-1}h) - 8L^2\rho\right]^{-1} p_{01}^2 + \frac{2}{\pi^2 L\rho}(1+L^{-2})^{1/2}$$

$$\mathrm{Tanh}\left[h(1+L^{-2})^{\frac{1}{2}}\right] P_{11}^2 + \frac{1}{2}\left[m - 8L\rho\,\mathrm{Tanh}(h)\right]^{-1} P_{\xi}^2$$
$$+\frac{1}{2}\left[m - 8L^2\rho\,\mathrm{Tanh}(L^{-1}h)\right]^{-1} P_{\zeta}^2 + \frac{4}{\pi}\left[m\,\mathrm{Coth}(h) - 8L\rho\right]^{-1} P_{10} P_{\xi}$$
$$+\frac{4}{\pi}\left[m\,\mathrm{Coth}(L^{-1}h) - 8L^2\rho\right]^{-1} P_{01} P_{\zeta} + \frac{\pi^2}{4} L\rho g\, x$$
$$x\left(q_{10}^2 + q_{01}^2 + \frac{1}{2} q_{11}^2\right) + \frac{m k^2}{2}\left(\xi^2 + \zeta^2\right) \tag{3.16}$$

Substituting H from Eq.(3.16) into Hamilton equations (Cf. Whittaker [11] , § 109)

$$\dot{q}_r = \frac{\partial H}{\partial P_r} \quad , \quad \dot{p}_r = -\frac{\partial H}{\partial q_r} \tag{3.17}$$

and defining the vector $\{\chi(t)\}^T = \{q_{10}\ \ q_{01}\ \ q_{11}\ \ \xi\ \ \zeta\ \ p_{10}\ \ p_{01}\ \ p_{11}\ \ p_{\xi}\ \ p_{\zeta}\}^T$, we obtain the following autonomous differential system

$$\{\dot{\chi}\} = [N]\{\chi\} \tag{3.18}$$

in which the elements of the matrix N are

$$n_{16} = \frac{2m}{\pi^2 L\rho}\left[m\,\mathrm{Coth}(h) - 8L\rho\right]^{-1}$$
$$n_{19} = \frac{4}{\pi}\left[m\,\mathrm{Coth}(h) - 8L\rho\right]^{-1}$$
$$n_{27} = \frac{2m}{\pi^2 L^2\rho}\left[m\,\mathrm{Coth}(L^{-1}h) - 8L^2\rho\right]^{-1}$$
$$n_{2,10} = \frac{4}{\pi}\left[m\,\mathrm{Coth}(L^{-1}h) - 8L^2\rho\right]^{-1}$$
$$n_{38} = \frac{4}{\pi^2 L\rho}(1+L^{-2})^{\frac{1}{2}}\,\mathrm{Tanh}\left[h(1+L^{-\frac{1}{2}})^{\frac{1}{2}}\right]$$
$$n_{46} = \frac{4}{\pi}\left[m\,\mathrm{Coth}(h) - 8L\rho\right]^{-1}$$
$$n_{49} = \left[m - 8L\rho\,\mathrm{Tanh}(h)\right]^{-1}$$
$$n_{57} = \frac{4}{\pi}\left[m\,\mathrm{Coth}(L^{-1}h) - 8L^2\rho\right]^{-1}$$

$$n_{5,10} = \left[m - 8L^2 \rho \, \text{Tanh}\,(L^{-1}h) \right]^{-1}$$

$$n_{61} = n_{72} = \frac{\pi^2}{2} L\rho g \quad , \quad n_{83} = \frac{\pi^2}{4} L\rho g$$

$$n_{94} = n_{10,5} = m\,k^2$$

and all other elements take the zero value.

The initial values of $\{\chi\}$ (i.e. $\{\vec{\chi}(0)\}$) are

$$\chi_1 = \chi_2 = \frac{8}{\pi^2}\varepsilon h \; , \; \chi_3 = -\frac{64}{\pi^4}\varepsilon h \quad , \; \chi_4 = \xi_0 \; , \; \chi_5 = \chi_6 =$$
$$= \chi_7 = \chi_8 = \chi_9 = 0 \quad \text{and} \quad \chi_{10} = m\,\zeta_1$$

that correspond to the following initial conditions of the dynamical system :

$$\begin{aligned} &\xi(0) = \xi_0 \quad , \; \zeta(0) = 0 \quad , \; \dot{\zeta}(0) = \zeta_1 \; , \\ &\eta\,(x_1, x_2\,; 0) = \varepsilon h\left(1 - \frac{4}{\pi^2 L}\, x_1 x_2\right), \\ &\nabla\phi\,(x_1, x_2, y\,; 0) = 0 \end{aligned} \tag{3.19}$$

in which ξ_0 and $k^{-1}\zeta_1$ are $0(\varepsilon h)$.

By applying Laplace transform to Eq.(3.18) , we obtain

$$s\{\bar{\chi}(s)\} - \{\vec{\chi}(0)\} = [N]\,\{\bar{\chi}(s)\} \tag{3.20}$$

in which the bar denotes the L . T . of a quantity.

Thus, Eq.(3.20) can take the form

$$\{\bar{\chi}(s)\} = [sE - N]^{-1}\,\{\vec{\chi}(0)\} \tag{3.21}$$

in which $[s\,E - N]^{-1}$ is the system transfer matrix and E is the identity matrix.

Evaluating the transfer matrix by partitioning method (Cf. Frazer et al [12]), the relevant transforms are

$$\overline{q}_{10}(s) = \frac{s(s^2+n_{49}n_{94})\chi_1(0) - n_{19}n_{94}\, s\, \chi_4(0)}{D_{10}(s)} \quad (3.22)$$

$$\overline{q}_{01}(s) = \frac{(s^2+n_{5,10}n_{10,5})(s\chi_2(0)+n_{2,10}\chi_{10}(0)) - n_{2,10}n_{5,10}n_{10,5}\chi_{10}(0)}{D_{01}(s)} \quad (3.23)$$

$$\overline{q}_{11}(s) = \frac{s\,\chi_3(0)}{D_{11}(s)} \quad (3.24)$$

and

$$D_{10}(s) = s^4 + (n_{16}n_{61} + n_{49}n_{94})s^2 + n_{16}n_{61}n_{49}n_{94} - n_{19}^2 n_{61} n_{94} \quad (3.25)$$

$$D_{01}(s) = s^4 + (n_{5,10}n_{10,5} + n_{27}n_{72})s^2 + n_{27}n_{72} \times n_{5,10}n_{10,5} - n_{2,10}^2 n_{72} n_{10,5} \quad (3.26)$$

$$D_{11}(s) = s^2 + n_{38}n_{83} \quad (3.27)$$

Having found the transforms $\overline{q}_{10}$, $\overline{q}_{01}$, $\overline{q}_{11}$, we are now in a position to investigate the stability of the resulting waves. By applying Hurwitz stability criterion (Cf. Hurwitz [13] or Hayashi [14],p.73) to Eqs.(3.25-27) and forming Hurwitz determinant for each of these equations, it is found that all principal diagonal minors of each Hurwitz determinant are non-negative. Thus, we can conclude that the occurring standing waves are stable.

Now, applying the inversion theorem for the Laplace transform (Cf. Sneddon [15],p.174) , we obtain

$$q_{10} = A \cos \omega t \quad (3.28)$$

$$q_{01} = B \cos \Omega t + C \sin \Omega t \quad (3.29)$$

$$q_{11} = D \cos \sigma t \quad (3.30)$$

in which the quantities ω , Ω , σ ; A , B , C , D are

$$\omega = \sqrt{g\, \mathrm{Tanh}(h)\, \frac{[k^2 + g(1 - \frac{8L\rho}{m}\mathrm{Tanh}(h))\,\mathrm{Tanh}(h)]}{[k^2 + g\, \mathrm{Tanh}(h)]}} \quad (3.31)$$

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$$\Omega = \sqrt{g L^{-1} \, \mathrm{Tanh}(L^{-1}h) \, \frac{\left[k^2 + g\left(1 - \frac{8L^2\rho}{m}\, \mathrm{Tanh}(L^{-1}h)\right) L^{-1}\, \mathrm{Tanh}(L^{-1}h)\right]}{\left[k^2 + g L^{-1}\, \mathrm{Tanh}(L^{-1}h)\right]}} \tag{3.32}$$

$$\sigma = g^{\frac{1}{2}} \left(1 + L^{-\frac{1}{2}}\right)^{\frac{1}{4}} \mathrm{Tanh}^{\frac{1}{2}}\left[h\left(1 + L^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right] \tag{3.33}$$

$$A = \frac{\frac{8}{\pi^2}\left\{k^2\left[m - 8L\rho\, \mathrm{Tanh}(h)\right]^{-1} - \frac{\omega^2}{m}\right\} \varepsilon h - \frac{4}{\pi}\left[m\, \mathrm{Coth}(h) - 8L\rho\right]^{-1} k^2 \xi_0}{\left[k^2 + g\, \mathrm{Tanh}(h)\right]\left[m - 8L\rho\, \mathrm{Tanh}(h)\right]^{-1} - 2\frac{\omega^2}{m}} \tag{3.34}$$

$$B = \frac{8}{\pi^2}\, \frac{\left\{k^2\left[m - 8L^2\rho\, \mathrm{Tanh}(L^{-1}h)\right]^{-1} - \frac{\Omega^2}{m}\right\} \varepsilon h}{\left[k^2 + g L^{-1}\, \mathrm{Tanh}(L^{-1}h)\right]\left[m - 8L^2\rho\, \mathrm{Tanh}(L^{-1}h)\right]^{-1} - 2\frac{\Omega^2}{m}} \tag{3.35}$$

$$C = -\frac{4}{\pi}\, \frac{\left[m\, \mathrm{Coth}(L^{-1}h) - 8L^2\rho\right]^{-1} \Omega \zeta_1}{\left[k^2 + g L^{-1}\, \mathrm{Tanh}(L^{-1}h)\right]\left[m - 8L^2\rho\, \mathrm{Tanh}(L^{-1}h)\right]^{-1} - 2\frac{\Omega^2}{m}} \tag{3.36}$$

$$D = -\frac{64}{\pi^4}\, \varepsilon h \tag{3.37}$$

Thus, substituting Eqs.(3.28 - 31) into Eqs.(3.1 & 2), the surface-wave profile and velocity-potential are

$$\begin{aligned} \eta(x_1, x_2; t) = {} & A \cos x_1 \cos \omega t + B \cos L^{-1}x_2 \cos \Omega t \\ & + C \cos L^{-1} x_2 \sin \Omega t + D \cos x_1 \cos L^{-1} x_2 \cos \sigma t \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} \phi(x_1, x_2, y; t) = {} & \omega A\, \mathrm{Csch}(h) \cos x_1\, \mathrm{Cosh}(y+h) \sin \omega t \\ & + \Omega L\, \mathrm{Csch}(L^{-1}h) \cos L^{-1} x_2\, \mathrm{Cosh}\left[L^{-1}(y+h)\right] \times \\ & \left(B \sin \Omega t - C \cos \Omega t\right) + \sigma D \left(1 + L^{-2}\right)^{-\frac{1}{2}} \mathrm{Csch}\left[h\left(1 + L^{-2}\right)^{\frac{1}{2}}\right] \\ & \times \cos x_1 \cos L^{-1} x_2\, \mathrm{Cosh}\left[\left(1 + L^{-2}\right)^{\frac{1}{2}}(y+h)\right] \sin \sigma t\, . \end{aligned} \tag{3.39}$$

For the determination of the pressure distribution , we note that the free surface is an equi-pressure one (with p = 0). Thus, from Eq.(2.5) , we get

$$g\eta = \left(\frac{\partial \phi}{\partial t}\right)_{y=0} - \ddot{\xi}\, x_1 - \ddot{\zeta}\, x_2 \tag{3.40}$$

Eliminating the base accelerations $\ddot{\xi}$, $\ddot{\zeta}$ between Eqs. (2.5) & (3.40) yields

$$p = \rho\left[g(\eta - y) + \frac{\partial\phi}{\partial t} - \left(\frac{\partial\phi}{\partial t}\right)_{y=0}\right] \tag{3.41}$$

Substituting for η , $\frac{\partial\phi}{\partial t}$ from Eqs. (3.38 & 39) , we obtain

$$\begin{aligned}\frac{p}{\rho} = {} & A\cos x_1\left(\left[\mathrm{Csch}(h)\,\mathrm{Cosh}(y+h) - 1\right]\omega^2 + g\right)\cos\omega t \\ & + \cos L^{-1}x_2\left(\left\{L\,\mathrm{Csch}(L^{-1}h)\,\mathrm{Cosh}\left[L^{-1}(y+h)\right] - 1\right\}\Omega^2 + g\right)\times \\ & \left(B\cos\Omega t + C\sin\Omega t\right) + D\cos x_1\cos L^{-1}x_2\left(\left\{(1+L^{-2})^{-\frac{1}{2}}\right.\right. \\ & \left.\left.\mathrm{Csch}\left[h(1+L^{-2})^{\frac{1}{2}}\right]\mathrm{Cosh}\left[(1+L^{-2})^{\frac{1}{2}}(y+h)\right] - 1\right\}\sigma^2 + g\right)\cos\sigma t - gy\end{aligned} \tag{3.42}$$

4. DISCUSSION

In the previous section , analytical solutions in closed forms have been obtained for the surface-wave profile, the velocity-potential and the pressure distribution inside the liquid. For discussing the influence of basin inertia and the restoring force modulus on the resulting waves characteristics, we recall that the frequency of standing waves in a rectangular stationary basin (Cf. Coulson [10] , § 49) is given by

$$\sigma_r^2 = g\ r\ \ \mathrm{Tanh}\ rh \tag{4.1}$$

$$r^2 = p^2 + L^{-2}\ q^2 \tag{4.2}$$

in which p , q are non-negative integers. Thus, taking advantage of Eqs. (4.1 & 2) , Eqs. (3.31 - 33) are rewritten in the forms

$$\left(\frac{\omega}{\sigma_{10}}\right)^2 = 1 - \frac{8}{\pi^2 gh}\ \frac{\sigma_{10}^{-2}}{\left[1+\frac{m_b}{m_l}\right]\left[1+\left(\frac{k}{\sigma_{10}}\right)^2\right]} \tag{4.3}$$

$$\left(\frac{\Omega}{\sigma_{0i}}\right)^2 = 1 - \frac{8L^2}{\pi^2 gh}\ \frac{\sigma_{0i}^{-2}}{\left[1+\frac{m_b}{m_l}\right]\left[1+\left(\frac{k}{\sigma_{0i}}\right)^2\right]} \tag{4.4}$$

$$\sigma = \sigma_{11} \tag{4.5}$$

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Eq.(4.5) reveals that the coupling mode is unaffected by the basin inertia and the elastic restoring force. For other modes, it is clearly seen that they are reduced by a reduction factor containing m_b , k and h as parameters. For example, Eq.(4.3) indicates that with the increase of the modulus k , the reduction factor decreases and the difference $|\sigma_{10} - \omega|$ decreases consequently. In the limiting case $k \uparrow \infty$, $\omega \rightarrow \sigma_{10}$. Also , a similar behaviour is shown with the increase of the basin mass until it becomes comparatively large with m_ℓ , the liquid motion is exactly like free standing waves in a stationary basin. This can be explained from Eqs.(2.6 , 2.7) that for $m_b \uparrow \infty$ and all forces on the R.H.S. are finite $\ddot{\xi} \downarrow 0$,resulting a stationary basin (or ,naturally, a basin with rectilinear uniform velocity) . Also, if $h \uparrow \infty$, then $\frac{\omega}{\sigma_{10}} \rightarrow 1$.

Now , let us consider the case of the ratio L becomes large. Letting , in Eq.(3.1) , $L \uparrow \infty$, applying the condition of constant volume of liquid i.e. $\int_0^{\pi} \eta \, dx_1 = 0$, and noting that q_{11} is now merged with q_{10} we obtain

$$\eta = A \cos x_1 \cos \omega t \tag{4.6}$$

in which A and ω are given by Eqs. (3.34) and (4.3) respectively. This gives us the case of standing gravity waves occurring on the surface of a liquid embodied in a two-dimensional finite rectangular basin vibrating along the x_1 - axis. This case has been treated by Sretenskiĭ [16][§] . Eq. (4.3) agrees exactly with Sretenskiĭ results. However , Sretenskiĭ approach depends on the direct treatment of the field equations (i.e., the force method). He has finally come up with two coupled differential

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§ I have been unable to obtain Sretenskiĭ paper but a good précis of it is given in Wehausen & Laitone [1], p. 628 . This précis is sufficient for the present discussion.

equations , one of them is of the third order ($\dddot{\xi}$ in the present notation) and to integrate them he was forced to make an ad hoc assumption in the initial conditions , that is $\frac{\partial \phi(x_1, y, 0)}{\partial t} = 0$. This third order differential equation implies that there exists a force-potential that is a function of the acceleration and his assumption also means that an initial value of the acceleration should be imposed. Since Hydrodynamics is based entirely on Newtonian mechanics , then this ad hoc assumption - despite its mathematical correctness - is not permissible in the realm of Newtonian mechanics. This point shows clearly that the Hamiltonian formalism applied in the present study offers significant advantages vis - à - vis the direct force method.

5. CONCLUSIONS

From the present study, the following points can be concluded .

1 - The occurring standing waves are characterised by being oscillatory stable.

2 - The increase of the basin inertia , restoring force modulus and liquid depth yields ,both individually and simultaneously, an increase in the frequency of the occurring standing waves. This increase is bounded by the frequency of standing waves in a stationary basin.

3 - The basin inertia and the restoring force modulus have no effect on the frequency of the coupled modes.

4 - The Hamiltonian formulation offers ,for the treatment of interaction liquid waves problems, certain advantages over the direct force method and it is strongly recommended for tackling similar problems.

5 - The present analysis is useful in studying the stability and response of airplanes and space vehicles containing liquid fuel - filled tanks. For example,integrating the

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pressure on basin (tank) walls and base yields the total force due to fuel motion. Thus, we have a transfer function relating the basin motion to the total liquid force. This transfer function can then be used in analysing the closed loop response motion of the airplane or space vehicle.

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