Bulletin of Faculty of Science ,Zagazig University (BFSZU) 2022
Bulletin of Faculty of Science, Zagazig University (BFSZU)
e-ISSN: 1110-1555
Volume-2022, Issue-3, pp-122-130
Research Paper DOI:10.21608/bfszu.2022.147965.1154

Information Measures for Record Values and their Concomitants under Haung-Kotz FGM Bivariate Distribution

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ABSTRACT : **The Fisher information (FI) about the shape-parameter vectors of the Huang-Kotz FGM (HK-FGM) is investigated. We study analytically and numerically the Fisher information matrix (FIM) and Shanon entropy related to the record values and their concomitants for HK-FGM.**

KEYWORDS Fisher information; Record values; Concomitants of record values; Haung-Kotz FGM distribution types.

Date of Submission: 29-06-2022 Date of acceptance: 09-08-2022

I. INTRODUCTION

Suppose F_X is a continuous distribution function (DF) of an RV X. If we kept the withdrawal observations from time to time from F_X, then an observation that is larger than all the drawn observations previously is called a record and its value is called an upper record value or a record value. Let $\{R_n\}$ be the sequence of the observed record values and let f_X be the probability density function (PDF) of X. Furthermore, we adopt $R_0 = X$. Then, the PDF of R_n is given by (cf. Ahsanullah, 1995 and Arnold et al., 1998)

$$g_n(x) = \frac{[-\log(1-F_X(x))]^n}{n!} f_X(x), -\infty < x < \infty, n = 1, 2, \ldots.$$

Note that $g_0(x) = f_X(x)$. Moreover, for any $1 \leq n < m$, the joint PDF of the mth and nth upper record values is given by (cf. Ahsanullah, 1995 and Arnold et al., 1998)

$$g_{m,n}(x_1, x_2) = \frac{(-\log(1-F_X(x)))^{n-1}}{\Gamma(n)} \frac{(-\log(\frac{1-F_X(x_2)}{1-F_X(x_1)}))^{m-n-1}}{\Gamma(m-n)}$$

$$\times \frac{f_X(x_1) f_X(x_2)}{1-F_X(x_1)}, -\infty < x_1 < x_2 < \infty,$$

Given a continuous bivariate distribution with PDF $f_{X,Y}(x,y)$, let $(X_i, Y_i), i = 1, 2, \ldots$ be a random sample of the paired RVs (X, Y). The second component connected to the first component's record value is known as the concomitant of that recorded value when the researcher is just interested in looking at the sequence of recordings of the first component X_i's. The concomitants of record values arise widely in practical experiments, see Arnold et al. (1998) and Bdair and Raqab (2013). Some properties from concomitants of record values were discussed in Ahsanullah (2009) and Ahsanullah and Shakil (2013). Let the sequence of record values in the sequence of X's is $\{R_n, n \geq 1\}$, while the corresponding concomitant in the sequence of Y's is $R_{[n]}$. The joint PDF of R_n and $R_{[n]}$ can be expressed as (cf. Houchens, 1984)

$$h_{[n]}(x,y) = f_{Y|X}(y|x) g_n(x), \tag{1.1}$$

where $f_{Y|X}(y|x)$ is the conditional PDF of Y given X. Consequently, the PDF of the concomitant $R_{[n]}$ is given by

$$h_{[n]}(y) = \int_0^\infty f_{Y|X}(y|x) g_n(x) dx.$$

The FI is a vital criterion in statistical inference especially in large sample studies in estimation theory. In this paper, we will define the form of the FI for the DF (see Kharazmi and Asadi, 2018). The FI related to the distribution parameters gives us feed back how much information about an unknown parameter from a sample and FI is connecting with the efficiency of an estimator and sufficiency of a statistic. When we have an unknown parameter in the DF from which the sample is drawn, and when the sample is sufficiently huge, the Knowing FI helps tracking down limits on the variance of a given estimator of that parameter and to rough the testing appropriation of this estimator. Also, FI based on censored samples is a helpful tool for designing the life-testing experiments as well as for assessing the effectiveness of estimators and tests. Abo-Eleneen and Nagaraja (2002) explored the properties of FI regarding the dependence parameter in the Farlie-Gumbel-Morgenstern (FGM) parent.

Consider a random vector (X, Y) of a PDF $f(x, y; \lambda)$, where $\lambda \in \Lambda$ is an unknown parameter in a parameter space Λ. The FI measure contained in the random vector (X, Y) about the parameter λ is given by (cf. Abo-Eleneen and Nagaraja, 2002)

$$I_\lambda(X,Y) = E\left(\frac{\partial \log f(x,y;\lambda)}{\partial \lambda}\right)^2 = -E\left(\frac{\partial^2 \log f(x,y;\lambda)}{\partial \lambda^2}\right). \tag{1.2}$$

When the parameter λ is a vector $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$, the FIM $II(X, Y; \underline{\lambda})$ is an $m \times m$ matrix, whose (i,j)th element is $I_{\lambda_i,\lambda_j}(X,Y) = -E\left(\frac{\partial^2 \log f(X,Y;\underline{\lambda})}{\partial \lambda i \lambda_j}\right)$.

The FGM distribution was originally proposed by Morgenstern (1956) for Cauchy marginals. Gumbel (1960) investigated the proposed structure for exponential marginals. Farlie (1960) proposed a novel generic form of a bivariate distribution for given arbitrary marginals in connection with his studies of the correlation coefficient, which were influenced by the works of Morgenstern (1956) and Gumbel (1960). Later, Johnson and Kotz (1975, 1977) extended the introduced bivariate distribution to the multivariate case and begat the term FGM DF. The FGM DF is given by $F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \lambda(1 - F_X(x))(1 - F_Y(y))], -1 \leq \lambda \leq 1$, where F_X and F_Y are the marginal DFs of some RVs X and Y. While the classical FGM distribution is a flexible family and significant in numerous applications, a notable impediment of the wide use of this family is the low dependence level it provides between its RVs, where the maximal positive correlation coefficient is 0.33. Therefore, the utilization of FGM is restricted to the cases that show low correlation. Huang and Kotz (1984) used successive iterations in the FGM distribution to increment the relationship between the parts, furthermore, demonstrated the way that only one single iteration can bring about significantly increasing the covariance for certain marginals. Later, this model was extensively studied by Alawady et al. (2020), Barakat and Husseiny (2021), and Barakat et al. (2020, 2021). One of the best and notable endeavors to improve the scope of relationship and give greater adaptability of the FGM distribution is due to Huang and Kotz (1999). Huang and Kotz (1999) proposed two analogous extensions, the first one, on which we focus in this study (denoted by HK-FGM(λ, p)), is defined by

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)\left[1 + \lambda(1 - F_X^p(x))(1 - F_Y^p(y))\right], p \geq 1,$$

with the PDF

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1 + \lambda((1+p)F_X^p(x) - 1)((1+p)F_Y^p(y) - 1)\right], \tag{1.3}$$

The admissible range of the shape-parameter vector (λ, p) is $\Omega = \{(\lambda, p): -p^{-2} \leq \lambda \leq p^{-1}, p \geq 1\}$ and the maximal positive correlation for this model is 0:375. The first aim of this paper is to evaluate the FIM

$$II(R_n, R_{[n]}) = \begin{pmatrix} I_{[\lambda]}(R_n, R_{[n]}) & I_{([\lambda,p])}(R_n, R_{[n]}) \\ I_{([\lambda,p])}(R_n, R_{[n]}) & I_{[p]}(R_n, R_{[n]}) \end{pmatrix} \tag{1.4}$$

connected with the concomitants of record values about the shape-parameter vector (λ, p), for HK-FGM. Since, our main aim is to study the FI in concomitants of record values about the unknown shape-parameter-vector (λ, p) our emphasis ought to be on the copula connected with the HK-FGM model. Copula is liberated from all obscure boundaries with the exception of the shape parameters, and it can be obtained by letting the two RVs X and Y have uniform DFs.

The second aim of this paper is studying the Shanon entropy of the concomitant of record values based on the HK-FGM model. The Shannon entropy is a mathematical measure of information that measures the average reduction of uncertainty or variability associated with a RV. The Shannon entropy of a continuous RV X having PDF $f_X(x)$, is defined by

$$H(X) = -\int_{-\infty}^{\infty} f_X(x) \log f_X(x)\, dx. \tag{1.5}$$

The measure $H(X)$ is maximal for uniform distributions, additive for independent events, increasing in the number of outcomes with non-zero probabilities and continuous. See Abd Elgawad et al. (2020, 2021), Alawady et al. (2021), Barakat and Husseiny (2021), and Pathria and Beale (2011) for further information on this measure.

The rest of the paper is organized as follows: In Section 2, a closed form for FIM for concomitants of record values according to HK-FGM is derived. In Section 3, the Shanon entropy for concomitants of record values based on HKFGM is derived. Finally, In Section 4, some numerical calculations are carried out for the considered information measures.

II. FIM IN RECORD VALUES AND THEIR CONCOMITANTS FOR HK-FGM

Throughout this section, define the sequences $A_i = (-1)^i \lambda_\ell^i$, $A_{ij} = (-1)^j (1+p)^j \binom{i+2}{j}$, $A_{ik} = (-1)^k (1+p)^k \binom{i+2}{k}$, $B_{ij} = (-1)^j (1+p)^j \binom{i}{j}$, $C_{ij} = (-1)^j (1+p)^j \binom{i+1}{j}$, $D_{kl} = (-1)^l \binom{kp}{l}$,

$and\ \ Q_{kt} = (-1)^t \binom{(k+1)p}{t}$, where the ranges of the subscripts of these sequences (such as $i, j, k, l, r, s, \ldots$) will be separably defined in Theorems 2.1. The copula density of (1.3) may be written in the form

$$f_{X,Y}(x,y) = [1 + \lambda C(x,y;p)], \quad 0 \le x, y \le 1, \tag{2.1}$$

where $C(x,y;p) = (1-(1+p)x^p)(1-(1+p)y^p)$. The FIM, $II(R_n, R_{[n]})$ defined in (1.4), for (2.1) is given in the next theorem.

Theorem 1 ***Let $(\lambda, p) \in \Omega^\star \cap \Omega$, where***

$$\Omega^\star = \{(\lambda, p) : |\lambda C(x,y,p)| < 1, \forall\ 0 \le x, y \le 1\}. \tag{2.2}$$

Then, the elements $I_{[\lambda]}(R_n, R_{[n]})$, $I_{([\lambda,p])}(R_n, R_{[n]})$ and $I_{[p]}(R_n, R_{[n]})$ of the FIM $II(R_n, R_{[n]})$ are

$$I_{[\lambda]}(R_n, R_{[n]}) = \sum_{i=0}^{\infty} \sum_{j=0}^{i+2} \sum_{k=0}^{i+2} \sum_{l=0}^{kp} \frac{A_i A_{ij} A_{ik} D_{kl}}{(jp+1)(l+1)^n}, \tag{2.3}$$

$$I_{[\lambda,p]}(R_n, R_{[n]}) = \sum_{i=0}^{\infty} A_i \Bigg[\sum_{j=0}^{i+1} \sum_{k=0}^{i} C_{ij} \frac{B_{ik}}{jp+1}$$

$$\times \left(\sum_{t=0}^{(k+1)p} Q_{kt} + (1+p) \int_0^1 x^{(k+1)p} \log(x) (-\log(1-x))^{n-1} dx\right)$$

$$+ \sum_{s=0}^{i+1} \sum_{l=0}^{i} C_{is} B_{il_1} \left(\frac{1}{(l_1+1)p+1} + (1+p)\frac{l}{(l_1+2)^2}\right) \sum_{r=0}^{sp} \frac{D_{sr}}{(r+1)^n}\Bigg], \tag{2.4}$$

and

$$I_{[p]}(R_n, R_{[n]}) = \lambda^2 \sum_{i=0}^{\infty} A_i \Bigg[\sum_{j=0}^{i+2} \sum_{k=0}^{i} \frac{A_{ij} B_{ik}}{jp+1} \int_0^1 x^{(k+2)p} (1 + (1+p)\log(x))^2 (-\log(1-x))^{n-1} dx$$

$$+2 \sum_{l=0}^{i+1} \sum_{l_1=0}^{i+1} C_{il} C_{il_1} \left(\frac{1}{(l+1)p+1} + \frac{(1+p)l}{(2+l)^2}\right) \int_0^1 x^{(l_1+1)p} (1 + (1+p)\log(x))^2 (-\log(1-x))^{n-1} dx$$

$$+ \sum_{r=0}^{i+2} \sum_{s=0}^{i} \sum_{s_1=0}^{rp} \frac{A_{ir} B_{is} D_{rs}}{(s_1+1)^n} \left(\frac{1}{sp+1} + 2(1+p)\frac{s}{(s+2)^2} + (1+p)^2\right) \int_0^1 y^{sp} (\log(y))^2 dy\Bigg]. \tag{2.5}$$

Proof. **By combining (1.1), (1.2), (1.4), and (2.1) we get**

$$\frac{\partial^2 \log h_{[n]}(x,y)}{\partial \lambda^2} = \frac{\partial^2 \log f_{X,Y}(x,y)}{\partial \lambda^2} = -\frac{C^2(x,y;p)}{(1+\lambda C(x,y;p))^2}. \tag{2.6}$$

The condition (2.2) allows us to expand $(1 + \lambda C(x,y;p))^{-1}$ by the binomial expansion. Thus, the relations (1.2) and (2.6) yield

$$I_{[\lambda]}(R_n, R_{[n]}) = \frac{1}{\Gamma(n)} \int_0^1 \int_0^1 \frac{C^2(x,y;p)}{(1+\lambda C(x,y;p))} (-\log(1-x))^{n-1} dx dy$$

$$= \frac{1}{\Gamma(n)} \sum_{i=0}^{\infty} (-1)^i \lambda^i \int_0^1 \int_0^1 C^{i+2}(x,y;p) (-\log(1-x))^{n-1} dx dy$$

$$= \frac{1}{\Gamma(n)} \sum_{i=0}^{\infty} (-1)^i \lambda^i \int_0^1 \int_0^1 (1-(1+p)x^p)^{i+2} (1-(1+p)y^p)^{i+2} (-\log(1-x))^{n-1} dx dy.$$

Upon using the binomial expansion for $(1-(1+p)x^p)^{i+2}$ and $(1-(1+p)y^p)^{i+2}$, and after some simple algebra, we get (2.3). On the other hand, a combination of (1.2) and (2.1) yields

$$\frac{\partial^2 \log h_{[n]}(x,y)}{\partial\lambda\partial p} = \frac{\frac{\partial C(x,y;p)}{\partial p}}{(1+\lambda C(x,y;p))^2}. \tag{2.7}$$

Thus, by using (1.2) and (2.7), we get

$$I_{[\lambda,p]}(R_n, R_{[n]}) = \frac{1}{\Gamma(n)}\int_0^1\int_0^1 \frac{\zeta(x,y)}{(1+\lambda C(x,y;p))}(-\log(1-x))^{n-1}dxdy, \tag{2.8}$$

where

$$-\zeta(x,y) = x^p(1-(1+p)y^p)(1+(1+p)\log x) + y^p(1-(1+p)x^p)(1+(1+p)\log y).$$

Again, the condition (2.2) allows us to expand $(1+\lambda C(x,y;p)))^{-1}$ by the binomial expansion. Thus, (2.8) can be written as

$$I_{[\lambda,p]}(R_n, R_{[n]}) = \sum_{i=0}^{\infty} A_i(I_{1;i} + I_{2;i}), \tag{2.9}$$

where

$$I_{1;i} = \frac{1}{\Gamma(n)}\int_0^1\int_0^1 x^p(-\log(1-x))^{n-1}(1-(1+p)x^p)^i(1-(1+p)y^p)^{i+1}(1+(1+p)\log x)dxdy \tag{2.10}$$

and

$$I_{2;i} = \frac{1}{\Gamma(n)}\int_0^1\int_0^1 y^p(-\log(1-x))^{n-1}(1-(1+p)y^p)^i(1-(1+p)x^p)^{i+1}(1+(1+p)\log y)dxdy. \tag{2.11}$$

By using the binomial expansion for the three middle terms of the integrand in $I_{1;i}$, the 3rd, 4th and 5th terms in $I_{2;i}$ and combining (2.9), (2.10) and (2.11) we get (2.4). Finally, on the bases of (1.2), we have

$$I_{[p]}(R_n, R_{[n]}) = E\left(\frac{\partial \log h_{[n]}(x,y)}{\partial p}\right)^2. \tag{2.12}$$

Therefore, (1.1) and (2.1) imply

$$I_{[p]}(R_n, R_{[n]}) = \frac{1}{\Gamma(n)}\int_0^1\int_0^1 \frac{\lambda^2\zeta^2(x,y)}{(1+\lambda C(x,y;p))}(-\log(1-x))^{n-1}dxdy, \tag{2.13}$$

where $\zeta(x,y)$ is defined in (2.8). Once more, the condition (2.2) allows us to expand $(1+\lambda C_1(x,y;p)))^{-1}$ by the binomial expansion. Thus, (2.13) after using the binomial expansion and some routine simplifications can be written as

$$I_{[p]}(R_n, R_{[n]}) = \sum_{i=0}^{\infty} A_i^{(1)}\lambda^2[J_{1;i} + J_{2;i} + J_{3;i}], \tag{2.14}$$

where

$$J_{1;i} = \sum_{j=0}^{i+2}\sum_{k=0}^{i} \frac{A_{ij}B_{ik}}{jp+1}\int_0^1 x^{(k+2)p}(1+(1+p)\log x)^2(-\log(1-x))^{n-1}dx, \tag{2.15}$$

$$J_{2;i} = 2\sum_{l=0}^{i+1}\sum_{l_1=0}^{i+1} C_{il}C_{il_1}\left(\frac{1}{(l+1)p+1} + (1+p)\frac{l}{(2+l)^2}\right)$$
$$\times \int_0^1 x^{(l_1+1)p}(1+(1+p)\log(x))^2(-\log(1-x))^{n-1}dx, \tag{2.16}$$

and

$$J_{3;i} = \sum_{r=0}^{i+2}\sum_{s=0}^{i}\sum_{s_1=0}^{rp} \frac{A_{ir}B_{is}D_{rs_1}}{(s_1+1)^n}\left(\frac{1}{sp+1} + 2(1+p)\frac{s}{(s+2)^2} + (1+p)^2\right)\int_0^1 y^{sp}(\log(y))^2dy. \tag{2.17}$$

A combination of (2.14) with (2.15)-(2.17) proves the relation (2.5). This completes the proof of the theorem.

III. Shannon entropy in concomitants of record values based on HK-FGM$(\lambda;p)$

Barakat et al. (2019) derived the PDF of the concomitant $R_{[n]}, n \geq 1$, under HK-FGM1 as

$$h_{[n]}(y) = f_Y(y)[1-\gamma((1+p)F_Y^p(y)-1)], \tag{3.1}$$

where $\gamma = \lambda(1-(1+p)\sum_{i=0}^{\aleph(p)} \frac{(-1)^i\binom{p}{i}}{(i+1)^n})$, $\aleph(x)=\infty$, if x is non-integer, and $\aleph(x)=x$, if x is integer. The next theorem gives the Shanon entropy of concomitants of record value R_n.

Theorem 2 *Let $R_{[n]}$ be the concomitant of the nth record value from HK-FGM$(\lambda;p)$, with PDF (3.1). Therefore, an explicit expression of the Shannon entropy of $R_{[n]}$ is given by*

$$H(R_{[n]}) := H_{[n]} = (1+\gamma)H(Y) + \gamma(1+p)\psi(p) - \log(1-p\gamma) - \gamma p(1+p) \times \left[\frac{a}{c(1+p)} + \frac{1}{c}\left(b-\frac{ab}{c}\right) - \frac{b}{c}\left(b-\frac{ab}{c}\right)\int_0^1 \frac{1}{cz^p+b}dz\right], \tag{3.2}$$

where $H(Y)$ is the Shannon entropy of Y, $\psi(p) = \int_{-\infty}^{\infty} f_Y(y)F_Y^p(y)\log f_Y(y)dy$, $a=-\gamma$, $b=1+\gamma$, and $c=a(1+p)$.

Proof. In view of (1.5) and (3.1), we get

$$H_{[n]} = -\int_{-\infty}^{\infty} h_{[n]}(y)\log h_{[n]}(y)dy \tag{3.3}$$

$$= -\int_{-\infty}^{\infty} f_Y(y)[1-\gamma((1+p)F_Y^p(y)-1)]\log f_Y(y)[1-\gamma((1+p)F_Y^p(y)-1)]dy$$

$$= -\int_{-\infty}^{\infty} f_Y(y)\log f_Y(y)dy + \gamma\int_{-\infty}^{\infty} f_Y(y)(1+p)F_Y^p(y)\log f_Y(y)dy$$

$$-\gamma\int_{-\infty}^{\infty} f_Y(y)\log f_Y(y)dy - \int_{-\infty}^{\infty} f_Y(y)[1-\gamma((1+p)F_Y^p(y)-1)]\log[1-\gamma((1+p)F_Y^p(y)-1)]dy,$$

$$= H(Y) + \gamma(1+p)\psi(p) + \gamma H(Y) + I = (1+\gamma)H(Y) + \gamma(1+p)\psi(p) + I,$$

where

$$I = -\gamma\int_{-\infty}^{\infty} f_Y(y)\log f_Y(y)dy - \int_{-\infty}^{\infty} f_Y(y)[1-\gamma((1+p)F_Y^p(y)-1)]\log[1-\gamma((1+p)F_Y^p(y)-1)]dy.$$

Using the substitution $u = \log[1-\gamma((1+p)F_Y^p(y)-1)]$, then,

$$I = -\log(1-p\gamma) - \gamma(1+p)p\int_{-\infty}^{\infty} \frac{(1+\gamma)F_Y^p(y)-\gamma F_Y^{2p}(y)}{1-\gamma((1+p)F_Y^p(y)-1)} f_Y(y)dy.$$

Let $z = F_Y(y)$ after some substitutions we get (3.2).

3.1 Numerical Study

We use MATHEMATICA Ver. 11.3, to evaluate $H_{[n]}$ by using (3.2), for the copula of HK-FGM. From Table 1, the following general property can be extracted: for every fixed n we have $H_{[n]}$ increases with increasing n. Moreover, Table 2 gives some numerical values for $II(R_n, R_{[n]})$, at $\lambda = 0.25, p = 1,2,3,4$.

Table 1: The Shanon Entropy $H_{[n]}$ for the Coupla of HK-FGM1 at $(\lambda = 0.25, p = 1,2,3,4)$

n	$p=1$	$p=2$	$p=3$	$p=4$

3	-0.0059	-0.0446	-0.1285	-0.557
5	-0.0092	-0.0800	-0.2580	-0.5758
7	-0.0102	-0.0919	-0.3082	-0.7313
10	-0.0104	-0.0957	-0.3254	-0.7972

Table 2: $II(R_n, R_{[n]})$, at $\lambda = 0.25$

n	$p = 1$	$p = 2$	$p = 3$	$p = 4$
1	0.1137,-0.1786,0.0482	0.7332,-0.0.0703,0.0121	0.6461,0.0563,0.0105	0.6890,0.1006,0.0305
2	0.1521,-0.0600,0.0197	0.4741,-0.0335,0.0066	0.5219,0.0397,0.0066	0.5794,0.0712,0.0193
3	0.2228,-0.0517,0.0186	0.5031,-0.0317,0.0065	0.5840,0.0397,0.0067	0.6580,0.0765,0.0198
4	0.2755,-0.0767,0.0219	0.4160,-0.0372,0.0067	0.3977,0.0342,0.0063	0.4221,0.0607,0.0184
5	0.3079,-0.0546,0.0205	0.5971,-0.0342,0.0072	0.7047,0.0444,0.0074	0.8012,0.0871,0.0219
6	0.3261,-0.0469,0.0160	0.3841,-0.0287,0.0057	0.4319,0.0348,0.0059	0.4870,0.0655,0.0175
7	0.3359,-0.0486,0.0147	0.2956,-0.0261,0.0049	0.2949,0.0260,0.0049	0.3121,0.0462,0.0141
8	0.3410,-0.0556,0.0222	0.6966,-0.0349,0.0077	0.8284,0.0468,0.0078	0.9497,0.0934,0.0232
9	0.3437,-0.0578,0.021	0.5870,-0.0361,0.0073	0.6905,0.0412,0.008	0.7814,0.0903,0.023
10	0.3450,-0.0536,0.0182	3.1427,-0.0333,0.0065	0.5212,0.0322,0.0068	0.5807,0.0782,0.0202
11	0.3457,-0.0437,0.0145	3.1509,-0.0269,0.0052	0.3538,0.0224,0.0055	0.3876,0.0596,0.0164
12	0.3461,-0.0325,0.0110	3.1550,-0.0197,0.0039	0.2265,0.0224,0.0042	0.2427,0.0404,0.0121

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