1 # A HIGHER ORDER BOUNDARY INTEGRAL EQUATIONS FOR 3D STEADY STATE HEAT CONDUCTION

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4 ## ABSTRACT

The manifest success of the Finite Element Method has led to progressively increased demands being made of it. In particular, there is increasing pressure to use sophisticated 3D models which result in large costly numerical systems. It has shown that the singular point of the fundamental solutions can validly be taken outside the domain of the problem thereby yielding regular boundary integral equations [1]. The location of the singular point outside the domain of the problem permits the use of harmonic functions with higher order singularities, as kernels in the boundary integral equations. This possibility is attractive because the higher order singularities decay rapidly away from the singular point thereby resulting in more diagonally dominant algebraic equations.

In this paper, a set of higher order functions are used as kernel functions, which still satisfy the governing equation (Lapalce's equation) with discretization acheived as continuous, discontinuous and partially discontinuous elements.

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INTRODUCTION

The computational methods most widely used at present tackle the governing differential equations directly in the form in which they were derived without any further mathematical manipulation. This is usually done either by approximating the differential operators in the equations by simpler, localised algebraic ones volid at a series of nodes within the region (e.g.Finite Difference Methods, FDM) or by representing the region itself by noninfinitesmal (i.e.Finite) elements of material which are assembled to provide an approximation to the real system (Finite Element method, FEM). All such methods (Domain methods) involve wholebody discretization schemes which require, finally, the solution of very large systems of algebraic equations. These methods unavoidably generate the solution at all the internal 'nodes' used, whether or not this information is required.

The Finite Element Method has in recent years reached such a stage of development that many workers would doubt whether any equivalent, let alone superior, technicque could ever appear. However , we could return to our set of differential equations and, as an alternative approach, try to integrate them analytically in some way before either proceeding to any discretization scheme or introducing any approximations. We are ,of course, attempting to integrate the differential equations to find a solution whatever method we use, but the essence of integral equation techniques is the transformation of the differential equations into equivalent sets of integral ones as the first step in their solution. Intuitively one would expect from such an operation a set of equations which would involve only values of the variable at the extremes of the range of integration (i.e. on the boundaries of the region) with the implication that any discretization scheme needed subsequently would only involve subdivision of the bounding surface of the body[2]. This exactly what happens and has let to the boundary integral equation methods (or the name Boundary Element Methods, BEM). Some of the attractive features of BEM are that, by using a boundary discretization scheme, the effective dimensionality of any problem is reduced by one, which, especially in three dimensions, leeds to an appreciable reduction in the number of algebraic equations generated for the solution, as well as much simplified data preparation. In addition, the solution variables vary continuously throughout the region and therefore detailed output can be obtained at subsequently chosen internal points. The solution is , in principle, exact, and approximations need only be introduced if they are required either to describe the boundary geometry or to evaluate the various integrals involved [3].At the very heart of the method is the fact that infinitely distant boundaries can be automatically accounted for without discretization, a property unique to BEM.

THEORY

The formulation of a potential problem (steady state heat conduction) using the BEM starts by considering a potential function (temperature) T over the domain Ω which satisfies the governing equation of the probem. Consider for instance, an equation such as Laplace's for simplicity;

$$\nabla^2 T = 0 \qquad \text{in } \Omega \tag{1}$$

The boundary Conditions for the problem are:

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(i) essential $\quad T = \bar{T} \quad$ on Γ_1

(ii) natural $\quad \dfrac{\partial T}{\partial n} = \overline{\dfrac{\partial T}{\partial n}} \quad$ on Γ_2 (2)

where $\bar{T}$ and $\overline{\frac{\partial T}{\partial n}}$ are given temperature and temperature normal derivative on the boundary. Γ_1 & Γ_2 partition the boundary Γ ($\Gamma = \Gamma_1 + \Gamma_2$) , and n is outward normal (see figure 1.)

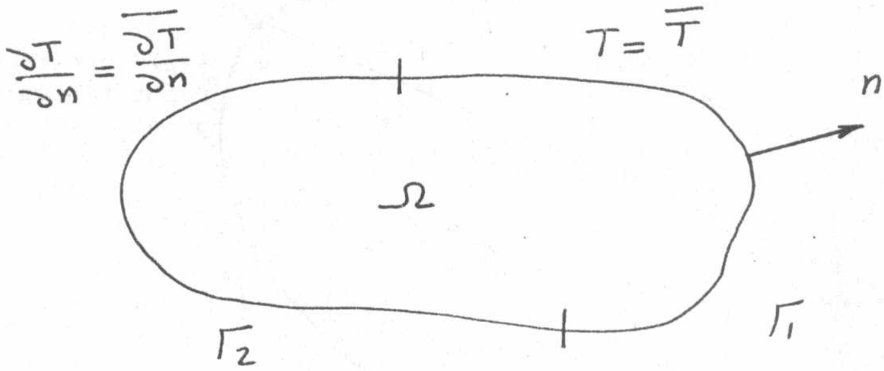

Figure.1. Problem definition

Now introducing a function T* for the numerical solution of the problem , one can write a weighted residual statement in which the Laplacian term is integrated by parts twice [4]. The resulting statement is then satisfied by the governing equation of the problem which is :

$$\nabla^2 T^* + \Delta(x) = 0 \qquad (3)$$

in which $\Delta(x)$ is the Dirac delta function and the function T^* depends upon two points, the 'source' point 'x' and the 'observation' point 'y'. To formulate the boundary problem, the source point 'x' is taken on the boundary of the domain Ω and knowing the boundary conditions for respective parts of the boundary, the general Boundary Integral Equation is written as:

$$C_{(x)}\, T_{(x)} + \sum_{j=1}^{N} \int T_{(y)} \frac{\partial T^*_{(x,y)}}{\partial n}\, d\Gamma_{(y)} = \sum_{j=1}^{N} \int \frac{\partial T}{\partial n}(y)\, T^*_{(x,y)}\, d\Gamma_{(y)}$$

$$\text{for all } y \in \Gamma \qquad (4)$$

where C(x) is an unknown coefficient and $T_{(x)}$ is the potential (temperature) at 'x' .

N = Number of boundary elements.

T*= is the fundamental solution (Kernel function)

which for 3D case is given as:

$$T^* = \frac{1}{4\pi\, r(x,y)} \qquad (5)$$

where 'r' is the distance between the 'source' and the 'observation' points and $\frac{\partial T^*}{\partial n}$ is the normal derivative of T* , given by

$$\frac{\partial T^*}{\partial n} = \frac{\partial T^*}{\partial n}(x,y) = \frac{1}{4\pi r^2} \sum_{\ell=1}^{3} \hat{n}_{\ell_y} \frac{x_\ell - y_\ell}{r} \qquad (6)$$

where $x = (x_1, x_2, x_3)$ is the source point
$y = (y_1, y_2, y_3)$ is the field observation point
$n = n_y$ is the normal at point 'y' (sec figure 2).

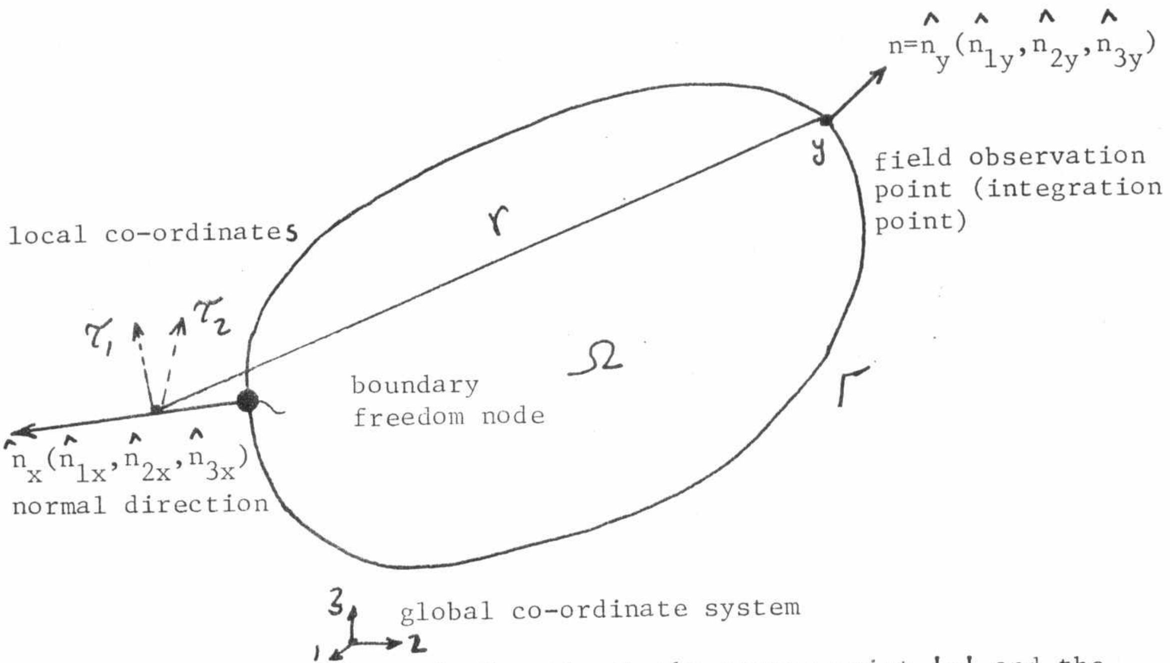

Figure 2. Normals at the source point 'x' and the field observation point 'y'.

The integrals in equation (4) can easily be evaluated numerically for all segments except the one containing the source point (x). The singular integrals is either computed analytically or by using a special integration scheme using higher order integration rules [5]. However, if the source point 'x' is located outside the domain Ω, the coefficient C(x) in equation (4) equals zero and we get:

$$\sum_{j=1}^{N} \int \frac{\partial T^*}{\partial n} T \, d\Gamma = \sum_{j=1}^{N} \int \frac{\partial T}{\partial n} T^* \, d\Gamma \qquad (7)$$

which gives a system of 'Regular boundary integral equations, one for each singular point corresponding to the boundary node 'x' under the Boundary element discretization and located outside the domain, Ω , of the given problem at an aribrary distance from 'x' and along the outward normal [6]. Now, not only is no special attention required regarding singular integrands but also higher order hormonic functions can be empleyed as kernels. In this paper, a set of higher order functionsare used as kernel functions, which still satisfies the Laplace's equation.

The first higher order function used is obtained by the differentiation of scalar field function T* with respect to the normal 'n_x'. Note that this differentiation with respect to 'n_x' ,actually ,is a linear combination of the differentiation of T* with respect to the global Co-ordinate axes (1,2,3) in which there is no obligation to use 'n_x' compents (n_{1x}, n_{2x}, n_{3x}). We can use any linear combination, for instance ($\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$) , to generate the same solution on the boundary provided that all compenents exist. The differentiation can be preformed as in the following:

$$T_I^* = \frac{\partial T^*}{\partial n_x} = \nabla T^* \cdot \hat{n}_x = \frac{1}{4\pi r^2} \sum_{\ell=1}^{3} n_{\ell_x} \frac{x_\ell - y_\ell}{r}$$

$$\frac{\partial T_I^*}{\partial n} = \frac{\partial T_I^*}{\partial n_y} = \nabla T_I^* \cdot \hat{n}_y = \frac{1}{4\pi r^3} \left[3\left\{ \sum_{\ell=1}^{3} n_{\ell_x} \frac{x_\ell - y_\ell}{r} \right\}\left\{ \sum_{\ell=1}^{3} n_{\ell_y} \frac{x_\ell - y_\ell}{r} \right\} - \sum_{\ell=1}^{3} n_{\ell_x} \cdot n_{\ell_y} \right]$$

where ∇ is the gradient operator $\left(\nabla = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right)$.

The remaining higher order functions can also be deduced in the same way. Let us put all these higher order functions in a more compact form, as in the following (table 1).

APPLICATIONS

Two 3D steady state heat conduction problems are analysed to test the validity of the proposed technique

Steady state heat conduction of a cube

The cube is subjected to the boundary conditions shown in figure 3. Numerical values for the problem are assumed as:

$a=6$, $T=T_o = 300$

The exact solution is :

$$T = \left(1 - \frac{x}{a}\right) T_o$$

$$\frac{\partial T}{\partial n} = \frac{\partial T}{\partial x} = -\frac{T_o}{a} = -50$$

whilst this is a planar problem, it is analysed in 3-space and it is interesting as a validity test on the coding and as an investigation of the suitable number of higher order kernel and its critcal radius.

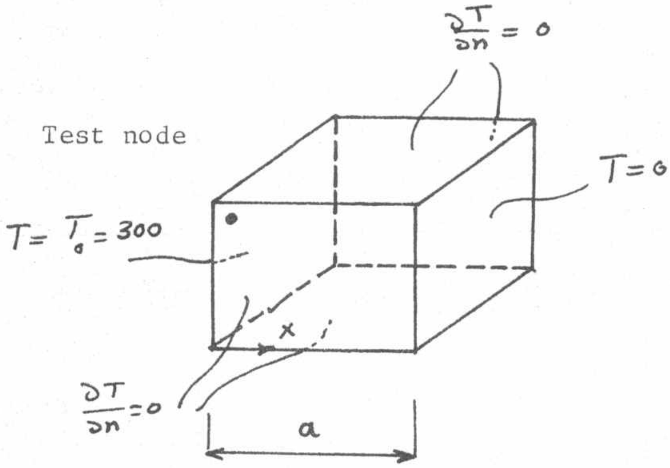

Fig.3 Boundary conditions.

For three-dimensional problems, using the higher order kernels as the fundamental solution, the order of intergrations λ_1, λ_2 and the precision of integration coefficient 'k' [5 ,7,8] will be higher than that of the corresponding values of the conventional kernels. The higher order kernels will be of the order $\frac{1}{r^{n+1}}$ and $\frac{1}{r^{n+2}}$ depending on the type of the kernel used, where $n \in \{1,2,3,4,5,6\}$ is the order of the new kernel. On the light of experience it was found that the location of the singular point, the precision of integration coefficient and the order of kernel functions are three related parameters in the program. Le us fix the location of the singular points as discussed before in [1]. After a systematic study, it was found that for the orders [1,2,3,4] and [5,6] of the kernel functions, the precision of integration coefficients 5.0 and 6.0 , respectively, are relatively suitable for the integration process (see Table 2).

Table 1 Higher order kernels

order	function (kernel)	Normal derivative of the function, where $(n = n_y)$
0	$\phi^* = \frac{1}{4\pi r}$ conventional kernel	$\frac{\partial\phi^*}{\partial n} = \frac{1}{4\pi r^2}\sum_y$
1	$\phi_1^* = \frac{1}{4\pi r^2}\sum_x$	$\frac{\partial\phi_1^*}{\partial n} = \frac{1}{4\pi r^3}(3\sum_x . \sum_y - \hat{n}_x.\hat{n}_y)$
2	$\phi_2^* = \frac{1}{4\pi r^3}[3(\sum_x)^2 - 1]$	$\frac{\partial\phi_2^*}{\partial n} = \frac{1}{4\pi r^4}[A\sum_y - 6\sum_x(\hat{n}_x.\hat{n}_y)]$ $A = \{5[3(\sum_x)^2 - 1] + 2\}$
3	$\phi_3^* = \frac{1}{4\pi r^4}\sum_x .(A - 6)$	$\frac{\partial\phi_3^*}{\partial n} = \frac{1}{4\pi r^5}\{3[5(\sum_x)^2 - 3][7\sum_x\sum_y - \hat{n}_x.\hat{n}_y]$ $- 6\sum_x[5\sum_x(\hat{n}_x.\hat{n}_y) - 3\sum_y]$
4	$\phi_4^* = \frac{3}{4\pi r^5}[35(\sum_x)^4 - 30(\sum_x)^2 + 3]$	$\frac{\partial\phi_4^*}{\partial n} = \frac{-3}{4\pi r^6}\{35(\sum_x)^3[4(\hat{n}_x.\hat{n}_y) - 9\sum_x\sum_y]$ $- 30(\sum_x)[2(\hat{n}_x.\hat{n}_y) - 7\sum_x\sum_y]$ $- 15\sum_y\}$
5	$\phi_5^* = \frac{15}{4\pi r^6}\{63(\sum_x)^5 - 70(\sum_x)^3 + 15\sum_x\}$	$\frac{\partial\phi_5^*}{\partial n} = \frac{15}{4\pi r^7}\{\sum_x\sum_y[693(\sum_x)^4 - 630(\sum_x)^2 + 105]$ $- 15(\hat{n}_x.\hat{n}_y)[21(\sum_x)^4 - 14(\sum_x)^2 + 1]\}$
6	$\phi_6^* = \frac{45}{4\pi r^7}\{231(\sum_x)^6 - 315(\sum_x)^4 + 105(\sum_x)^2 - 5\}$	$\frac{\partial\phi_6^*}{\partial n} = \frac{45}{4\pi r^8}\{231(\sum_x)^5[13\sum_x\sum_y - 6(\hat{n}_x.\hat{n}_y)]$ $- 315(\sum_x)^3[11\sum_x\sum_y - 4(\hat{n}_x.\hat{n}_y)]$ $+ 105(\sum_x)[9\sum_x\sum_y - 2(\hat{n}_x.\hat{n}_y)] - 35\sum_y\}$

where, $\sum_x = \sum_{l=1}^{3} n_{l_x}\frac{x_l - y_l}{r}$

$$\sum_y = \sum_{l=1}^{3} n_{l_y}\frac{x_l - y_l}{r}$$

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Table 2. Order of the kernel function test

Order of kernel functions	Execution time (sec)	error $\left\|\frac{\delta x}{x}\right\|$, where $x = \frac{\partial T}{\partial n}$	Other parameters
0	245	0.10×10^{-3} %	- location of singularity coefficient ≡ 1.0 -precision of integration coefficient ≡ 5.0 -discontinuous elements are used -matched mesh -quadratic freedom functions -one subdomain -the aspect ratio of an element ≡ 1.0 -Test example:the same example as shown in figure 3 is used. -exact value at the test node ($\frac{\partial T}{\partial n}$) =50.0
1	"	0.70×10^{-3} %	
2	"	0.80×10^{-2} %	
3	"	0.70×10^{-1} %	
4	"	0.65 %	
5	"	5.2 %	
6	"	27.6 %	

This method seems to be more expensive than using the conventional kernels. However it is not,as these higher order functions decay more quickly with distance than does the fundamental solution, and thus improve diagonal dominance in the algebraic system. More importantly, they offer the possiblity that the equations need only be evaluated up to some distance from the singularity, the remaining contributions effectively vanishing. This distance will be called the 'critical radius'-see figure 4.

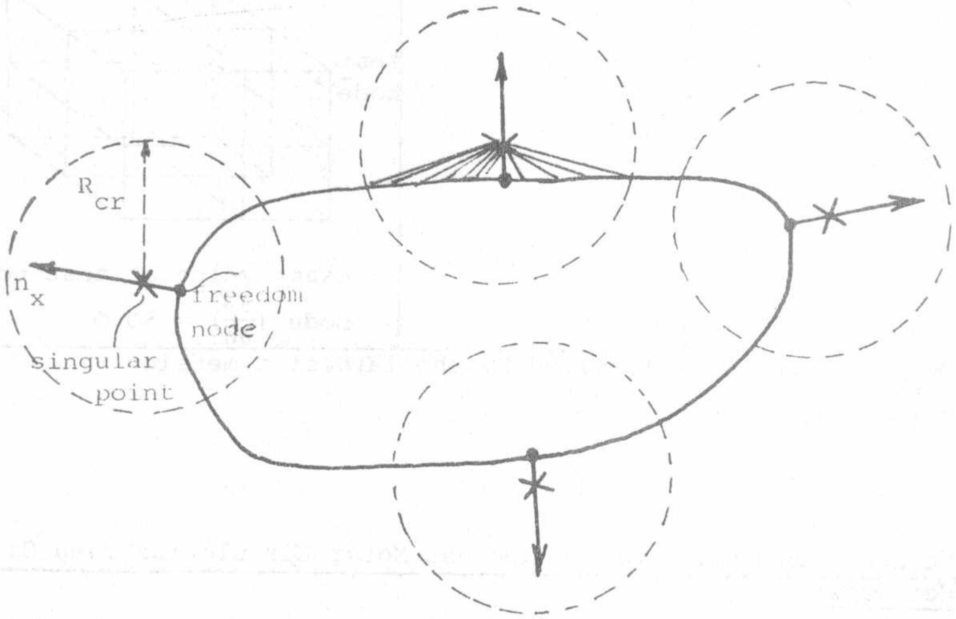
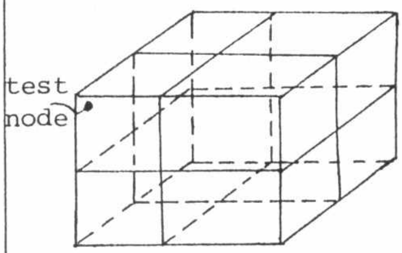


Figure 4. Critical radius for higher order kernels.

This incomplete integration procedure reduces the labour required to set up the algebraic equation (central processing time) depending on the required accuracy. On the light of experience it was found that the critical radius, equal to half of the greatest dimension of the current subdomain ,is relatively suitable for the integration process with a significant 15% reduction of the execution time (third order kernels). However this is without appreciable deterioration of the accuracy compared to the boundary values of the conventional kernels (see Table 3).

Table 3 Critical radius coefficient test

Critical radius coefficient (R_{cr})	Execution time (sec)	error $\frac{\lvert\delta_x\rvert}{x}$, where $x \equiv \frac{\partial T}{\partial n}$	Other parameters
0.3	653	2.1%	- location of singularity coefficient $\equiv$ 1.0 - precision of integration coefficient $\equiv$ 3.0 - partially discontinuous elements [9]. - matched mesh - quadratic freedom functions are used. - one subdomain. - the aspect ratio of an element $\equiv$ 1.0 - Test example: heat conduction of a cube  - exact value at test node node $(\frac{\partial T}{\partial n})$ = 50.0
0.4	704	0.85%	
0.5	774	0.54%	
0.6	822	0.42%	
0.8	896	0.16%	
1.0	913	0.11%	

where R_{cr} is a factor multiplied by the largest dimension of the domain (subdomain)

Steady State Heat Conduction in a Glandless Motor Circulating Pump Case Including Hot Neck

The glandless motor circulating pump case with its hot neck to be analysed is the pump which was subject to research and study by [10,11]. The glandless motor circulating pump is used in modern electrical power generating stations where steam, the product of combustion of a fossil or nuclear fuel, drives the turbo-generators (see figure 5).

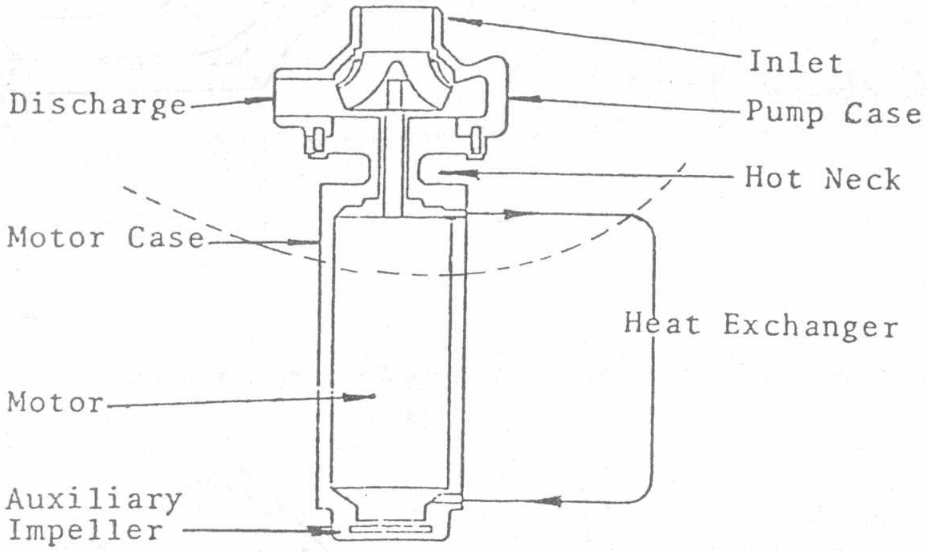


Figure 5. G.M.P. Pump and Motor Unit Assembly.

The pump case with the hot neck arrangement is symmetric with respect to its own axis, so only a 30 degree sector (say) needs to be considered here. The dimensions of the problem, and the boundary element discretization employed are given in figure (6 and 7). Ninety six 8-freedom nodes of discontinuous and partially discontinuous elements, in which three are degenerate (triangular), that is 625 freedom nodes for seven subdomains, are taken using a mismatched mesh. The assumed boundary values of temperature and of its normal derivative are shown in figure 8. In reality, there were radii at most of the corners of the arrangement, but some of them could not be modelled numerically. The problem is analysed by the Regular Boundary Method using discontinuous and partially discontinuous elements in conjunction with conventional and higher order kernels. Although it is relatively suitable to solve this problem by the domain methods (finite elements), since the surface/volume ratio is relatively high, and the band width is relatively small. In the analysis by the Boundary Element Method, most of the computing time is spent integrating for the matrix, whereas from previous experiences of finite element analysis, the reduction of the system of equations is the longest calculation.However, by using the higher order kernels in the Regular Boundary Element Method, computing time needed for integration to form the matrix will be reduced according to the required accuracy, and the problem of high surface/volume ratio will be suitable for the demonstration of such an advantage. In the present analysis a precision of integration 5'0 is used for quadratic functional variation in the case of the conventional kernels, while a precision of integration 6.0 is used for the 3rd order kernels. The contour plotting of temperatures (isothermal lines) on a 15 degree plane of this problem is shown in figure 9. It is interesting to note that the same isothermal lines are obtained by using the 3rd order kernels , with change of temperatures by not more than 5.0 %, from the results obtained by using the conventional kernels. The execution time, however, decreases by about 15%.

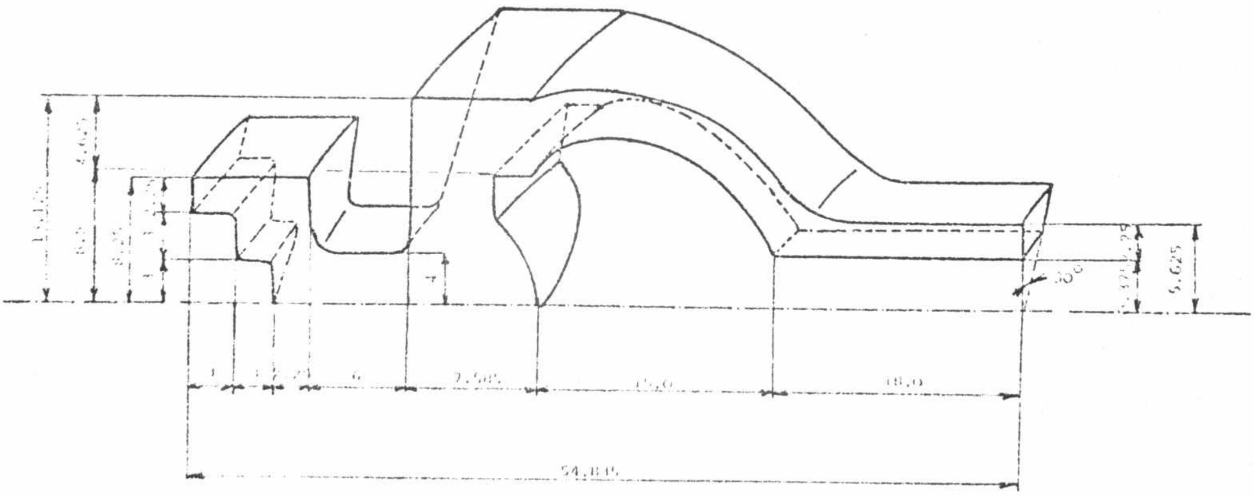
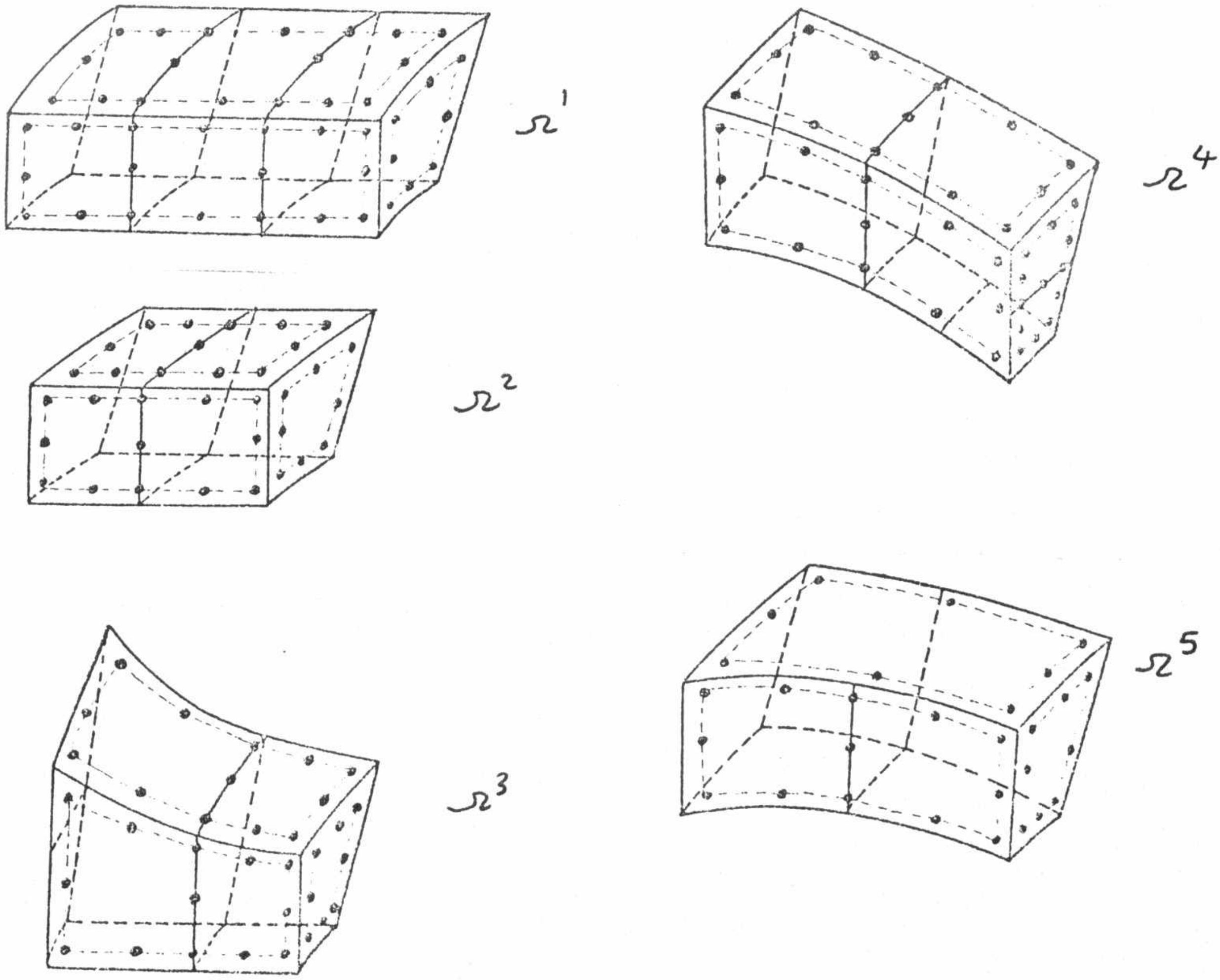


Fig.6 Pump Case Including Hot Neck-Problem Definition.



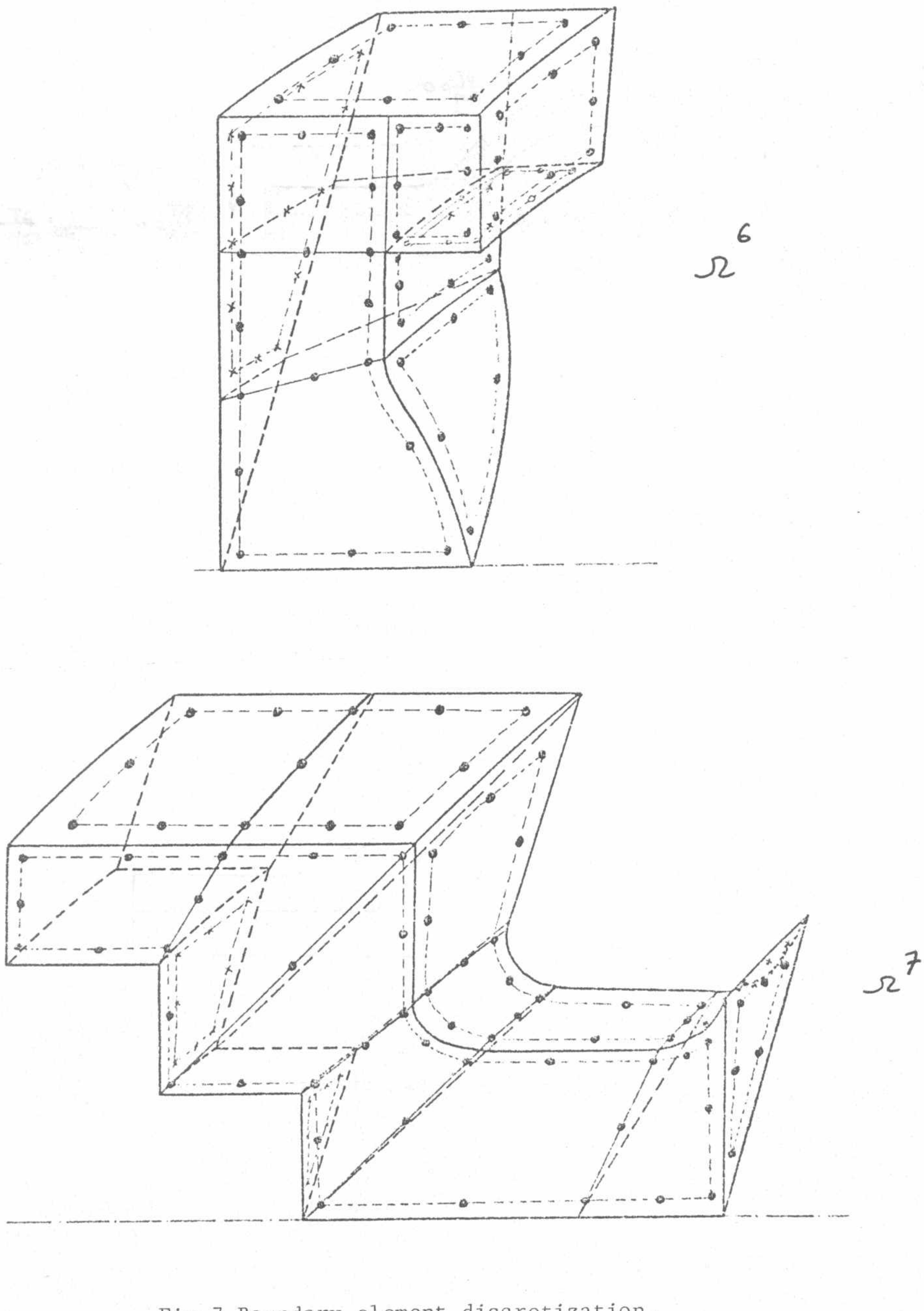


Fig.7 Boundary element discretization.

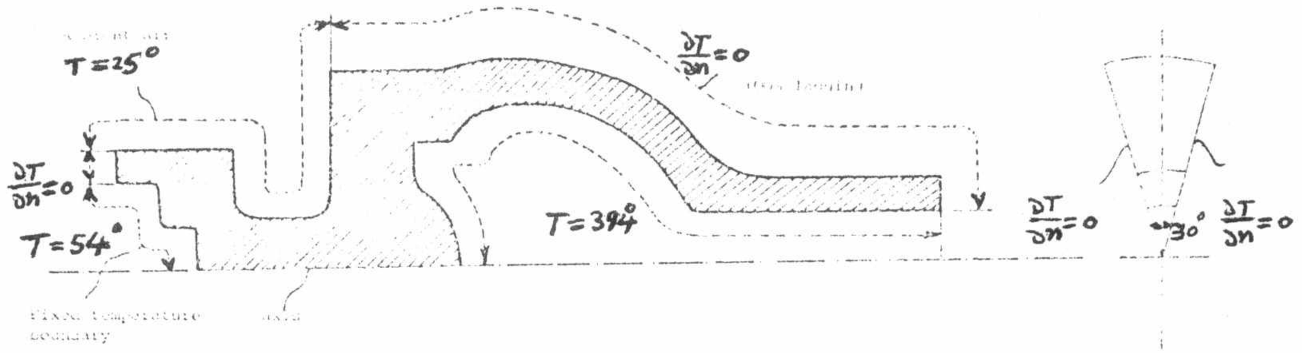


Fig.8.Boundary conditions.

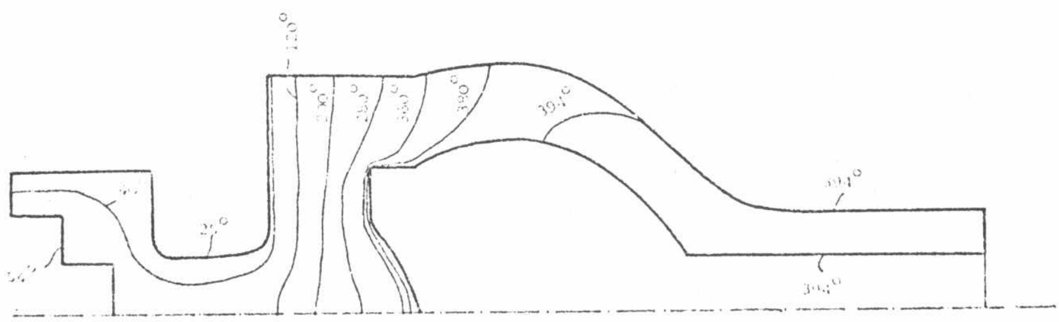


Fig.9 Isothermal lines for a 15 degree plane.

DISCUSSION AND CONCLUSIONS

In this paper a Higher order BEM has been presented. Here a set of higher order Fundamental solutions are used as kernel functions with its singularity taken outside the domain of the problem, giving an infinite system of higher order regular integral equations. On disretizing the system using continuous, discentinous and partially-Discontinuous elements [9]. The resulting higher order kernels are everywhere regular over the boundary. The case study results presented here, for 3D steady state heat conduction problems, show that the conventional and higher order kernels have similar convergence characteristics. On the boundary, variations of around 3% were found using quaddratic elements , but the computed interior results were in close agreeement. It should be noted that for the 'Higher order regular method', these results correspond to the best location of the singular point which was found to be the shortest dirtance between any two neighbouring freedoms away from the element along the outward normal [4] .

For these higher order kernels as fundemental solutions the precision of integration (the number of integration points in the Gaussian integration formula [8]) will be higher than that of the corresponding value for the conventional kernels . This method seems to be more expensive than the use of conventional kernels. However, it may not be so, as these higher order functions are of potentially great interest because they decay more quickley with distance than does the fundamental solution (see figure 4) and thus improve the diagnnal dominance in the algebraic system. Moreover, they offer the possibility that the equations need only be evaluated up to some distance from the singularity, the remaining contributions effectively vanishing . If this is so, then the labour required to set up the algebraic equations (central processing time) can be reduced. The results are promising. We should not forget in the Regular BEM to generate solutions only on the boundary. After that, we can use the conventional kernels to find the internal solution of the problem using Grean's identity [7]. In conclusion: A higher order boundary Method has been presented and used in the analysis of two test problems. Results show that the method is validly be employed for the regular method. The use of higher order kernels in the boundary integral equations involved no increased computing time for a given quality of solution and is potentially advantageous because of improved diagonal dominance in the algebraic equations.

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