EFFECT OF PRESSURE DIFFERENCE ON STATIONARY AND ROTATING ANISOTROPIC DISCS

E.E.El SOALY*, N.A. EL SEBAI**

ABSTRACT

The stress analysis of pressurized and rotating isotropic discs is a classical problem which can be found in literature. In a recent paper, Kookson and Sathianathan indicated that the effect of anistropy, on the stress distribution within a rotating disc, is considerable. In our proposed work, the effect of pressure on stationary and steady rotating discs is studied. The stress distribution in stationary discs, under inner and outer pressures, is analyzed for different values of anisotropic constant $\lambda = \sqrt{E_\theta / E_r}$, where E_θ and E_r are the material elastic moduli in tangential and radial directions respectively. The effect of variable parameters, radius ratio, anisotropic constant λ, pressure difference, is determined for the case of orthotropic steady rotating annular discs. Our results indicate that the effect of anisotropy, on the stress distribution within a disc, is considerable.

INTRODUCTION

The analysis of thin elastic annular discs is of great importance for many engineering applications such as gas-turbines, steam turbines, compressors, flywheels and computer stores. Hence, it is often necessary to determine their deformations and the stress distributions within them.

A considerable amount of literature exists for the case where isotropic materials are used in fabrication of discs. However, much interest has been shown, in recent years, in the possibility of constructing discs from composite materials. These materials always have some kind of anisotropy, in which the elastic modulus and the strength differ from one direction to another. Composite materials are used in such a manner that the direction of their greatest strength corresponds to the direction of principal load acting above the device. In this way, it can be possible to construct light weight components of optimum load carrying capacity.

Cookson and Sathianathan |1| determined the stress distribution within anisotropic discs which are rotating at steady speed. They indicated that the effect of anisotropic constant λ is considerable, while the effect of varying the value of Poisson's ratio $\nu_{\theta r}$ is slight.

In the present work, the stress analysis of stationary annular anisotropic discs, under the effect of internal and external pressures, is deduced. The

* Col. Dr., ** Lt. Col. Dr., Dpt. of Mechanics and Elasticity, Military Technical College, Cairo, Egypt.

method of superposition is used to determine the stresses within pressurized steady rotating orthotropic discs. The effect of variable parameters, anisotropic constant radius ratio and pressure difference, has been analysed for the case of the orthotropic annular disc.

During the analysis, the following assumptions are made:

(1) the material is macroscopically homogeneous and cylindrically anisotropic,
(2) a condition of plane stress exists.

THEORETICAL ANALYSIS

Under the condition of plane stress and for cylindrically anisotropic material, the stress-strain relationships in polar co-ordinates can be expressed as follows:

$$\sigma_r = \frac{E_r}{N} (\varepsilon_r + \nu_{\theta r} \varepsilon_\theta)$$

$$\sigma_\theta = \frac{E_\theta}{N} (\varepsilon_\theta + \nu_{r\theta} \varepsilon_r) \qquad (1)$$

$$\tau_{r\theta} = 0$$

where $N = 1 - \nu_{r\theta} \nu_{\theta r}$.

Due to rotational symmetry of disc problems, the strain-displacement relationships reduce to ,

$$\varepsilon_r = \frac{du}{dr} , \quad \varepsilon_\theta = \frac{u}{r} \quad \text{and} \quad \gamma_{r\theta} = 0 \qquad (2)$$

and the equilibrium equation, for the stationary case, can be written as

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (3)$$

This equation may be written in terms of radial displacement u. Substituting equations (2) and (1) into equation (3), the following differential equation is obtained :

$$r^2 . \frac{d^2u}{dr^2} + kr . \frac{du}{dr} - L u = 0 \qquad (4)$$

where $k = (1 + \nu_{\theta r}) - \nu_{r\theta} \lambda^2$

$L = \lambda^2$

$\lambda = \sqrt{E_\theta / E_r}$: anisotropic constant.

The solution of differential equation (4) is

$$u = A r^C + B r^D \qquad (5)$$

where

$$C = \tfrac{1}{2} [\lambda^2 \nu_{r\theta} - \nu_{\theta r} + \sqrt{(\lambda^2 \nu_{r\theta} - \nu_{\theta r})^2 + 4 \lambda^2}] ,$$

$$D = \tfrac{1}{2} [\lambda^2 \nu_{r\theta} - \nu_{\theta r} - \sqrt{(\lambda^2 \nu_{r\theta} - \nu_{\theta r})^2 + 4 \lambda^2}] ,$$

A and B are two integration constants.
Substituting expression of radial displacement u into equations (1) and (2) the stress components are obtained in the following form:

$$\sigma_r = \frac{E_r}{N}\left[A\ (\nu_{\theta r} + C)\ r^{C-1} + B\ (\nu_{\theta r} + D)\ r^{D-1}\right]$$

$$\sigma_\theta = \frac{E_\theta}{N}\left[A\ (1 + C\,\nu_{r\theta})\ r^{C-1} + B\ (1 + D\nu_{r\theta})\ r^{D-1}\right] \tag{6}$$

For a simple constant thickness, annular anisotropic disc, which is acted upon by internal pressure P_a and external pressure P_b , Fig.1, the boundary conditions at the inner and outer radii are,

$$\sigma_r = -P_a \quad \text{at} \quad r = a$$

$$\sigma_r = -P_b \quad \text{at} \quad r = b \tag{7}$$

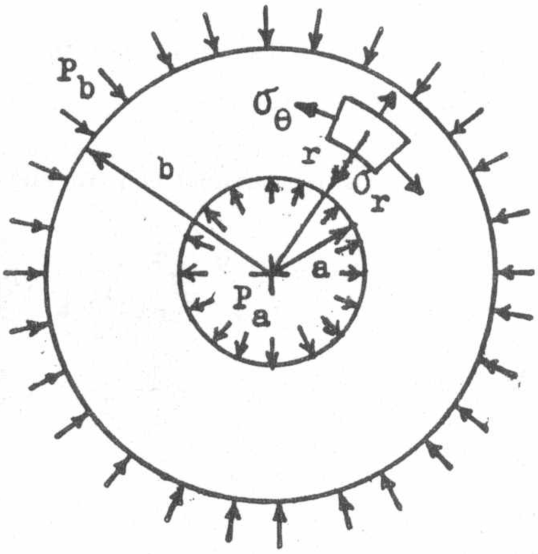


Fig.1 Cylind rically anisotropic disc under inner and outer pressure.

Substituting the boundary conditions (7) into expression (6), the integration constants A and B can be determined. Hence, the radial and tangential stress components in a stationary pressurized cylindrically anisotropic annular disc, are given as

$$\sigma_r = \frac{1}{M}\left[(P_a b^{D-1} - P_b a^{D-1})\ r^{C-1} + (P_b a^{C-1} - P_a b^{C-1})\ r^{D-1}\right]$$

$$\sigma_\theta = \frac{\lambda^2}{M}\left[\left(\frac{1+C\nu_{r\theta}}{\nu_{\theta r}+C}\right)(P_a b^{D-1} - P_b a^{D-1})\ r^{C-1} + \right.$$

$$\left. + \left(\frac{1+D\nu_{r\theta}}{\nu_{\theta r}+D}\right)(P_b a^{C-1} - P_a b^{C-1})\ r^{D-1}\right] \tag{8}$$

where $M = (a^D b^C - a^C b^D) / ab$

For cylindrically orthotropic annular discs, i.e. where

$$\frac{\nu_{r\theta}}{E_r} = \frac{\nu_{\theta r}}{E_\theta} \qquad \text{or} \qquad \lambda^2 \nu_{r\theta} = \nu_{\theta r} \quad , \tag{9}$$

the stress distribution reduces to the following form:

$$\sigma_r = \frac{1}{M^*} \left[(P_a b^{-\lambda-1} - P_b a^{-\lambda-1}) r^{\lambda-1} + (P_b a^{\lambda-1} - P_a b^{\lambda-1}) r^{-\lambda-1} \right]$$

$$\sigma_\theta = \frac{\lambda}{M^*} \left[(P_a b^{-\lambda-1} - P_b a^{-\lambda-1}) r^{\lambda-1} - (P_b a^{\lambda-1} - P_a b^{\lambda-1}) r^{-\lambda-1} \right] \tag{10}$$

where

$$M^* = \left[\left(\frac{a}{b}\right)^{-\lambda} - \left(\frac{a}{b}\right)^{\lambda} \right] / ab$$

For the case of isotropic material, that is where $\lambda = 1, \nu_{\theta r} = \nu_{r\theta} = \nu$, equation (10) reduces to the classic isotropic solution.

Cookson and Sathianathen |1| investigated the steady stress distribution within annular orthotropic disc, which is rotating with an angular velocity ω, in the forms:

$$\sigma_r = \left[C_1 \left(\frac{r}{b}\right)^{\lambda-1} + C_2 \left(\frac{r}{b}\right)^{-\lambda-1} - \left(\frac{\nu_{\theta r} + 3}{9 - \lambda^2}\right) \left(\frac{r}{b}\right)^2 \right] \rho \omega^2 b^2$$

$$\sigma_\theta = \left[\lambda C_1 \left(\frac{r}{b}\right)^{\lambda-1} - \lambda C_2 \left(\frac{r}{b}\right)^{-\lambda-1} - \left(\frac{3\nu_{\theta r} + \lambda^2}{9 - \lambda^2}\right) \left(\frac{r}{b}\right)^2 \right] \rho \omega^2 b^2 \tag{11}$$

where

$$C_1 = K \left[\left(\frac{a}{b}\right)^{\lambda+3} - 1 \right]$$

$$C_2 = K \left[\left(\frac{a}{b}\right)^{\lambda-3} - 1 \right] \left(\frac{a}{b}\right)^{\lambda+3}$$

$$K = \frac{\nu_{\theta r} + 3}{(9 - \lambda^2) \left[\left(\frac{a}{b}\right)^{2\lambda} - 1 \right]}$$

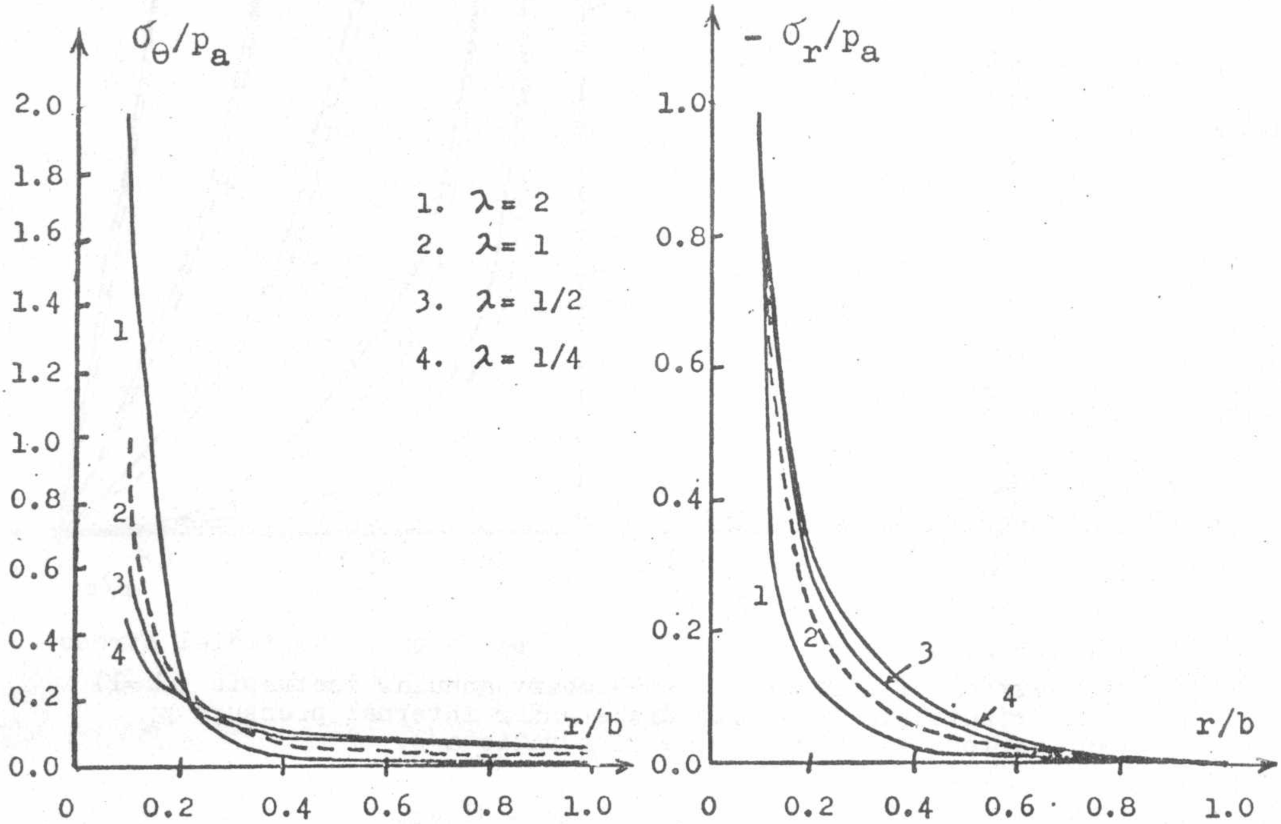
For the case of pressurized rotating cylinderically orthotropic discs, the method of superposition can be applied and the two expressions (10) and (11) are added to each other to obtain the resultant stress components.

RESULTS AND DISCUSSIONS

For the case of stationary orthotropic discs, non dimensional radial and tangential stress distributions were computed according to equation (10). Annular discs with different radius ratios (a/b), under external and internal pressure, were considered. Computations were made for various values of anisotropic constant λ. The computations are illustrated graphically in Figures 2-6.

Fig. 2. shows the nondimensional radial and tangential stress distributions, within a cylindrically orthotropic annular disc acted upon by internal pressure P_a, for various values of anisotropic constant λ= 0.25 , 0.5 , 1 and 2.

The maximum value of the hoop stress σ_θ always occurs at the inner bore for all values of anisotropic contant. The value of the hoop stress at the inner bore reduces as λ decreases, i.e. as the material becomes more stronger in radial direction.



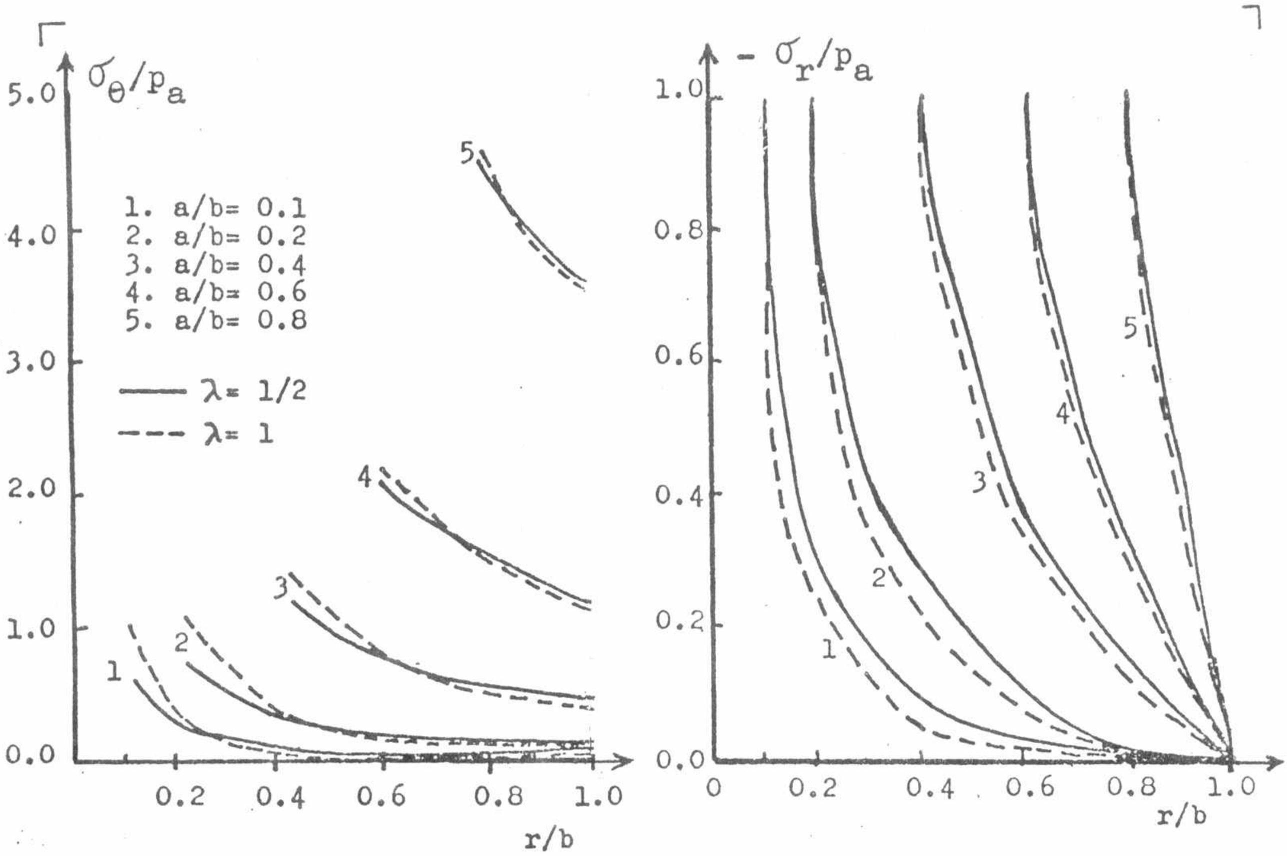
(a) Tensile hoop stress (b) Compressive radial stress

Fig. 2 Stress distributions in a stationary orthotropic annular disc with radius ratio a/b = 0.1 under the effect of internal pressure p_a for various values of anisotropic const. λ

Fig.3 illustrates a nondimentional radial and tangential stress distributions, within anuular disc under internal presure, for various values of the radius ration a/b. The stress distributions are represented for an isotropic case , λ= 1, and an orthotropic case with λ= 0.5 . It may be observed that the value of the hoop stress σ_θ always increases with the redius ratio a/b for both isotropic and orthotropic cases.

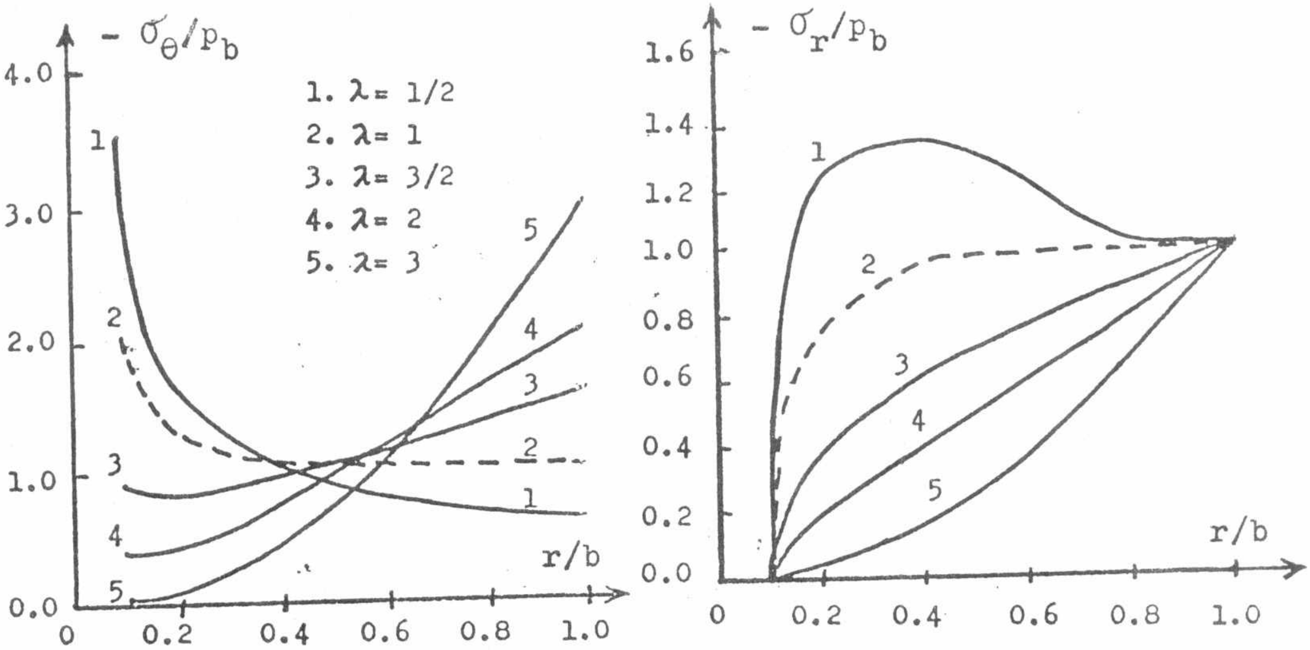
Fig.4 shows the stress distributions in a stationary orthotropic case under the effect of external pressure for various values of anisotropic constant λ= 0.5 , 1,1.5 ,2 and 3 . With increasing λ, the value of the hoop stress at the inner bore reduces while it increases at the outer radius. For λ=1.5, the hoop stress is approximately contsant through the disc radius. For values of λ<1, the radial stress σ_r may reach values greater than the outer pressure.

For the case of a steady rotating disc under external pressure, Fig. 5 illustrates the effect of variation of anisotropic constant λ in the stress distributions within a disc, with radius ratio a/b =0.1. The figure shows



(a) Tensile hoop stress (b) Compressive radial stress

Fig. 3 Comparison of stresses in stationary annular isotropic (λ=1) and orthotropic (λ = 1/2) discs under internal pressure p_a for various values of the radius ratio a/b.



(a) Compressive hoop stress (b) Compressive radial stress

Fig. 4 Stress disributions in a stationary orthotropic annular disc with radius ratio a/b = 0.1 under the effect of external pressure p_b for various values of anisotropic const. λ

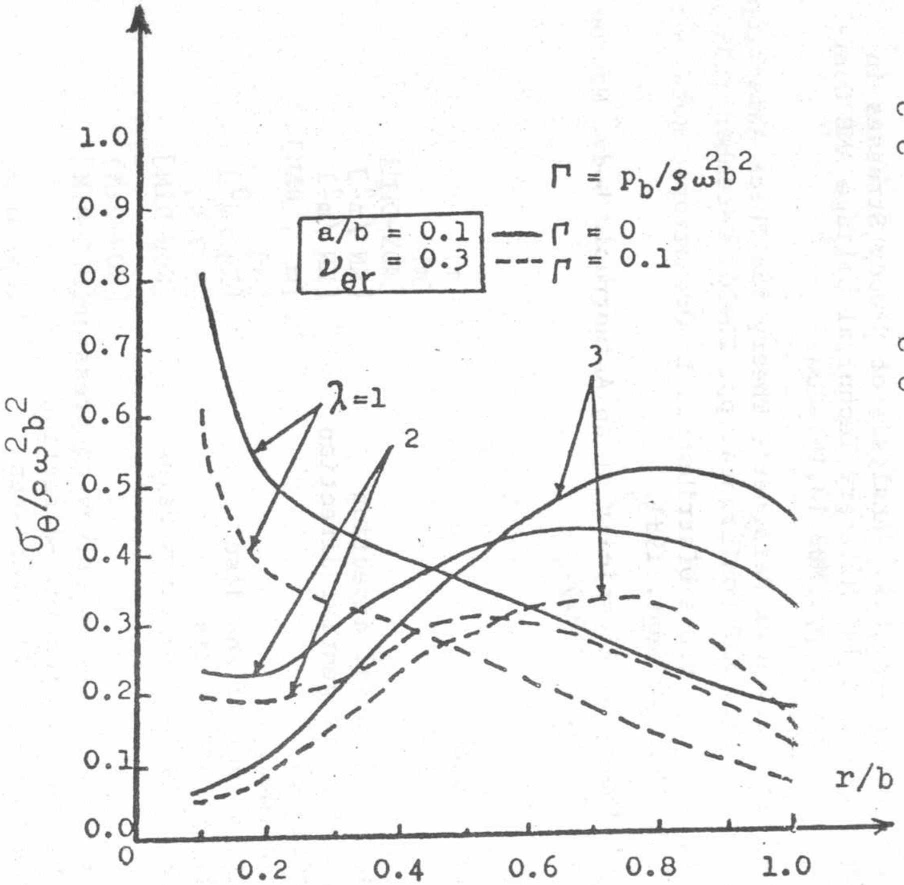


Fig. 5 Hoop stress distribution in a rotating orthotropic disc under external pressure with radius ratio a/b = 0.1 for various values of anisotropic const. λ

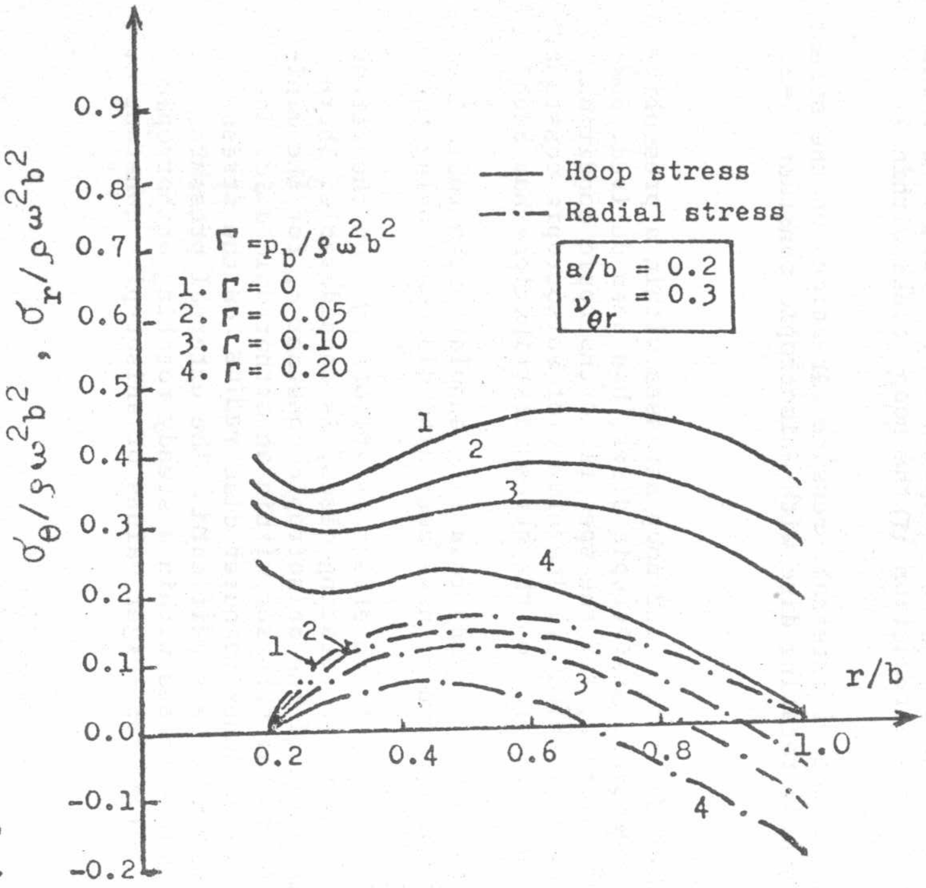


Fig. 6 Stress distributions in a rotating orthotropic disc with radius ratio a/b = 0.2 and anisotropic const. λ = 2 for various values of external pressure p_b.

also the effect of external pressure P_b of a magnitude $P_b = 0.1 \rho \omega^2 b^2$. The external pressure always causes a reduction in the hoop stress within a steady rotating annular disc.

Finally, Fig.6 shows the effect of external pressure variations on the stress distributions within a steady rotating disc with anisotropic constant $\lambda=2$.

CONCLUSIONS

An analytical solution for the radial and hoop stresses within a pressurized, constant thickness, annular, anisotropic discs, has been derived. Results obtained using this solution, for the specific orthotropic condition, are deduced and represented graphically. The effect of anisotropic constant, radius ratio and pressure difference on the stress distributions has been considered.

The stress distributions within a steady rotating annular orthotropic disc, under the effect of external pressure, have been also discussed using the superposition principle.

The computed results indicate that the effect of anisotropy , on the stress dsitribution within stationary and rotating discs, is considerable. There appears to be an optimum value of the anisotropic constant λ, for the minimum value of the hoop and radial stresses within an orthotropic disc. The influence of the ratio of the inner to outer disc radius on the stress distribution, has been shown to be significant. The external pressure reduces the hoop and radial stresses within a steady rotating orthotropic disc and this effect increases for larger values of anisotrpic constant λ.

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NOMENCLATURE

a	inner radius of disc	[m]
b	outer radius of disc	[m]
a/b	radius ratio	[NON-DIM]
E_r	elastic modulus in radial direction	[KN /m^2]
E_θ	elastic modulus in tangential direction	[KN /m^2]
r,θ	Polar co-ordinates	[m , RAD]
u	radial displacement	[m]
ρ	mass per unit volume of the disc	[Kg /m^3]
ω	angular velocity of the disc	[RAD/S]
$\varepsilon_r, \varepsilon_\theta, \gamma_r$	radial, tangential and shear strains	[NON-DIM]
$\lambda=\sqrt{E_\theta/E_r}$	anisotropic constant	[NON-DIM]
$\nu_{\theta r}$	Poisson's ratio which characterises compression in r-direction for tension in θ-direction	[NON-DIM]
$\sigma_r, \sigma_\theta, \tau_{r\theta}$	radial, tangential and shear stresses	[KN/ m^2]