MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

1 # STUDIES ON THE PLATE-BENDING ELEMENT OF MORLEY

2

3 HAMED ABDALLA*, KAMAL HASSAN**

4 ## ABSTRACT

A quantitative study of shape effect on the constant moment plate-bending element of Morley is presented. The so-called eccentricity coefficients were used to express in a very simple form the shape functions and the stiffness matrix of this triangular finite element. Eigenvalues of different shapes of triangles were calculated and the result of this study reaffirms the influence of the smallest triangle angle on the numerical deterioration of the element's properties.

---

5 * Lt.Col.Dr.,Chair of Rockets,Military Technical College, Cairo,Egypt, ** Dr.Eng.,Investigator,Ministry of Higher Education,Cairo,Egypt.

FIRST A.S.A.T. CONFERENCE

14-16 May 1985 , CAIRO

INTRODUCTION

The constant-moment plate-bending element of Morley [1] is the 'simplist' finite element for the solution of thin plates [2]. Its fields of application are the areas where two dimensional biharmonic equations appear [3], so it can be used for the solution of shell problems [4]. The convergence properties of this element have already been proven, both theoretically [5],[6] and numerically [1]. This paper presents a quantitative study of the stiffness matrix of this element by showing the influence of the shape of the triangle on its spectral properties. For this purpose, the shape functions of the element are formulated using the so-called eccenticity coefficients [7],[8],[9]. These coefficients express the deviation of the shape of a triangle from an equilateral one. They simplify the expressions of the stiffness matrix and facilitate the study of shape influence.

MORLEY'S ELEMENT

Given a triangle T (Figure 1) with vertices A_i whose coordinates (x_i, y_i), $1 \leq i \leq 3$. Let L_i = area coordinates relative to its vertices, where

$$L_1 = (a_1 + b_1 x + c_1 y)/\ 2A$$

$$a_1 = x_2 y_3 - x_3 y_2$$

$$b_1 = y_2 - y_3$$

$$c_1 = x_3 - x_2$$

with the remaining expressions following by cyclic permutation of the subscripts. The area of the triangle is denoted by A where

$$\begin{aligned} 2A &= c_3 b_2 - c_2 b_3 \\ &= c_1 b_3 - c_3 b_1 \\ &= c_2 b_1 - c_1 b_2 \end{aligned}$$

Let l_i = the length of the edge opposite to the vertex A_i, where

$$l_1^{\ 2} = b_1^{\ 2} + c_1^{\ 2}$$

and e_i = the eccentricity coefficients, where

$$e_1 = (l_3^2 - l_2^2)/ l_1^2$$

and as before cyclic permutation is used for the remaining expressions.

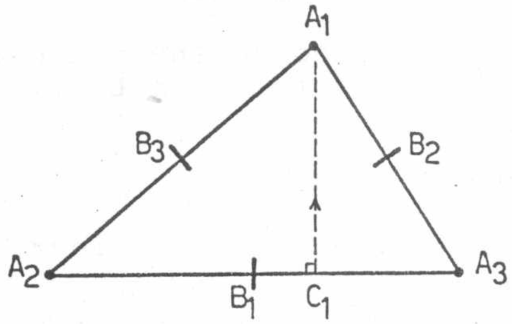

Figure 1. Morley's element

The degrees of freedom of this finite element [5] are the values of the function at the vertices A_i of the triangle: $w(A_i)$, and the values of its derivatives at the mid-points B_i of the edges of the triangle in the direction of normals $\overrightarrow{C_iA_i}$: $Dw(B_i).\overrightarrow{C_iA_i}$. So the expression for the deflection w is written as follows:

$$w = w(A_1)N_1 + w(A_2)N_2 + w(A_3)N_3 + (Dw(B_1).\overrightarrow{C_1A_1})N_4 + (Dw(B_2).\overrightarrow{C_2A_2})N_5 + (Dw(B_3).\overrightarrow{C_3A_3})N_6$$

where the quadratic shape functions $N_1,\ldots,N_6$ are given by:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & (1+e_2)/2 & (2+e_2-e_3)/2 & (1-e_3)/2 \\ 0 & 1 & 0 & (1-e_1)/2 & (1+e_3)/2 & (2+e_3-e_1)/2 \\ 0 & 0 & 1 & (2+e_1-e_2)/2 & (1-e_2)/2 & (1+e_1)/2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_1L_2 \\ L_2L_3 \\ L_3L_1 \end{bmatrix}$$

where

$$Dw(B).\overrightarrow{CA} = \frac{\partial w}{\partial x}(B)\ XCA + \frac{\partial w}{\partial y}(B)\ YCA$$

and XCA is the x-component of CA while YCA is its y-component.

ELEMENT STIFFNESS MATRIX

Using Morley's element to solve the plate-bending problem, the elementary stiffness matrix can be written as follows [1] :

$$\mathbf{K} = \iint \mathbf{C}^T \mathbf{D}\ \mathbf{C}\ dx\ dy$$

The flexural rigidity matrix **D** is defined for the isotropic plate as:

$$\mathbf{D} = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & 0 & \nu \\ 0 & 2(1-\nu) & 0 \\ \nu & 0 & 1 \end{bmatrix}$$

where E is the Young's modulus, t the thickness, and ν the Poisson's ratio. The matrix **C** is given by:

$$\mathbf{C}^T = \frac{1}{2A^2} \begin{bmatrix} b_1^2-\frac{1}{2}(1+e_2)b_2^2-\frac{1}{2}(1-e_3)b_3^2 & b_1c_1-\frac{1}{2}(1+e_2)b_2c_2-\frac{1}{2}(1-e_3)b_3c_3 & c_1^2-\frac{1}{2}(1+e_2)c_2^2-\frac{1}{2}(1-e_3)c_3^2 \\ b_2^2-\frac{1}{2}(1+e_3)b_3^2-\frac{1}{2}(1-e_1)b_1^2 & b_2c_2-\frac{1}{2}(1+e_3)b_3c_3-\frac{1}{2}(1-e_1)b_1c_1 & c_2^2-\frac{1}{2}(1+e_3)c_3^2-\frac{1}{2}(1-e_1)c_1^2 \\ b_3^2-\frac{1}{2}(1+e_1)b_1^2-\frac{1}{2}(1-e_2)b_2^2 & b_3c_3-\frac{1}{2}(1+e_1)b_1c_1-\frac{1}{2}(1-e_2)b_2c_2 & c_3^2-\frac{1}{2}(1+e_1)c_1^2-\frac{1}{2}(1-e_2)c_2^2 \\ -b_1^2 & -b_1c_1 & -c_1^2 \\ -b_2^2 & -b_2c_2 & -c_2^2 \\ -b_3^2 & -b_3c_3 & -c_3^2 \end{bmatrix}$$

Writing the symmeric stiffness matrix:

$$\mathbf{K} = \frac{Et^3}{24(1-\nu^2)A}\ S$$

6 and making the necessary calculations one obtains the following simple expressions:

$$S(1,1) = -S(1,4) + (1+e_2)S(1,5)/2 + (1-e_3)S(1,6)/2$$

$$S(1,2) = -S(1,5) + (1+e_3)S(1,6)/2 + (1-e_1)S(1,4)/2$$

$$S(1,3) = -S(1,6) + (1+e_1)S(1,4)/2 + (1-e_2)S(1,5)/2$$

$$S(1,4) = -(1-\nu)(2+e_2-e_3)$$

$$S(1,5) = (1-\nu)(1+e_3)$$

$$S(1,6) = (1-\nu)(1-e_2)$$

$$S(2,2) = -S(2,5) + (1+e_3)S(2,6)/2 + (1-e_1)S(2,4)/2$$

$$S(2,3) = -S(2,6) + (1+e_1)S(2,4)/2 + (1-e_2)S(2,5)/2$$

$$S(2,4) = (1-\nu)(1-e_3)$$

$$S(2,5) = -(1-\nu)(2+e_3-e_1)$$

$$S(2,6) = (1-\nu)(1+e_1)$$

$$S(3,3) = -S(3,6) + (1+e_1)S(3,4)/2 + (1-e_2)S(3,5)/2$$

$$S(3,4) = (1-\nu)(1+e_2)$$

$$S(3,5) = (1-\nu)(1-e_1)$$

$$S(3,6) = -(1-\nu)(2+e_1-e_2)$$

$$S(4,4) = l_1^4/2A^2 = 8(1-e_3)/(\ (1+e_1)(3-e_1+e_3+e_1e_3)\ )$$
$$= 8(1+e_2)/(\ (1-e_1)(3+e_1-e_2+e_1e_2)\ )$$

$$S(4,5) = 2\nu + (1-e_1)^2\ S(4,4)/4$$

$$S(4,6) = 2\nu + (1+e_1)^2\ S(4,4)/4$$

$$S(5,5) = l_2^4/2A^2 = 8(1-e_1)/(\ (1+e_2)(3-e_2+e_1+e_2e_1)\ )$$
$$= 8(1+e_3)/(\ (1-e_2)(3+e_2-e_3+e_2e_3)\ )$$

$$S(5,6) = 2\nu + (1-e_2)^2\ S(5,5)/4$$

$$S(6,6) = l_3^4/2A^2 = 8(1-e_2)/(\ (1+e_3)(3-e_3+e_2+e_3e_2)\ )$$
$$= 8(1+e_1)/(\ (1-e_3)(3+e_3-e_1+e_3e_1)\ )$$

Therefore, for an isotropic plate of constant thickness, one can easily see that the stifness of the element depends on its eccentricity coefficients, and is inversely proportional to its area.

EFFECTS OF ELEMENT SHAPE

To study the effect of the shape of the element, it is convenient to classify triangles into three main types: isosceles, right angled, and obtuse angled triangles (see Figure 2). For each type, the eigenvalues of the stiffness matrix were calculated as function of eccentricity coefficients.

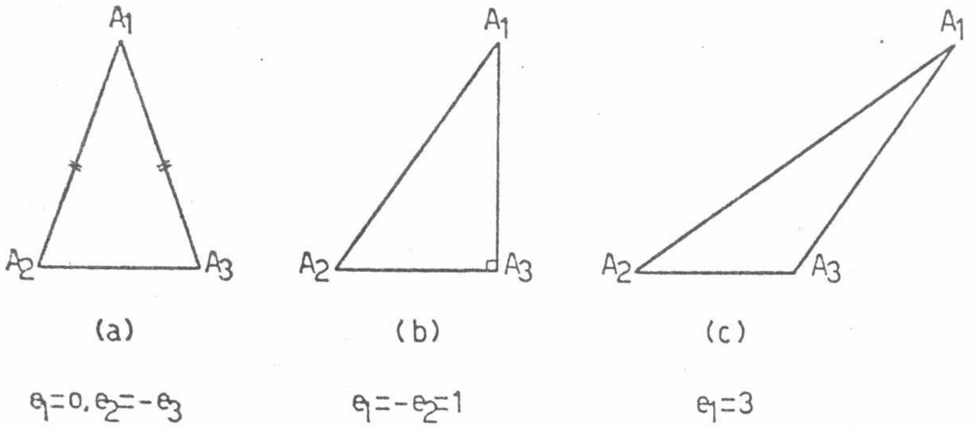

Figure 2. (a) isosceles; (b) right angled; (c) obtuse angled triangles.

It is obvious that each matrix has three zero eigenvalues corresponding to the rigid body motions. The remaining non-zero eigenvalues will be ordered as $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Figure 3 shows the behaviour of the largest eigenvalue λ_1 for different triangular shapes. One can easily see that the minimum value of λ_1 occurs for the equilateral triangle ($e_i = 0$, $i = 1,2,3$). Any deviation from this shape increases the value of λ_1. This increase of λ_1 implies an increase of the largest eigenvalue of the global assembled stiffness matrix [10]. It is clear that an increase of $|e_i|$ corresponds to a decrease of the smallest angle of the triangle. Therefore, one concludes that as the triangle 'deteriorates', i.e. its smallest angle decreases, the largest eigenvalue increases. Figure 4 shows that this 'deterioration' corresponds not only to an increase of λ_1, but also to a decrease of λ_3.

6

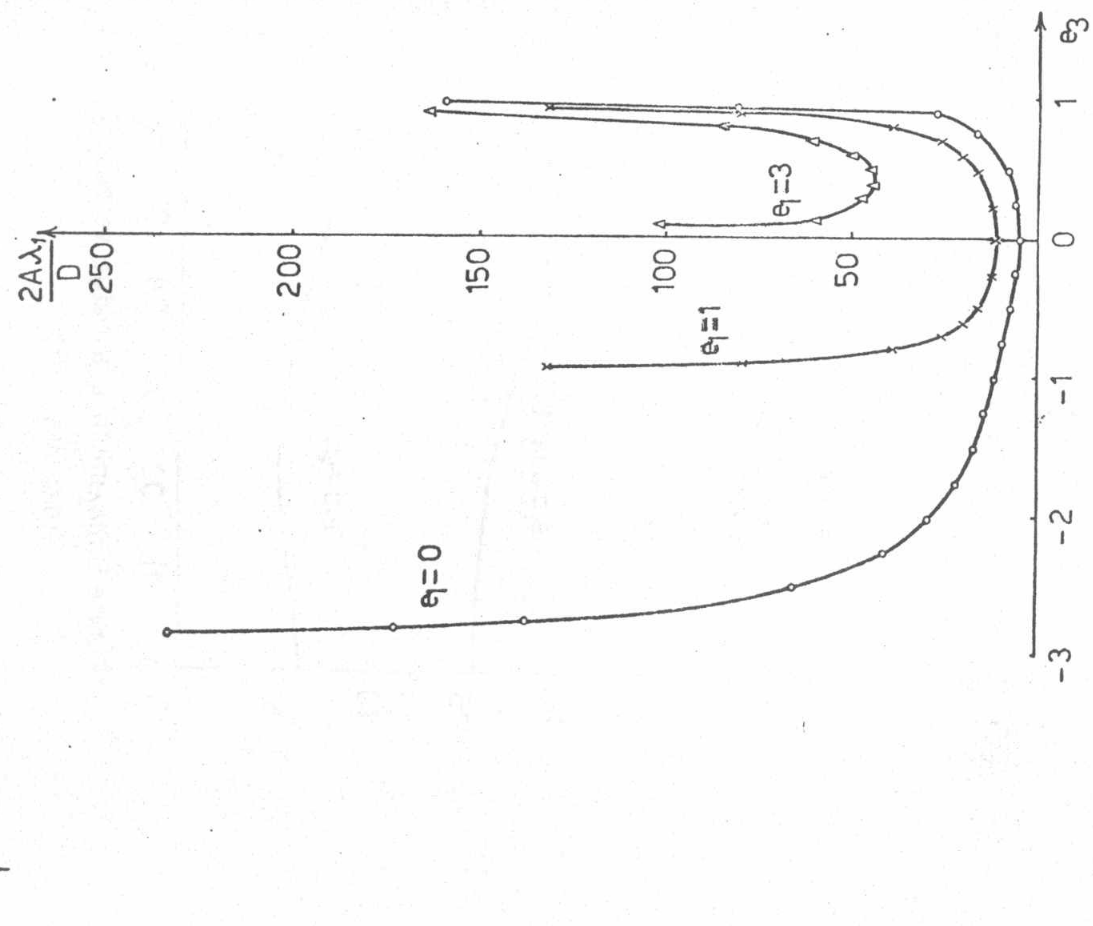

Figure 3. Maximum eigenvalue versus eccentricity coefficients ($D=Et^3/12(1-\nu^2)$)

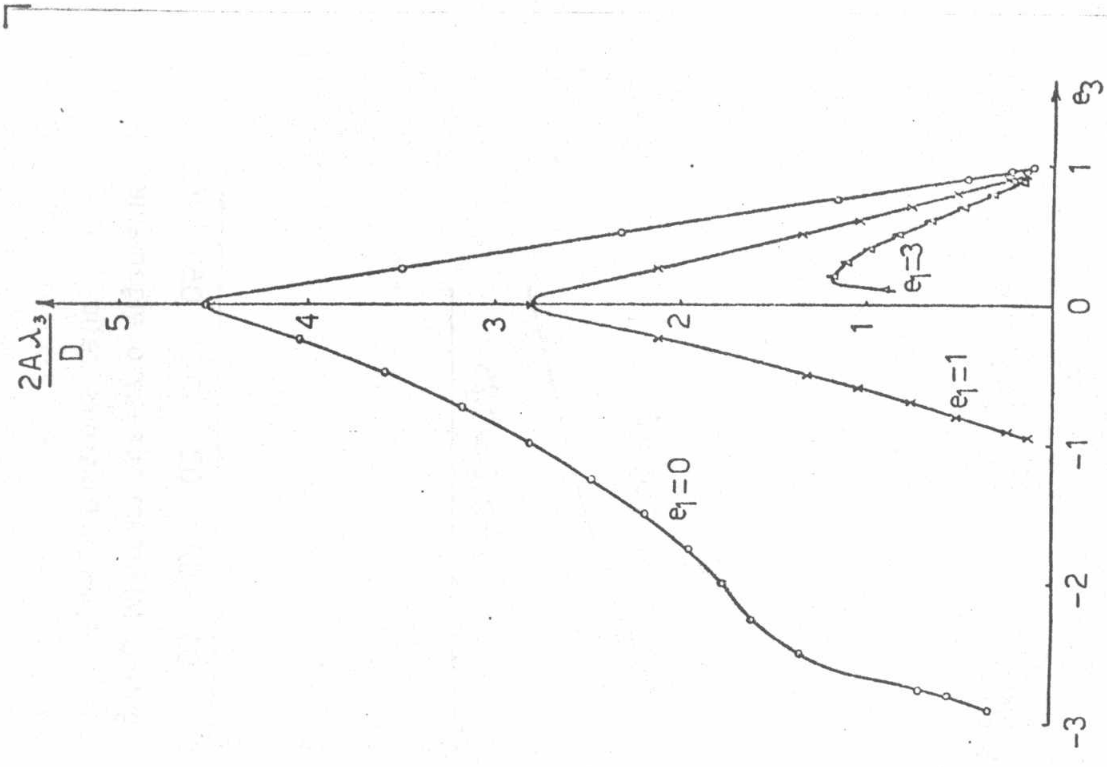

Figure 4. Minimum non-zero eigenvalue versus eccentricity coefficients ($D=Et^3/12(1-\nu^2)$)

FIRST A.S.A.T. CONFERENCE
14-16 May 1985 , CAIRO

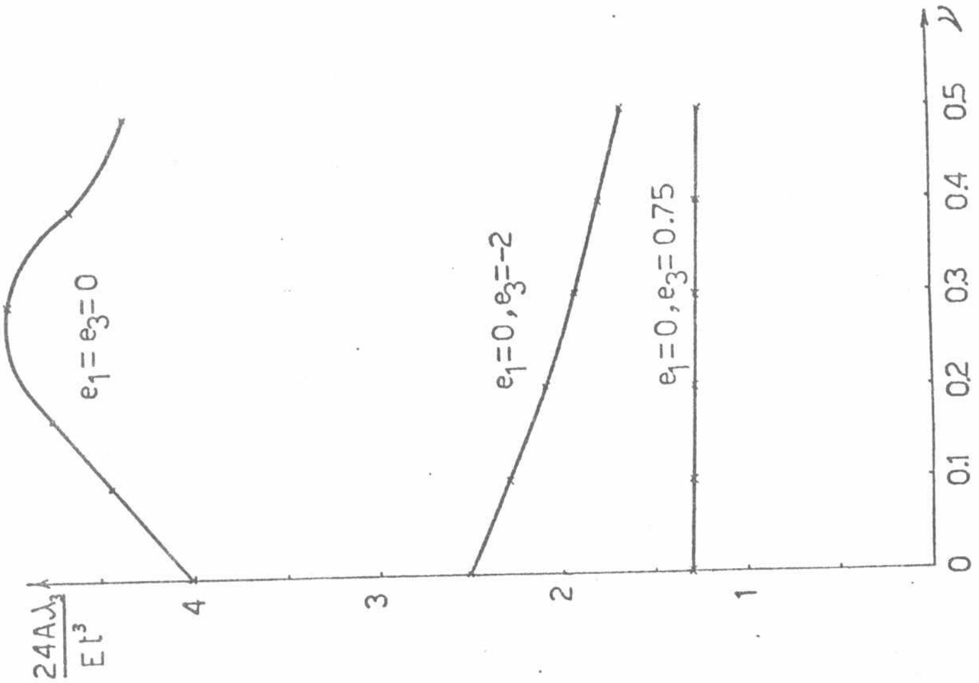

Figure 6. Minimum non-zero eigenvalue versus Poisson's ratio

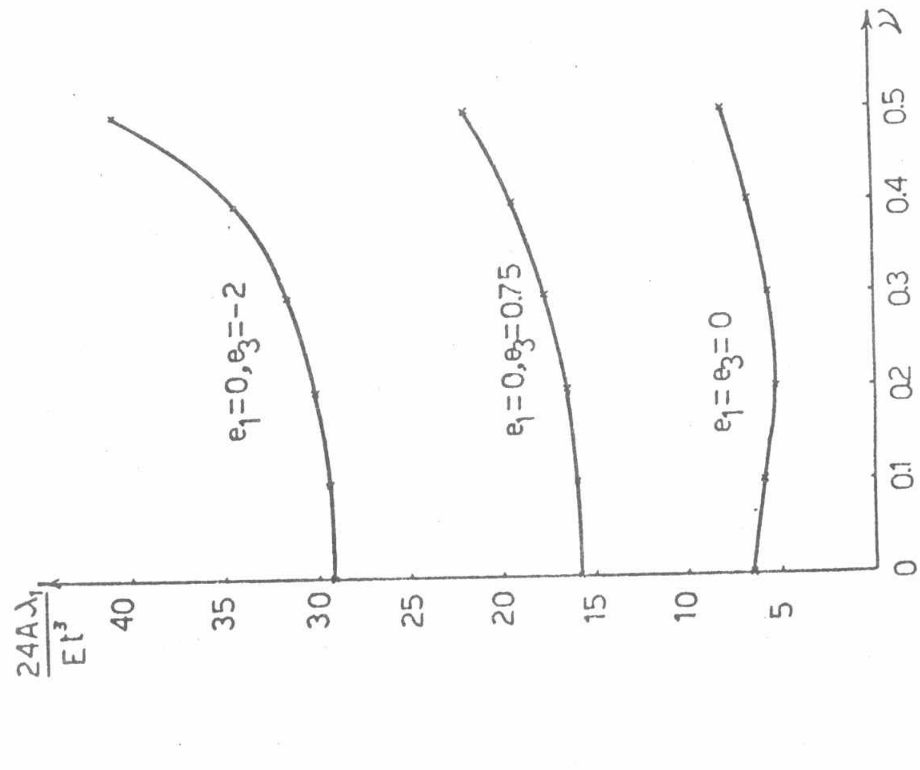

Figure 5. Maximum eigenvalue versus Poisson's ratio.

In addition, one can prove that if the area is not changed, at least one of the diagonal coefficients increases as the triangle 'deteriorates'. According to Cook [11] this implies an increase of the condition number of the global stiffness matrix.

All the previous results were made for Poisson's ratio $\nu = 0.3$. In fact, it is found that ν is of little effect as seen from Figures 5 and 6.

CONCLUSIONS

1. The stiffness of the constant-moment plate-bending element of Morley is inversely proportional to its area if the shape and thickness are not changed.
2. The smallest angle of a triangle is responsible, to a large extent, for the possible bad performance of the element. A decrease of this angle leads to an increase of the maximum eigenvalue and the condition number of the global stiffness matrix.

REFERENCES

1. Morley, L.S.D., "The Constant-Moment Plate Bending Element", J.Strain Analysis,6,20-24 (1971).
2. Zienkiewicz, O.C., "The Finite Element Method", 3rd ed., McGraw-Hill,(1977).
3. Scheffler, S. and Whiteman, J.R., "Two Dimensional Biharmonic Problems", in a Survey of Numerical Methods for Partial Differential Equations, Ed.I.Gladwell and R.Wait, Clarendon Press,Oxford (1979).
4. Thomas, G.R. and Gallagher, R.H., "A Triangular Element Based on Generalized Potential Energy Concepts", in Finite Elements for Thin Shells and Curved Members, Ed. D.G. Ashwell and R.H. Gallagher, John Wiley & Sons,Inc., London,(1976).
5. Ciarlet, P.G., "The Finite Element Method for Elliptic Problems", North-Holland, Amsterdam (1978).
6. Lascaux, P. and Lesaint, P., "Some Non-Conforming Finite Elements for the Plate Bending Problem", R.A.I.R.O.,R-1, 9-53 (1975).
7. Argyris, J.H., "Continua and Discontinua", Proc.Conf.on Matrix Methods in Structural Mechanics, Wright-Patterson A.F.B., Ohio (1965).
8. Argyris, J.H., Hasse, M. and Malejannakis, G.A., "Natural Geometry of Surfaces with Specific Reference to the Matrix Displacement Analysis of Shells. I,II, and III", Proc. Kon.Ned.Akad.Wetensch.,B,Vol.76, 361-410 (1973).

9. Bernadou, M. and Hassan, K., "Basis Functions for General Hsieh-Clough-Tocher Triangles, Complete or Reduced", Int. J.Num.Meth.Engng.,17,784-789 (1981).
10. Fried, I., "Condition of Finite Element Matrices Generated from Non-Uniform Meshes", AIAA J.,10,219-221 (1972).
11. Cook, R.D., "Concepts and Applications of Finite Element Analysis", 2nd Edn, John Wiley & Sons,(1981).