

## AXISYMMETRIC VIBRATION OF STEPPED CIRCULAR

 PLATES WITH ELASTICALLY BUILT-IN BOUNDARY CONDITIONM.G. SHEBL*

## ABSTRACT

This paper deals with the axisymmetric motion of stepped circular plates in which its boundary is considered elastically built -in. The solution of uniform plate is applied to each step of the plate. Continuity conditions at each step must be satisfied besides the boundary conditions in order to obtain the solution of such plates.

The boundary of the plate is considered to be elastically built-in in a manner that prevents transverse edge motion and provides a restoring edge moment linearly related to edge rotation. Thus limiting cases are a clamped plate and a simply supported plate.

The eigenvalues of the axisymmetric modes are obtained by using a constructed computer program written with FORTRAN-IV language. The program is applied to circular plates of different step radil. At the same time two different forms of the plate are considered.

Finally, a discussion of the results obtained is presented in order to show the effect of the fixity parameter ( proportionality factor between tine moment and slope of the plate at the boundary) on the values of the natural frequency of the plate at different step radil and different symmetric modes.

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Axisymmetric vibration, circular plates, stepped plates,elastically boundary condition, natural frequency, frequency determinant.

## INTRODUCTION

Although problems of transverse vibrations of thin plates have received considerable attention in the last century, there are relatively few exact solutions known. The general methods of solution are well developed but there is considerable labor in carrying through the details of the thickness has been studied by some authors. Such of these plates which are interest to engineers are the stepped plates. The vibration of these plates has been treated by Juarez [1].

The previous paper[2] has aimed to obtain the frequency of symmetric modes of the stepped circular plates for each of clamped and simply supported boundary conditions.

In this paper, other forms of boundary conditions are dealt with.
Consequently a definite relation between the moment and the slope of the plate is presented to be satisfied at the boundary. This relation is verified also at the two extreme cases of simply and clamped supported boundary conditions. In view of these boundary conditions, the hatural prequencies are obtained at different step radii for some axisymmetric modes.

## THEORETICAL ANALYSIS

The govering equation of the transverse motion of the circular plate with uniform thickness is expressed here with the polar coordinates system r and $\theta$. In the axisymmetric vibration, the natural modes are independent of the coordinate $\theta$, therefore, the equation of motion has the form,
where

$$
\begin{equation*}
\nabla^{4} w(r)-B^{4} w(r)=0 \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
B^{4}=12\left(1-v^{2}\right) \quad \rho \omega^{2} / E h^{2} \tag{2}
\end{equation*}
$$

The general solution of eqn. (1) has the following form which is expressed in Bessel functions

$$
\begin{equation*}
W(r)=A J_{0}(B r)+B Y_{0}(B r)+C I_{0}(B r)+F K_{0}(B r) \tag{3}
\end{equation*}
$$

where $A, B, C$ and $F$ are constants.
Fig. 1 shows the forms of such stepped circular plate. At $r=a$, the boundary is supported in a manner that prevents transverse edge motion and in addition, provides a restoring edge moment proportional to the edge rotation. The plate shown can be divided into two zones, zone I represents a plate

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with uniform thickness $h_{1}$, while zone II represents annular plato with uniform thickness $h_{2}$. The common radius $r_{1}$ between the two zones $1 s$ called the step radfus.


Fig. 1. Cross section of the two forms of stepped circular plates a-Stepped circular plate with central region raised. b-Stepped circular plate with central region unralsed.

The solution given in eqn. (3) can be written for each of zone I and II respectively as follows:

$$
\begin{align*}
& W_{1}(r)=A_{1} J_{0}\left(B_{1} r\right)+C_{1} I_{0}\left(B_{1} r\right)  \tag{4}\\
& W_{2}(r)=A_{2} J_{0}\left(B_{2} r\right)+B_{2} Y_{0}\left(B_{2} r\right)+C_{2} I_{0}\left(B_{2} r\right)+F_{2} K_{0}\left(B_{2} r\right) \tag{5}
\end{align*}
$$

It is seen in eqn. (4) that the coefficients $B_{1}$ and $F_{1}$ are zero because the Bessel functions $Y_{0}$ and $K_{0}$ are infinite at $r=0$.
The expressions of $B_{1}$ and $B_{2}$ can be given according to eqn. (2) as follows

$$
\begin{align*}
& B_{1}^{4}=12\left(1-v^{2}\right) \cdot \rho \omega^{2} / E h_{1}^{2}  \tag{6}\\
& B_{2}^{4}=12\left(1-v^{2}\right) \cdot \rho w^{2} / E h_{2}^{2} \tag{7}
\end{align*}
$$

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## CONTINUITY CONDITIONS

A continuity of some parameters must be satisfied at the step radius $r_{1}$. The following are four conditions for the continuity in each of the deflection, slope, shear force and radial bending moment:

$$
\begin{align*}
W_{1}\left(r_{1}\right) & =W_{2}\left(r_{1}\right)  \tag{8}\\
d W_{1}\left(r_{1}\right) / d r & =d W_{2}\left(r_{1}\right) / d r  \tag{9}\\
Q_{1}\left(r_{1}\right) & =Q_{2}\left(r_{1}\right)  \tag{10}\\
M_{r l}\left(r_{1}\right) & =M_{r 2}\left(r_{1}\right) \tag{11}
\end{align*}
$$

The expressions of $Q$ and $M_{r}$, [ 5 ]are given by,

$$
\begin{align*}
& Q=-D \quad d\left(d^{2} W / d r^{2}+\frac{1}{r} d W / d r\right) / d r  \tag{12}\\
& M_{r}=-D \quad\left(d^{2} W / d r^{2}+\frac{V}{r} d W / d r\right. \tag{13}
\end{align*}
$$

where $D$ is the flexural rigidity of the plate, given by

$$
\begin{equation*}
D=E h^{3} / 12\left(1-v^{2}\right) \tag{14}
\end{equation*}
$$

The equations of $W_{1}(r)$ and $W_{2}(r)$ given in eqn. (4) and (5) are written at $r=r_{1}$ as follows:

$$
\begin{align*}
& W_{1}\left(r_{1}\right)=A_{1} J_{0}\left(\beta_{1} r_{1}\right)+C_{1} I_{0}\left(\beta_{1} r_{0}\right)  \tag{15}\\
& W_{2}\left(r_{1}\right)=A_{2} J_{0}\left(\beta_{2} r_{1}\right)+B_{2} Y_{0}\left(\beta_{2} r_{1}\right)+C_{2} I_{0}\left(\beta_{2} r_{1}\right)+F_{2} K_{0}\left(B_{2} r_{1}\right) \tag{16}
\end{align*}
$$

From condition (8) we obtain :

$$
\begin{align*}
A_{1} J_{0}\left(\beta_{1} r_{1}\right)+C_{1} I_{0}\left(\beta_{1} r_{1}\right) & -A_{2} J_{0}\left(\beta_{2} r_{1}\right)-B_{2} Y_{0}\left(\beta_{2} r_{1}\right) \\
& -C_{2} I_{0}\left(\beta_{2} r_{1}\right)-F_{2} K_{0}\left(\beta_{2} r_{1}\right)=0 \tag{17}
\end{align*}
$$

According to the differentiation of the Bessel's functions, [6] , condition (9) becomes:

$$
\begin{align*}
& A_{1} \beta_{1} J_{1}\left(\beta_{1} r_{1}\right)-C_{1} \beta_{1} I_{1}\left(\beta_{1} r_{1}\right)-A_{2} \beta_{2} J_{1}\left(\beta_{2} r_{1}\right) \\
- & B_{2} \beta_{2} Y_{1}\left(\beta_{2} r_{1}\right)+C_{2} \beta_{2} I_{1}\left(\beta_{2} r_{1}\right)-F_{2} \beta_{2} K_{1}\left(\beta_{2} r_{1}\right)=0 \tag{18}
\end{align*}
$$

Also each of condition (10) and (11) will have the following forms,

$$
\begin{gathered}
A_{1} q_{1} J_{1}\left(\beta_{1} r_{1}\right)+C_{1} q_{1} I_{1}\left(\beta_{1} r_{1}\right)-A_{2} q_{2} J_{1}\left(\beta_{2} r_{1}\right) \\
-B_{2} q_{2} Y_{1}\left(\beta_{2} r_{1}\right)+C_{2} q_{2} I_{1}\left(\beta_{2} r_{1}\right)+F_{2} q_{2} K_{1}\left(\beta_{2} r_{1}\right)=0 \\
A_{1} m_{1}\left[-\beta_{1} J_{0}\left(\beta_{1} r_{1}\right)+L J_{1}\left(\beta_{1} r_{1}\right)\right]+C_{1} m_{1}\left[\beta_{1} I_{0}\left(\beta_{1} r_{1}\right)-L I_{1}\left(\beta_{1} r_{1}\right)\right] \\
-A_{2} m_{2}\left[-\beta_{2}{ }_{0}\left(\beta_{2} r_{1}\right)+L J_{1}\left(\beta_{2} r_{1}\right)\right]+B_{2} m_{2}\left[\beta_{2} Y_{0}\left(\beta_{2} r_{1}\right)-L Y_{1}\left(\beta_{2} r_{1}\right)\right] \\
-C_{2} m_{2}\left[\beta_{2} I_{0}\left(\beta_{2} r_{1}\right)+L I_{1}\left(\beta_{2} r_{1}\right)\right]-F_{2} m_{2}\left[\beta_{2} K_{0}\left(\beta_{2} r_{1}\right)+L K_{1}\left(\beta_{2} r_{1}\right)\right]=0
\end{gathered}
$$

where

$$
\begin{align*}
& q_{1}=\beta_{1}^{3} h_{1}^{3}, \quad q_{2}=\beta_{2}^{3} h_{2}^{3}  \tag{21}\\
& m_{1}=\beta_{1} h_{1}^{3}, \quad m_{2}=\beta_{2} h_{2}^{3} \\
& L=(l-v) / r_{1}
\end{align*}
$$

## BOUNDARY CONDITIONS

The boundary conditions at $r=a$ will be

$$
\begin{align*}
& \quad W_{2}(a)=0  \tag{22}\\
& -D\left[d^{2} W_{2}(a) / d r^{2}+\frac{v}{a} d W_{2}(a) / d r\right]=\delta \quad d W_{2}(a) / d r \tag{23}
\end{align*}
$$

By substituting expression $W_{2}$, eqn. of (5) in eqn. (22), we have,

$$
\begin{equation*}
A_{2} J_{0}\left(\beta_{2} a\right)+B_{2} Y_{0}\left(\beta_{2} a\right)+C_{2} I_{0}\left(\beta_{2}^{a}\right)+F_{2} K_{0}\left(\beta_{2}^{a}\right)=0 \tag{24}
\end{equation*}
$$

In condition given by eqn. (23), $\delta$ is representing the proportionality factor between the moment and slope of the plate at the boundary.

Consequently by substituting the derivatives required in the same eqn. We have:

$$
\begin{align*}
& A_{2}\left[-\phi J_{1}\left(\beta_{2} a\right)+R_{1}\right]+B_{2}\left[-\phi Y_{1}\left(\beta_{2} a\right)-R_{2}\right] \\
& C_{2}\left[\phi I_{1}\left(B_{2} a\right)+R_{3}\right]+F_{2}\left[-\phi K_{1}\left(B_{2} a\right)+R_{4}\right]=0 \tag{25}
\end{align*}
$$

Where

$$
\begin{equation*}
\phi=\delta a / D \tag{26}
\end{equation*}
$$

The dimensionless ratio $\delta a / D$ will be referred to as the edge fixity parameter.

and

$$
\left.\begin{array}{l}
R_{1}=-\beta_{2} J_{o}\left(\beta_{2}{ }^{a}\right)+p J_{1}\left(\beta_{2}{ }^{a}\right) \\
R_{2}=\beta_{2} Y_{0}\left(\beta_{2}\right)^{a}-p Y_{1}\left(\beta_{2}^{a}\right)  \tag{27}\\
\left.R_{3}=\beta_{2} I_{0}\left(\beta_{2} 2^{a}\right)-p I_{1}\left(\beta_{2}\right)^{a}\right) \\
R_{4}=\beta_{2} K_{0}\left(\beta_{2}^{a}\right)+p K_{1}\left(\beta_{2}\right)
\end{array}\right\}
$$

where

$$
\begin{equation*}
p=(1-v) / a \tag{28}
\end{equation*}
$$

The four continuity conditions represented in eqns. (17),(18),(19) and (20) besides the two boundary conditions represerted in.eqns.(24) and (25) will be arranged in a matrix form as:

$$
\begin{equation*}
[\mathrm{Y}][\mathrm{C}]=[0] \tag{29}
\end{equation*}
$$

where
$[\mathrm{Y}]$ is $6 \times 6$ square matrix
$[C]$ is the column matrix containing the constant $A_{1}, B_{1}, A_{2}, B_{2}, C_{2}$ and $F_{2}$.

In order to obtain the values of the natural frequencies of the plate, the determinant of the matrix $[\mathrm{Y}]$ must equal zero.

Special Cases in the Bending Conditions
The second boundary equation can be particularized to the usual limiting cases of a clamped and simply supported plates.

Simply supported plate: $\delta a / D \longrightarrow$ o
Clamped plate : $\delta a / D \longrightarrow \infty$

## NUMERICAL RESULTS AND DISCUSSIONS

The computation processes of the natural frequencies are applied to the two forms of the plates shown in Fig.1. In the first one, the two thicknesses $h_{1}$ and $h_{2}$ are 2 and 1 mm . respectively. For the other one, the two thicknesses ${ }^{2} h_{1}$ and $h_{2}$ are 1 and 2 mm . respectively. For the two models, the outer diameter is 300 mm at the boundary support. Six step radil $r_{1}=0,30,60,90,120$ and 150 mm are used. The step radil o and 150 mm are pointing to a circular plates with uniform thicknesses.

The material of the test models is a Pirespex with Young's modulus equal to $262 \times 10^{7} \mathrm{~N} . \mathrm{m}^{-2}$, Poisson's ratio $v=0.38$ and a density $1.237 \times 10^{3}$ $\mathrm{kg} . \mathrm{m}^{-3}$.

A FORTRAN IV Computer program was constructed to determine the natural frequencies by solving the frequency determinant. Fig. 2 shows the results


Fig. 2 Frequencies of Stepped Plates at Mode(00).


Fig. 3 Frequencies of Stepped Plates at Mode (01).
for the stepped circular plates at first symmetric mode 00 and at different values of edge fixity parameter. Similarly Fig. 3 shows the results at the second symmetric mode 01.

Table 1 shows the values of natural frequencies of the plates with uniform thicknesses for each of clamped and simply supported boundary conditions. These values are obtained by substituting in eqn. (2) with the given material data $E, \rho, v$, geometric parameters $h, a$ and with the eigenvalues of $B \mathrm{mn}$ [4]. Consequently, the results obtained can be considered excellent.

Table 1 Frequencies of uniform plates

| Mode | Thick. | Frequency $(\mathrm{Hz})$ |  |
| :---: | :---: | :---: | :---: |
| mn | $(\mathrm{mm})$ | Simply <br> support | Clamped <br> support |
| 00 | $\{$ | 1 | 16.20 |
|  | 2 | 32.40 | 65.78 |
|  |  | 1 | 95.71 |

## CONCLUSIONS

The vibration analysis of stepped circular plates is performed to shows the effect of variation of step radius and fixity parameter on the frequency results. In the case of central region raised, the frequency increased as the step radius increases, while in the case of central region unraised, the frequency decreases as the step radius increases. Also it can be seen that, at each step radius the frequency increases as the 1 ix ity parameter increases.

## REFERENCES

1. Juarze, J.A.," Axisymmetric Vibrations of Circular Plates with Stepped Thickness", Journal of Sound and Vibration, Vol.26,411-416 (1973).
2. Shebl , M.G." Axisymmetric Vibration of Stepped Circular Plates", 2 nd Alexandria Univ. PEDAC Conf., Alexandria, Egypt, (1983).
3. Leissa, A.W.," Vibration of Plates ", NASA, Sp-(1969).
4. Shebl, M.G." Stresses and Deflection in Circular Plates due to Dynamic Loading", Ph.D; thesis. Helwan University, Egypt (1982).
5. Timoshenko, S.," Theory of Plates and Shells",Mc Graw-Hill, 2 nd edition (1959).
6. Luck, " Integrals of Bessel Functions ",(1962).

NOMENCLATURE
a
E Modulus of elasticity.
$h \quad$ Thickness of plate
$I_{o}, I_{1}$
$J_{0}, J_{1}$
$K_{o}, K_{1}$
M
m
n
Q
r
w
$Y_{0}, Y_{1}$
B
$\rho$
$\nu$
$\theta$
$\omega$
Radial bending moment.
Number of nodal diameters:
Number of nodal circles.
Transverse shearing force.
Deflection of the plate.

Density of the plate.
Poisson's ratio.
Angular displacement. Natural frequency

Outer radius of the circular plate.

Modified Bessel functions of the first kind of order 0 .and 1.
Bessel functions of the first kind of order 0 and 1.
Modified Bessel functions of the second kind of order 0 and 1.

Radial distance from the centre of the circular plate.
Bessel functions of the second kind of order 0,1 .
Eigenvalue of the frequency equation.


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