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NUMERICAL SOLUTION OF INWARD CYLINDERICAL AND SPHERICAL PHASE CHANGE PROBLEM

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ABSTRACT

The characteristic feature of the phase change problems is the coupling of the temperature field with the rate of propagation of the phase boundary between the solid and the liquid phases. With regarding to the nonlinear nature of these problems, the solutions are expected to be obtained by analytical approximations and numerical methods. An analytical approximate method for one dimensional phase change utilizing a new series for the temperature field in terms of the interface position is presented. Simple and accurate formulas were obtained and solved numerically by a computer program for a wide range of parameters which cover all ranges of practical systems. The resulting diagrams of the melting or solidification times for the cylinderical and spherical bodies initially at the fusion temperature are given. The boundary conditions of the first, second and third kind are considered. Evaluation of error and comparison with other approximations are presented.

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INTRODUCTION

The phase change problems are of great practical importance. They are encountered in applications such as casting thermoplastices and metals, freezing of foods, freezing of soil, thermal storage devices for space vehicles etc. The characteristic feature of such problems is the coupling of the temperature field with the rate of propagation of the phase boundary between solid and liquid phases. With regard to the nonlinear nature of these problems, solutions are expected to be obtained by analytical approximations and numerical methods. Among analytical methods the approach of Brovman and Surin [1] appeared to be superior in simplicity and accuracy to any approximations known as yet. This method was however used for plane cases. An approximate relation in Lin [4] to obtain the solution for cylinderical and spherical problems from the corresponding plate cases was applied in the previous works [2-3] . This method does not yield so accurate results as for plane problems. In the present work the approach [1] is extended to cylinderical and spherical bodies. The example which is used to demonstrate the method refers to the same kind of the problem as in Stefan [3] and Tao [5] .

PROBLEM FORMULATION

The inward phase change in the plane, cylinder and sphere regions is considered by introducing the following assumptions:

a-All properties of the material are constant.

b-Initially $t'=0$ the old phase, occupying the entire space $0 \leqslant r' \leqslant R$ is at fusion temperature Ts'.

The energy equation for the new phase is then;

$$\frac{\partial T'}{\partial t'} = a \, r'^{-l} \frac{\partial}{\partial r'} \left(r'^{l} \frac{\partial T'}{\partial r'} \right) \qquad y' \leqslant r' \leqslant R \qquad (1)$$

With $l=0,1,2$ for plane, cylinder and sphere respectively.

At the moving boundary $r' = y'(t')$ the conditions are;

$$T'(y', t') = Ts' \qquad (2)$$

$$r' = y'(t')$$

$$\lambda \frac{\partial T'(y', t')}{\partial r'} = h \, \rho \, \frac{dy'}{dt'} \qquad (3)$$

6 Where a, λ and ρ are the thermal diffusivity, the thermal conductivity and density respectively, and h is the latent heat of fusion. The initial condition has the form;

$$y'(0) = R \quad , \quad t' = 0 \tag{4}$$

Introducing the following dimensionless variables and parameters;

$$t = (Ja)\frac{at'}{R^2} \quad , \quad r = \frac{r'}{R} \quad , \quad y = \frac{y'}{R}$$

$$T = \frac{T' - Ts'}{To' - Ts'} \quad , \quad (Ja) = \frac{c\,(To' - Ts')}{h}$$

The above equations can be transformed to the dimensionless forms as follows;

$$(Ja)\frac{\partial T}{\partial t} = r^{-1}\frac{\partial}{\partial r}\left(r^{1}\frac{\partial T}{\partial r}\right) \qquad y \leqslant r \leqslant 1 \tag{5}$$

$$T(y\,,\,t) = 0 \qquad r = y \tag{6}$$

$$\frac{\partial T(y\,,\,t)}{\partial r} = -\dot{y} \tag{7}$$

$$y(0) = 1 \qquad t = 0 \tag{8}$$

External boundary conditions of the first, second and third kind, treated in common, are;

$$\left.\frac{\partial T}{\partial r}\right|_{r=1} = Q - B\,T_{(1,t)} \qquad r = 1 \tag{9}$$

Where;

$$Q = (q) = \frac{q'\,R}{\lambda(To' - Ts')}$$ and $B = 0$ for the condition of the second kind and $Q = B = (Bi) = \alpha R / \lambda$ for the condition of the third kind. The symbols are as, q' for the heat flux, T' for the surrounding temperature and α for the heat transfer coefficient.

The asymptotic case $(Bi) \longrightarrow \infty$ corresponds to the constant wall temperature.

An interesting physical approximation, which is frequently used consists of assuming $(Ja) = 0$. Then the heat flow is a steady type as indicated by Carslow and Jaeger [6] .

TEMPERATURE FIELD

In the spirit of Hill [7] Eqs. (5-7) can be regarded as a specific example of the non-characteristic Cauchy problem for the heat Eq. (5) in which the Cauchy data Eq. (6) and Eq. (7) are asigned along the non-characteristic curve $r = y(t)$. The final formula from which the function $T(r, t)$ for $l = 0, 1, 2$ will be determined as;

$$T(r,t) = \sum_{j=0}^{\infty} \frac{\partial^j}{\partial t^j} \left[\frac{\partial T}{\partial r}\bigg|_{r=y(t)} S_j(x,r)\bigg|_{x=y(t)} \right] \qquad (10)$$

Where S_j are the coefficients of expansion of the fundamental solution for Eq. (5) in the Laurent series. The coefficients S_j, are derived explicitly by Hill [7] for $l=0$, in conjunction with Eq. (10) yield;

For plane $l = 0$;

$$T(r,t) = \sum_{j=1}^{\infty} \frac{(Ja)^{j-1}}{(2j)!} \frac{\partial^j}{\partial t^j} (r - y)^{2j} \qquad (11)$$

But the coefficients S_j for $l = 1, 2$ can be derived and in conjunction with Eq. (10) it can be written as follows;

For cylinder $l = 1$;

$$T(r,t) = \sum_{j=0}^{\infty} \frac{(Ja)^j}{(j!)^2} \left(\frac{r}{2}\right)^{2(j+1)} \frac{\partial^j}{\partial t^j} \left\{ \dot{w} \left[\ln w \sum_{k=0}^{j} (j_k)^2 \; w^{j-k} + 2 R_j(w) \right] \right\} \qquad (12)$$

Where;

$$w = \left(\frac{y(t)}{r}\right)^2 , \quad \dot{w} = \frac{dw}{dt} = \frac{2}{r} y \dot{y} \quad \text{and}$$

$$R_0 = 0 , \quad R_1 = 1 - w , \quad R_2 = \frac{3}{2}(1 - w^2) ,$$

$$R_3 = \frac{11}{6}(1 - w^3) + \frac{9}{2} w (1 - w) \quad \text{etc.}$$

For sphere $l = 2$;

$$T(r,y) = -\frac{1}{r} \sum_{j=1}^{\infty} \frac{(Ja)^j}{(2j+1)!} y \dot{y} (r - y)^{2j+1} \qquad (13)$$

The solution of Eq. (2) was given by Hill [7], but the results Eq. (12) and Eq. (13) are new.

6 The series Eq. (11) and Eq. (13) can be written in the more useful form as a series expansion in space about the moving face $r = y(t)$ as follows;

For plane $l = 0$;

$$T(r,t) = \sum_{k=1}^{\infty} f_k (r - y)^k \tag{14}$$

Where:

$$f_k = \frac{(-1)^k}{k!} \sum_{j=\text{entire}(\frac{k}{2}+1)}^{j=k} (Ja)^{j-1}\, j! \sum_{\substack{\beta_1+\dots+\beta_j=2j-k \\ 1\beta_1+\dots+j\beta_j=j}} \prod_{i=1}^{\infty} \frac{(y^{(i)})^{\beta_i}}{\beta_i!\,(i!)^{\beta_i}} \tag{15}$$

For sphere $l = 2$;

$$T(r,t) = \frac{1}{r} \sum_{k=1}^{\infty} u_k (r - y)^k \tag{16}$$

Where:

$$u_k = \frac{(-1)^k}{k!} \sum_{j=\text{entire}(\frac{k}{2})} (Ja)^j\, j! \sum_{\substack{\beta_1+\dots+\beta_j=2j+1-k \\ 1\beta_1+\dots+j\beta_j+\gamma_1+\gamma_2=j}} \frac{y^{(\gamma_1)}\, y^{(1+\gamma_2)}}{\gamma_1!\,\gamma_2!} \prod_{i=1}^{\infty} \frac{(y^{(i)})^{\beta_i}}{\beta_i!\,(i!)^{\beta_i}} \tag{17}$$

Where, symbols:

$$\sum_{\substack{\beta_1+\dots+\beta_j=2j-k \\ 1\beta_1+\dots+j\beta_j=j}} \quad \text{and} \quad \sum_{\substack{\beta_1+\dots+\beta_j=2j+1-k \\ 1\beta_1+\dots+j\beta_j+\gamma_1+\gamma_2=j}}$$

denote to the summation over all sequences of nonnegative integers $\beta_1, \beta_2, \dots, \beta_j$ and $\beta_1, \beta_2, \dots, \beta_j, \gamma_1, \gamma_2$ respectively, which satisfy the system of Diofantic equations given below the summation sign.

It should be noted that the functions f_k and u_k are expressed in the close form. It differs from the previous solutions [2 - 4] obtained only for plane $l = 0$ where the functions f_k are to be found by a recursive method.

In the series Eq. (11) , Eq. (13) and Eq. (14) , Eq. (16) the

interface position $y(t)$ is unknown function of time. These expressions can be applicable to the solution of the two category of phase change problems in which either $r = y(t)$ is assumed to be known a primarily or as a alternative where a condition at a fixed boundary $r = 1$ is prescribed. In the first case is a simpler one, the above formulas are directly provide, whenever this make sense, explicit solution fot $T(r,t)$. It means, they can be considered as the general solutions of the inverse Stefan problem [3], for $l = 0, 1, 2$. The present work considered the later case in which $y(t)$ is to be determined as a part of the solution from the boundary condition Eq. (9) at $r = 1$.

APPROXIMATE ANALYTICAL AND NUMERICAL SOLUTION

Substitution of Eq. (14), Eq. (12) and Eq. (16) into Eq. (9) yields a nonlinear ordinary differential equations of infinite order, which determine the function $y(t)$, for $l = 0, 1, 2$ respectively. It is not easy to obtain an exact analytical solution for equations of this type. Therefore, resort must be had to an approximate analytical procedure.

Here, having Eq. (12) and Eq. (16), we can extend the truncation method of Brovman [1] for the cylinderical and spherical symmetry. Let us assume that the temperature field in the cylinder and sphere can be approximated by the first two terms of the series Eq. (12)(without the term containing $\ddot{w}$) and Eq. (16). We thus obtain:

For cylinder $l = 1$;

$$\tilde{T} = \left(\frac{r}{2}\right)^2 \dot{w} \ln w + (Ja)\left(\frac{r}{2}\right)^4 \dot{w}^2 \left(\ln w - 1 + \frac{1}{w}\right) \qquad (18)$$

For sphere $l = 2$;

$$\tilde{T} = \frac{1}{r}\left[- y \dot{y} (r - y) + \frac{1}{2} y \dot{y}^2 (r - y)^2\right] \qquad (19)$$

These expressions directly satisfy the boundary conditions Eq. (6) and Eq. (7), but the energy Eq. (5) is only satisfied at the phase interface $r = y(t)$. This should be quite relevant, particularly for the derivation of $\frac{dy}{dt}$, which strongly depends on the internal energy stored near the interface and not

so much on the energy stored in more distant volume elements. Substitution of Eq. (18) and Eq. (19) into Eq. (9) yields an equation of the following type:

$$\frac{dz}{dt} = F(z, Q, B) \tag{20}$$

for $l = 1, 2$, with a new variable $z = 1 - y$. Direct integration of this relation with the initial condition $z = 0$ for $t = 0$ yields the final results:

For cylinder $l = 1$;

- the second boundary condition

$$\frac{1}{(Ja)} t = \frac{1}{2W}\left\{\frac{1}{2} z(2-z) + \frac{1}{2}\left[1-(1-z)\sqrt{(1-z)^2 - 2Wz(1-z)} + \frac{N}{W}\right]\right\} \tag{21}$$

Where:

$$N = \frac{1}{\sqrt{1-2W}} \ln\left[\frac{\sqrt{(1-z)^2 + 2Wz(2-z)} + (1-z)\sqrt{1-2W}}{1-\sqrt{1-2W}}\right] \quad \text{for } W < \frac{1}{2} \tag{22}$$

$$N = \frac{1}{2W-1}\left[\arcsin\sqrt{1-\frac{1}{2W}} - \arcsin(1-z)\sqrt{1-\frac{1}{2W}}\right] \quad \text{for } W \geqslant \frac{1}{2} \tag{23}$$

and $W = (Ja)(q)$

- the third boundary condition

$$t = \frac{1}{2(Bi)}\left\{\frac{1}{4}\left[(2+(Bi))z(2-z) + 2(Bi)(1-z^2)\ln(1-z)\right] + \int_0^z \sqrt{X_1[x,(Bi),(Ja)]}\,dx\right\} \tag{24}$$

With:

$$X_1 = (1-x)^2\left[1-(Bi)\ln(1-x)\right]^2 + (Ja)(Bi)\left[(2+(Bi))(2-x)x + 2(Bi)(1-x)^2\ln(1-x)\right]$$

The integral $\int_0^z \sqrt{X_1}\,dx$ can be determined by the numerical integration using a computer program.

For sphere $l = 2$:

$$t = \frac{1}{2Q}\left\{z\left[1-z+\frac{1}{3}z^2 + Bz\left(\frac{1}{2}-\frac{1}{3}z\right)\right] + \int_0^z \sqrt{X_2[x,B,Q,(Ja)]}\,dx\right\} \tag{25}$$

With:

$$X_2=(1-x)^2\left[(1-x)+Bx\right]^2+2(Ja)Qx(1-x)\left[(2-x)+Bx\right]$$

The integral $\int_0^z \sqrt{X_2}\,dx$ for the constant wall temperature $(B = Q = (Bi) \to \infty)$ is an elementary type. For other cases it can be expressed in terms of standard elliptic functions.

It is seen that the approximate solutions Eq. (12) and Eq. (13) are convergent to the quasistationary solution if $(Ja) = 0$. Putting $z = 0$ into Eq. (12) and Eq. (13) we obtain the expressions for calculating the time t^F required to complete the phase transformation in the sphere and the cylinder.

RESULTS AND DISCUSSION

Figs. (1 - 2) show the time required to complete the phase change in the cylinder and sphere, calculated from Eq. (24) and Eq. (25), for the boundary condition of the third kind. The obtained results cover all ranges of practical systems with (Ja) up to 6 and (Bi) from 0.2 to ∞. It is clear that for $(Ja) \geqslant 3$ the required time t^F can be twice greater than that based on neglecting the heat capacity of the new phase ($(Ja) = 0$). Therefore, the quasistationary approach cannot be applied to such materials as steel, nickel and other metals. The very accurate difference solution of Tao [5] can be used to estimate the relative errors of the discussed solutions, for values of (Ja) and $1/(Bi)$ up to 3 and 5, respectively. Percent relative errors of the time required to complete the phase change in the sphere ϵ % are shown in Fig. (3). Where $\epsilon = [(t^F - t_o^F) / t_o^F]\,100\%$. The error of t^F for the results of Stefan [3] represented by a continuous lines in Fig. (3). The agreement with Tao [5] is sufficient, maximum error is about 6%. It can be observed in Fig. (3) that for examined values of (Ja) and (Bi), the errors of t^F, calculated from Eq. (25) are about fourfold smaller than those of Stefan [3]. Maximum deviation of Stefan [3] is about 32%. In the case of the cylinder, results of the comparison are similar to that of the sphere. It is clear that the present analytical approximation and numerical computations yields a satisfactory results.

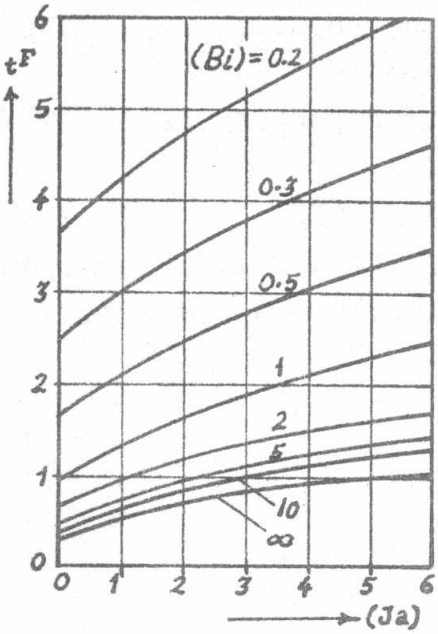

Fig.1. Time required to complete the phase change in the cylinder.

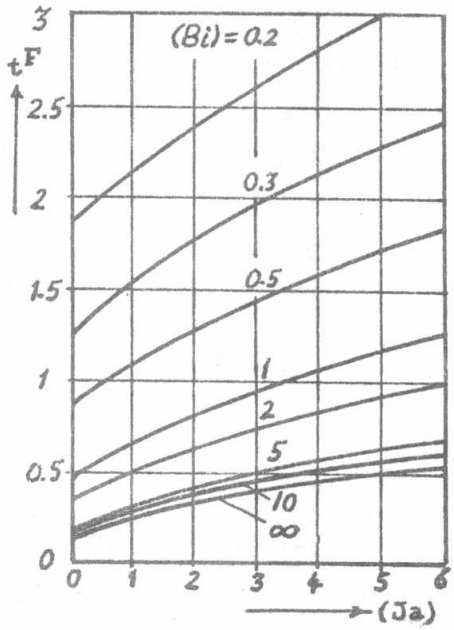

Fig.2. Time required to complete the phase change in the sphere.

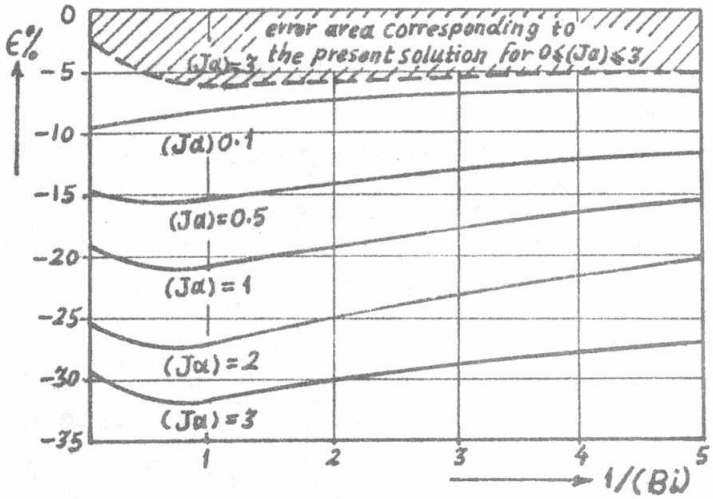

Fig.3. Percent relative error of the time required to complete the phase change in the sphere.

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NOMENCLATURE

a - thermal diffusivity.
(Bi) - Biot number.
B , Q - constant connected with Bi and q .
c - specific heat.
h - latent heat of fusion.
(Ja) - Jacob number.
l - configuration parameter.
q´- heat flux.
(q) - heat flux parameter.
r´, r - position coordinate.
R - radius, half-width of the plane.
t´, t - time.
T´, T - temperature.
t^F - time required to complete the phase change.
Ts´- fusion temperature.
To´- surrounding temperature.
y´, y - coordinate of the phase boundary.
α - coefficient of heat transfer.
λ - thermal conductivity.
ρ - density.

Note: Primed symbols as T´, To´, Ts´, r´, y´ denote to dimensional quantities and non-primed symbols denote to dimensionless quantities.