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NEW CONDITIONS FOR OUTPUT FEEDBACK DECOUPLING

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ABSTRACT

New necessary and sufficient conditions for decoupling of multivariable systems by constant precompensation and output feedback are derived in both the frequency and time domains. The conditions, given in terms of the open loop parameters of the system, are simple and do not require prior knowledge of a state feedback decoupling compensator. Together with these conditions a complete characterization of the decoupling compensators is given.

1. INTRODUCTION

The problem of decoupling a multivariable system by means of constant precompensation and constant output feedback has been considered by several authors, both in state space and in the frequency domain. In state space the problem was first considered by Falb and Wolovich [1] and by Gilbert [2]. Their approach involved first decoupling the system by state feedback and then obtaining a correspondence between state and output feedback. A similar approach was later used by Hazlerigg and Sinha [3] and by Sinha [4]. A more direct approach was employed by Howze [5] but his derivation and the resulting expression for the feedback compensator are rather involved.

In the frequency domain, Wang and Davison [6] presented a decoupling algorithm which utilizes a direct expression of the transfer function while Wolovich [7] and Bayoumi and Duffield [8] used an inverse formulation of the transfer function matrix. Using the inverse formulation also, Argoun and Van de Vegte [9] derived necessary and sufficient conditions for output feedback decoupling in terms of the open loop parameters of the system.

In this paper a simple solution to the output feedback decoupling problem in both state space and frequency domains is presented. Using the inverse formulation, new conditions are derived for output feedback decoupling in the frequency domain. Those conditions are then used to derive equivalent conditions in state space for controllable-observable systems with no integrators. The new conditions are in terms of the open loop parameters of the system $[\underline{A},\underline{B},\underline{C}]$, and do not involve the inversion

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of the system transfer function. For systems which satisfy the decoupling conditions a complete characterization of the decoupling compensators is also given. Several numerical examples are solved to illustrate the features of the approach.

2. DEFINITIONS AND PROBLEM FORMULATION

Consider the linear multivariable time-invariant dynamic system described by

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \qquad , \underline{y} = \underline{C}\,\underline{x} \tag{1}$$

where $\underline{x} \epsilon R^n$, $\underline{u} \epsilon R^m$, $\underline{y} \epsilon R^m$ and $\underline{A}$, $\underline{B}$, and $\underline{C}$ are real constant matrices of appropriate dimensions. It is required to find the conditions under which this system can be decoupled by the control law

$$\underline{u} = \underline{G}\,\underline{v} + \underline{K}\,\underline{y} \tag{2}$$

where $\underline{G}$ and $\underline{K}$ are constant mxm matrices and $v \epsilon R^m$ is some new input. The closed loop system is given by

$$\dot{\underline{x}} = (\underline{A} + \underline{B}\,\underline{K}\,\underline{C})\underline{x} + \underline{B}\,\underline{G}\,\underline{v} \quad , \underline{y} = \underline{C}\,\underline{x} \tag{3}$$

In the frequency domain, the open loop and closed loop systems are described by

$$\underline{y}(s) = \underline{T}(s)\underline{u}(s) \tag{4}$$

and

$$\underline{y}(s) = \underline{T}_c(s)\underline{v}(s) \tag{5}$$

respectively, where

$$\underline{T}_c(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} \tag{6}$$

$$\underline{T}_c(s) = (\underline{I} - \underline{T}(s)\underline{K})^{-1}\underline{T}\,(s)\underline{G} \tag{7}$$

3. THE INVERSE FORMULATION

The inverted closed loop transfer function matrix is given by

$$\underline{T}_c^{-1}(s) = \underline{G}^{-1}[\underline{T}^{-1}(s) - \underline{K}] \tag{8}$$

If

$$\underline{T}(s) = \underline{N}(s)/d(s) \tag{9}$$

where $d(s) = \det(s\underline{I} - \underline{A})$, then the inverted transfer function matrix $\underline{T}(s)$ can be written as:

$$\underline{T}^{-1}(s) = \underline{P}(s)/\Delta(s) \tag{10}$$

where $\underline{P}(s) = d(s)\ \mathrm{adj}\ \underline{N}(s)$ and $\Delta(s) = \det\ \underline{N}(s)$.

Let

$$\underline{P}(s) = \underline{P}_p s^p + \underline{P}_{p-1} s^{p-1} + \ . \ . \ . + \underline{P}_1 s + \underline{P}_0 \tag{11}$$

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and $$\Delta(s) = \Delta_d s^d + \Delta_{d-1} s^{d-1} + \ldots + \Delta_1 s + \Delta_0 \tag{12}$$

where $\underline{P}_p, \ldots, \underline{P}_0$ are constant mxm matrices and $\Delta_d, \ldots, \Delta_0$ are scalers. The degree p of $\underline{P}(s)$ is generally higher than the degree d of $\Delta(s)$ and therefore, the expression for $\underline{T}^{-1}(s)$ is, in general, improper in the powers of s. Using (10), (11) and (12) the expression (8) can be written as:

$$\underline{T}_c^{-1}(s) = \frac{1}{\Delta(s)}\underline{G}^{-1}[\underline{P}_p s^p + \underline{P}_{p-1} s^{p-1} + \ldots + (\underline{P}_d - \underline{K}\Delta_d)s^d$$
$$+ \ldots + (\underline{P}_1 - \underline{K}\Delta_1)s + (\underline{P}_0 - \underline{K}\,\Delta_0)] \tag{13}$$

This expression is used in the following section to develop the decoupling conditions in the frequency domain.

4. DECOUPLING IN THE FREQUENCY DOMAIN

The following theorem establishes necessary and sufficient conditions for output feedback decoupling.

Theorem 1

The system given by (1) is decouplable by output feedback compensator $\underline{K}$ and precompensator $\underline{G}$ if and only if there exists a nonsingular matrix $\underline{\hat{G}}$ such that the following conditions are all satisfied:

$$\underline{\hat{G}}\,\underline{P}_\ell = \underline{\Lambda}_\ell \qquad , \ell = p, \ldots, d+1$$
$$\underline{\hat{G}}[\underline{P}_\ell/\Delta_\ell - \underline{P}_{\ell-1}/\Delta_{\ell-1}] = \underline{\Lambda}_\ell \qquad , \ell = d, \ldots, 1 \tag{14}$$

where $\underline{\Lambda}_\ell$, $\ell = p, \ldots, 1$ are constant diagonal matrices and $\underline{P}_\ell$, Δ_ℓ, are defined by (11) and (12). In this case the output feedback compensator $\underline{K}$ can be obtained from any of the expressions

$$\underline{K} = \underline{P}_\ell/\Delta_\ell - \underline{G}\,\underline{\Lambda} \qquad , \ell = d, \ldots, 0 \tag{15}$$

where $\underline{\Lambda}$ is an arbitrary constant diagonal matrix and

$$\underline{G} = [\underline{\hat{G}}]^{-1}. \tag{16}$$

Proof

Necessity: For decoupling, $\underline{T}_c^{-1}(s)$ must be diagonal and finite, i.e., the matrix coefficients of all powers of s in (13) must be diagonal. The expression $\underline{T}_c^{-1}(s)$ is diagonal only if there exists a matrix $\underline{G}$ such that

$$\underline{\hat{G}}\,\underline{P}_\ell = \underline{\Lambda}_\ell \qquad , \ell = P, \ldots, d+1$$

and $$\underline{\hat{G}}\,[\underline{P}_\ell - \underline{K}\Delta_\ell] = \underline{\Lambda}_\ell \qquad , \ell = d, \ldots, 0 \tag{17}$$

where Λ_ℓ, $\ell = P, \ldots, 0$ are constant diagonal matrices. From the last expression in (17),

$$\underline{K} = \frac{1}{\Delta_d}(\underline{P}_d - \hat{\underline{G}}^{-1}\underline{\Lambda}_d) = \frac{1}{\Delta_{d-1}}(\underline{P}_{d-1} - \hat{\underline{G}}^{-1}\underline{\Lambda}_{d-1}) = \ldots = \frac{1}{\Delta_o}(\underline{P}_o - \hat{\underline{G}}^{-1}\underline{\Lambda}_o)$$

which results in the following equation:

$$(P_\ell/\Delta_\ell - P_{\ell-1}/\Delta_{\ell-1}) = \hat{\underline{G}}^{-1}\ (\underline{\Lambda}_\ell' - \underline{\Lambda}_{\ell-1}') = \hat{\underline{G}}^{-1}\ \underline{\Lambda}_\ell, \qquad \ell = d, \ldots, 1 \quad (18)$$

which yields the second expression in (14). A precompensator $\underline{G}$ will exist only if $\underline{G}$ is nonsingular since $\underline{G} = [\underline{G}]^{-1}$. This completes the necessity part of the proof.

<u>Sufficiency</u>: If the above conditions are satisfied then $\underline{G}$ is obtained from $\underline{G}^{-1}$ and $\underline{K}$ can be obtained from any of the last $(d + 1)$ equations in (17).

$$\underline{K} = \underline{P}_\ell/\Delta_\ell - \underline{G}\ \underline{\Lambda}_\ell \qquad , \ell = d, \ldots, 0 \qquad (19)$$

Since the matrices $\underline{\Lambda}_\ell$ are required only to be diagonal, $\underline{\Lambda}_\ell$ in (19) can be taken to be an arbitrary constant diagonal matrix $\underline{\Lambda}$, hence expression (15). This completely characterizes the output feedback compensator and completes the proof of the theorem. ■

<u>Remarks</u>

1. The decoupling matrix $\underline{G}$ is related to the decoupling matrix B* [1] through the relationship $\underline{G} = \underline{\Lambda}\ \underline{B}^*$ where $\underline{\Lambda}$ is a constant diagonal matrix. However, $\underline{G}$ was derived here directly from the decoupling conditions and therefore its meaning is more clear.

2. The coefficients $\underline{\Lambda}_p, \ldots, \underline{\Lambda}_0$ appearing in (17) determine the stability and the location of the poles of the closed loop decoupled system. The improper coefficients $\underline{\Lambda}_p, \underline{\Lambda}_{p-1}, \ldots, \underline{\Lambda}_{d+1}$ are not affected by output feedback. Hence the following theorem:

<u>Theorem 2</u>

A necessary condition for the stability of the closed loop system is that the coefficients $\underline{\Lambda}_p, \ldots, \underline{\Lambda}_{d+1}$ satisfy Routh-Hurwitz conditions. ■

3. $\underline{K}$ is determined within a free diagonal matrix $\underline{\Lambda}$. If $\underline{K}$ is chosen such that $\underline{K} = \underline{P}_\ell/\Delta_\ell$, then the corresponding $\underline{\Lambda}_\ell$ in the closed loop system will be equal to zero. On the other hand, the coefficients of $s^d, \ldots, s^o$ in the closed loop system can be arbitrarily changed by appropriately setting $\underline{\Lambda}$ in equation (19).

4. The elements of $\underline{K}$ can be chosen one row at a time, i.e.,

$$\underline{K}_j = [\underline{P}_\ell]_j/\Delta_\ell - \underline{G}_j\lambda_j \qquad (20)$$

Remarks:

1. Using the output feedback compensator $\underline{K}_I$ given by (30) yields an integrator decoupled system in the sense that each of the resulting subsystems has a pole at s = 0. As seen from (13), with this compensator the inverted closed loop transfer function contains an $s\underline{I}_{mxm}$ factor, i.e., $\underline{T}_c(s)$ contains m integrators. This system can be modified using (25), i.e., by adding the feedback compensator $\underline{K}_1 = -\underline{G}\ \underline{\Lambda}$ where $\underline{\Lambda}$ is an arbitrary diagonal matrix. $\underline{\Lambda}$ will be chosen based on single input-single output considerations.

2. The case where all the matrices

$$[\underline{C}\ \underline{R}_\ell \underline{B}] \qquad , \ell = n-1, \ldots, 1 \tag{32}$$

are singular is treated in the following theorem. Denote by $[\underline{C}\ \underline{R}_\ell \underline{B}]_j = \underline{L}_{\ell,j}$ the matrix obtained from $[\underline{C}\ \underline{R}_\ell \underline{B}]$ with the j-th row deleted. Let $\underline{\Gamma}_j$ be the matrix,

$$\underline{\Gamma}_j = \begin{bmatrix} \underline{L}_{n-1,j} \\ \underline{L}_{n-2,j} \\ \vdots \\ \underline{L}_{1,j} \end{bmatrix} \tag{33}$$

Then we have the following result.

Theorem 4

The system given by (1) is decouplable by constant precompensator $\underline{G}$ and constant output feedback $\underline{K}$ if and only if

$$\text{Rank } \underline{\Gamma}_j < m \text{ for all } j \qquad , j = 1, 2, \ldots, m \tag{34}$$

In this case, the j-th column $\underline{g}_j$ of the decoupling compensator $\underline{G}$ is obtained such that it lies in the Kernel space of the above matrix, i.e.,

$$\underline{g}_j \in \text{Ker } \underline{\Gamma}_j \tag{35}$$

and $\underline{K} = -[\underline{C}\ \underline{A}^{-1}\underline{B}]^{-1} - \underline{G}\ \underline{\Lambda}$ (36)

where $\underline{\Lambda}$ is an arbitrary diagonal matrix.

Proof:

Considering the set of equations (23) one column at a time, the system is decoupled if and only if all the elements of the j-th column, but the j-th element, are zeros, i.e.,

$$[\underline{C}\ I_n\ \underline{B}]_j\ \underline{g}_j = [0\ 0\ .\ .\ 0]^T \qquad \text{m-1 elements}$$

$$[\underline{C}\ \underline{R}_{n-2}\underline{B}]_j\ \underline{g}_j = [0\ 0\ .\ .\ 0]^T \qquad \text{m-1 elements} \qquad (37)$$

$$\vdots$$

$$[\underline{C}\ \underline{R}_o\ \underline{B}]_j\ \underline{g}_j = [0\ 0\ .\ ..\ 0]^T \qquad \text{m-1 elements}$$

This can be written as

$$\begin{bmatrix} (\underline{C}\ \underline{I}_n\ \underline{B})_j \\ (\underline{C}\ \underline{R}_{n-2}\underline{B})_j \\ \vdots \\ (\underline{C}\ \underline{R}_o\underline{B})_j \end{bmatrix}_{n(m-1)xm} \underline{g}_j = \underline{\Gamma}_j\ \underline{g}_j = \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \qquad \begin{matrix} \text{n(m-1) elements} \\ j = 1, \ldots m \end{matrix} \qquad (38)$$

There exists a set of vectors $\underline{g}_j$, $j = 1, \ldots, m$ which satisfies the above equation if and only if

$$\text{Rank } \underline{\Gamma}_j < m \qquad (39)$$

The columns $\underline{g}_j$ of the matrix $\underline{G}$ are obtained such that they satisfy (38), i.e., they lie in the Kernel space

$$\underline{g}_j \in \text{Ker } \underline{\Gamma}_j \qquad (40)$$

$\underline{K}$ is computed as before. ■

6. EXAMPLES

The following examples illustrate the results of this paper.

Example 1. Falb and Wolovich [1] and Sinha [4].

$$\underline{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{K}_I = \begin{bmatrix} -0.5 & 1.5 \\ -0.5 & 4.5 \end{bmatrix}, \quad \hat{\underline{A}} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix}, \quad \underline{R}_2 = \underline{I}$$

$$a_2 = -\text{ tr } \hat{\underline{A}} = -5 \qquad \underline{R}_1 = -5\ I_n + \hat{\underline{A}} = \begin{bmatrix} -5 & 1 & 3 \\ 0 & -3 & 6 \\ 0 & 1 & -2 \end{bmatrix}, \quad \hat{\underline{A}}\ \underline{R}_1 = 0$$

where $[.]_j$ indicates the j-th row and λ_j is an arbitrary scaler. λ_j can be used to alter the response characteristics of the j-th sub-system in much the same way as is done in single-input single-output system design.

5. DECOUPLING CONDITIONS IN STATE SPACE

Consider the system given by (1) where $\underline{A}$ is nonsingular. In state space the closed loop transfer function of the system given by (3) is

$$\underline{T}_c(s) = \underline{C}[s\underline{I} - (\underline{A} + \underline{B}\ \underline{K}\ \underline{C})]^{-1}\underline{B}\ \underline{G} \tag{21}$$

It is required to find conditions under which $\underline{T}_c(s)$ is decouplable. The following theorem gives necessary and sufficient conditions for decoupling system (1) by constant precompensation and constant output feedback and characterizes these compensators. First, let $\underline{R}_{n-1}$, $\underline{R}_{n-2}$, . . ., $\underline{R}_0$ be the coefficients of powers of s in the Souriau-Frame Faddeev expansion of $[s\underline{I} - \underline{A}^{-1}]$, [10], i.e.,

$$\begin{aligned}
\underline{R}_{n-1} &\triangleq \underline{I}_n &, \quad a_{n-1} &= -tr(\hat{\underline{A}}\ \underline{R}_{n-1}) \\
\underline{R}_{n-2} &= a_{n-1}\underline{I}_n + \hat{\underline{A}}\ \underline{R}_{n-1} &, \quad a_{n-2} &= -\tfrac{1}{2}tr(\hat{\underline{A}}\ \underline{R}_{n-2}) \\
&\vdots & &\vdots \\
\underline{R}_0 &= a_1\underline{I}_n + \hat{\underline{A}}\ \underline{R}_1 &, \quad a_0 &= \frac{-1}{n}\ tr(\hat{\underline{A}}\ \underline{R}_0)
\end{aligned} \tag{22}$$

Then we have the following theorem:

Theorem (3): The system given by

$$\dot{\underline{x}} = \underline{A}x + \underline{B}\ \underline{u} \qquad , \ \underline{y} = \underline{C}\ \underline{x}$$

is decouplable by constant precompensation $\underline{G}$ and constant output feedback $\underline{K}$ if and only if there exists a constant matrix $\underline{G}$ such that

$$[\underline{C}\ \underline{R}_\ell\ \underline{B}]\ \underline{G} = \Lambda_\ell, \qquad \ell = n-1, \ldots, 1 \tag{23}$$

where $\underline{\Lambda}_{n-1}$, . . ., $\underline{\Lambda}$ are constant diagonal matrices, $\underline{R}_{n-1}$, . . ., $\underline{R}_1$ are defined by (22) and where

$$\hat{\underline{A}} = \underline{A} - \underline{B}[\underline{C}\ \underline{A}^{-1}\underline{B}]^{-1}\underline{C} \tag{24}$$

If these conditions are satisfied then the decoupling compensators are given by $\underline{G}$ and

$$\underline{K} = -[\underline{C}\ \underline{A}^{-1}\underline{B}]^{-1} - \underline{G}\ \underline{\Lambda} \tag{25}$$

where $\underline{\Lambda}$ is an arbitrary diagonal matrix.

Proof:

Necessity: The closed loop transfer function of system (3) is

$$\underline{T}_c(s) = \underline{C}[s\underline{I} - (\underline{A} + \underline{B}\ \underline{K}\ \underline{C})]^{-1}\underline{B}\ \underline{G} \tag{26}$$

Let $$\hat{\underline{A}} = \underline{A} + \underline{B}\ \underline{K}\ \underline{C} \tag{27}$$

Then $\underline{T}_c(s)$ is given by:

$$\underline{T}_c(s) = \frac{1}{\Delta(s)}[\underline{C}\ \underline{B}\ s^{n-1} + \underline{C}\ \underline{R}_{n-2}\underline{B}\ s^{n-2} + \ .\ .\ .\ + \underline{C}\ \underline{R}_1\underline{B}\ s + \underline{C}\ \underline{R}_0\underline{B}]\underline{G} \tag{28}$$

where $\Delta(s) = s^n + a_{n-1}s^{n-1} + \ .\ .\ .\ + a_1 s + a_0$

and $a_{n-1}, \ .\ .\ .\ ., a_0$ and $\underline{R}_{n-2}, \ .\ .\ .\ \underline{R}_0$ are given by (22). The problem in obtaining decoupling conditions from equation (28) is that the expressions for $\underline{R}_{n-2}, \ .\ .\ ., \underline{R}_0$ contain the unknown compensator $\underline{K}$. Recalling from theorem (1) that the decoupling output feedback compensator $\underline{K}$ is given by equation (15), one particularly useful form of (15) is that which does not include $\underline{G}$, i.e., $\underline{\Lambda} = \underline{0}$ and

$$K = \underline{P}_\ell / \Delta_\ell \qquad , \ell = d, \ .\ .\ ., 0 \tag{29}$$

Moreover, if $\ell = 0$, $\underline{K}$ can be obtained explicitly in terms of the open loop parameters of the system. Thus, if $\underline{A}^{-1}$ exists

$$\underline{K} = \underline{P}_0/\Delta_0 = \underline{T}^{-1}(o) = [\underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B}]^{-1}_{s=0} = -[\underline{C}\ \underline{A}^{-1}\underline{B}]^{-1}$$

This compensator will be called K_I, i.e.,

$$\underline{K}_I = -[\underline{C}\ \underline{A}^{-1}\underline{B}]^{-1} \tag{30}$$

Substituting (30) in (27) yields the form of $\underline{A}$ given by (24). The necessary conditions for decoupling the system $[\underline{A}, \underline{B}, \underline{C}]$ by means of a constant precompensator $\underline{G}$ are those of (23). This completes the necessary part of the proof.

Sufficiency: $\underline{G}$ is obtained by inverting any nonsingular matrix of the matrices appearing in (23), i.e.,

$$\underline{G} = [\underline{C}\ \underline{R}_\ell\underline{B}]^{-1}\underline{\Lambda} \qquad , \ell = n-1, \ .\ .\ ., 1 \tag{31}$$

where $\underline{\Lambda}$ is an arbitrary diagonal matrix. If none of these matrices is nonsingular, $\underline{G}$ can still be obtained as shown by theorem 4 below. The general form of $\underline{K}$ is given by (25) which is directly obtained from (19).

■

$$a_1 = -\frac{1}{2} \text{ tr } \hat{\underline{A}} \, R_1 = 0 \qquad \underline{k}_0 = \underline{0}$$

$$\underline{C}\,\underline{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \underline{C}\,\underline{R}_1\underline{B} = \begin{bmatrix} -6 & -4 \\ -1 & 1 \end{bmatrix}, \quad \underline{C}\,\underline{R}_2\underline{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\Gamma}_1 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{rank } \underline{\Gamma}_1 = 1 \quad , \quad \underline{\Gamma}_2 = \begin{bmatrix} 1 & 1 \\ -6 & -4 \\ 0 & 0 \end{bmatrix}, \quad \text{rank } \underline{\Gamma}_2 = 2 = m$$

i.e., the system cannot be decoupled by constant $\underline{K}$ and $\underline{G}$, which is the same conclusion reached in the above references.

Example 2.

$$\underline{A} = \begin{bmatrix} -1 & 0 & 4 & -2 \\ 0 & -2 & 0 & 0 \\ 1 & -1 & -4 & 0 \\ -2 & 2 & 0 & -5 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 2 & -1 \\ -4 & 2 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

$$\underline{K}_I = -(\underline{C}\,\underline{A}^{-1}\underline{B}) = \begin{bmatrix} 2 & 3.1111 \\ 3 & 5.1111 \end{bmatrix}$$

$$\hat{\underline{A}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1.7778 & -1.1111 \\ 0 & 0 & -4.444 & -2.7778 \end{bmatrix}$$

$$\underline{C}\,\underline{B} = \begin{bmatrix} -1 & 1 \\ 8 & -4 \end{bmatrix}, \quad \underline{C}\,\underline{R}_2\underline{B} = \begin{bmatrix} -6.5556 & 6.5556 \\ 52 & -26 \end{bmatrix}$$

$$\underline{C}\,\underline{R}_1\underline{B} = \begin{bmatrix} -9.1111 & 9.1111 \\ 72 & -36 \end{bmatrix}, \quad \underline{C}\,\underline{R}_0\underline{B} = \underline{0}$$

$$\underline{\Gamma}_1 = \begin{bmatrix} 8 & -4 \\ 52 & -26 \\ 72 & -36 \end{bmatrix}, \quad \text{rank } \underline{\Gamma}_1, \; \underline{g}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\underline{\Gamma}_2 = \begin{bmatrix} -1 & 1 \\ -6.5556 & 6.5556 \\ -9.1111 & 9.1111 \end{bmatrix}, \text{ rank } \underline{\Gamma}_2 = 1, \ \underline{g}_2 = \begin{bmatrix} 1 \\ +1 \end{bmatrix}$$

i.e., the system is decouplable by $\underline{G} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

and $\underline{K} = \begin{bmatrix} 2 & 3.1111 \\ 3 & 5.1111 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

7. CONCLUSION

In this paper new necessary and sufficient conditions for decoupling by means of constant precompensation and constant output feedback in both the frequency domain and state space are presented. In both cases the conditions are in terms of the open loop parameters of the system and do not require any knowledge of the state feedback compensator. Moreover, explicit computation of the decoupling matrix B* is not required and its meaning is made clear by relating it to the decoupling precompensator derived here.

Theorem 3, the main result of this paper, has the advantage of being easy to check because of the particularly simple form of $\underline{K}_I$. In addition, complete characterization of all possible feedback compensators is given. The problem of stability of the decoupled system is also addressed. The results of this paper can be extended to the important problem of approximate decoupling by output feedback which is the subject of current work.

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where $[.]_j$ indicates the j-th row and λ_j is an arbitrary scaler. λ_j can be used to alter the response characteristics of the j-th sub-system in much the same way as is done in single-input single-output system design.

5. DECOUPLING CONDITIONS IN STATE SPACE

Consider the system given by (1) where $\underline{A}$ is nonsingular. In state space the closed loop transfer function of the system given by (3) is

$$\underline{T}_c(s) = \underline{C}[s\underline{I} - (\underline{A} + \underline{B}\ \underline{K}\ \underline{C})]^{-1}\underline{B}\ \underline{G} \tag{21}$$

It is required to find conditions under which $\underline{T}_c(s)$ is decouplable. The following theorem gives necessary and sufficient conditions for decoupling system (1) by constant precompensation and constant output feedback and characterizes these compensators. First, let $\underline{R}_{n-1}$, $\underline{R}_{n-2}$, ..., $\underline{R}_0$ be the coefficients of powers of s in the Souriau-Frame Faddeev expansion of $[s\underline{I} - \underline{A}^{-1}]$, [10], i.e.,

$$\begin{aligned}
\underline{R}_{n-1} &\overset{\Delta}{=} \underline{I}_n &&, \quad a_{n-1} = -tr(\hat{\underline{A}}\ \underline{R}_{n-1}) \\
\underline{R}_{n-2} &= a_{n-1}\underline{I}_n + \hat{\underline{A}}\ \underline{R}_{n-1} &&, \quad a_{n-2} = -\tfrac{1}{2}tr(\hat{\underline{A}}\ \underline{R}_{n-2}) \\
&\ \vdots && \qquad \vdots \\
\underline{R}_0 &= a_1\underline{I}_n + \hat{\underline{A}}\ \underline{R}_1 &&, \quad a_0 = \frac{-1}{n} tr(\hat{\underline{A}}\ \underline{R}_0)
\end{aligned} \tag{22}$$

Then we have the following theorem:

Theorem (3): The system given by

$$\dot{\underline{x}} = \underline{A}x + \underline{B}\ \underline{u} \qquad , \quad \underline{y} = \underline{C}\ \underline{x}$$

is decouplable by constant precompensation $\underline{G}$ and constant output feedback $\underline{K}$ if and only if there exists a constant matrix $\underline{G}$ such that

$$[\underline{C}\ \underline{R}_\ell\ \underline{B}]\ \underline{G} = \Lambda_\ell, \qquad \ell = n-1, ..., 1 \tag{23}$$

where Λ_{n-1}, ..., $\underline{\Lambda}$ are constant diagonal matrices, $\underline{R}_{n-1}$, ..., $\underline{R}_1$ are defined by (22) and where

$$\hat{\underline{A}} = \underline{A} - \underline{B}[\underline{C}\ \underline{A}^{-1}\underline{B}]^{-1}\underline{C} \tag{24}$$

If these conditions are satisfied then the decoupling compensators are given by $\underline{G}$ and

$$\underline{K} = -[\underline{C}\ \underline{A}^{-1}\underline{B}]^{-1} - \underline{G}\ \underline{\Lambda} \tag{25}$$

where $\underline{\Lambda}$ is an arbitrary diagonal matrix.

Proof:

Necessity: The closed loop transfer function of system (3) is

$$\underline{T}_c(s) = \underline{C}[s\underline{I} - (\underline{A} + \underline{B}\ \underline{K}\ \underline{C})]^{-1}\underline{B}\ \underline{G} \tag{26}$$

Let $$\hat{\underline{A}} = \underline{A} + \underline{B}\ \underline{K}\ \underline{C} \tag{27}$$

Then $\underline{T}_c(s)$ is given by:

$$\underline{T}_c(s) = \frac{1}{\Delta(s)}[\underline{C}\ \underline{B}\ s^{n-1} + \underline{C}\ \underline{R}_{n-2}\underline{B}\ s^{n-2} + \ldots + \underline{C}\ \underline{R}_1\underline{B}\ s + \underline{C}\ \underline{R}_0\underline{B}]\underline{G} \tag{28}$$

where $\Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$

and $a_{n-1}, \ldots, a_0$ and $\underline{R}_{n-2}, \ldots \underline{R}_0$ are given by (22). The problem in obtaining decoupling conditions from equation (28) is that the expressions for $\underline{R}_{n-2}, \ldots, \underline{R}_0$ contain the unknown compensator $\underline{K}$. Recalling from theorem (1) that the decoupling output feedback compensator $\underline{K}$ is given by equation (15), one particularly useful form of (15) is that which does not include $\underline{G}$, i.e., $\underline{\Lambda} = \underline{0}$ and

$$K = \underline{P}_\ell/\Delta_\ell \qquad , \ell = d, \ldots, 0 \tag{29}$$

Moreover, if $\ell = 0$, $\underline{K}$ can be obtained explicitly in terms of the open loop parameters of the system. Thus, if $\underline{A}^{-1}$ exists

$$\underline{K} = \underline{P}_0/\Delta_0 = \underline{T}^{-1}(o) = [\underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B}]^{-1}_{s=0} = -[\underline{C}\ \underline{A}^{-1}\underline{B}]^{-1}$$

This compensator will be called K_I, i.e.,

$$\underline{K}_I = -[\underline{C}\ \underline{A}^{-1}\underline{B}]^{-1} \tag{30}$$

Substituting (30) in (27) yields the form of $\hat{\underline{A}}$ given by (24). The necessary conditions for decoupling the system $[\underline{A}, \underline{B}, \underline{C}]$ by means of a constant precompensator $\underline{G}$ are those of (23). This completes the necessary part of the proof.

Sufficiency: $\underline{G}$ is obtained by inverting any nonsingular matrix of the matrices appearing in (23), i.e.,

$$\underline{G} = [\underline{C}\ \underline{R}_\ell\underline{B}]^{-1}\underline{\Lambda} \qquad , \ell = n-1, \ldots, 1 \tag{31}$$

where $\underline{\Lambda}$ is an arbitrary diagonal matrix. If none of these matrices is nonsingular, $\underline{G}$ can still be obtained as shown by theorem 4 below. The general form of $\underline{K}$ is given by (25) which is directly obtained from (19).

■