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MODEL REDUCTION OF LINEAR MULTIVARIABLE SYSTEMS

By

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ABSTRACT

Reduced order models of multivariable systems obtained by available model reduction techniques often fail to preserve the interaction properties of the original systems they represent, and therefore lead to unreliable control systems. A method is presented which addresses the interaction aspects of multivariable system model reduction and provides insight into the reasons of failure of previous methods.

1. INTRODUCTION

Model reduction is often used to reduce the complexity of linear systems and to obtain lower order models which can be used more readily for control system analysis and design. The approach has been successfully employed in aircraft control, nuclear reactor control and process control application among others. However, most of the methods currently available for multivariable model reduction are extensions of methods originally developed for SISO systems [1,2] and thus do not address the problem of interaction between the multivariable system components. Some of these methods may even produce unstable models when the original system is stable, or visa versa [3]. The need therefore exists for an engineering approach to model reduction which takes into account the particular purpose for which the model is intended.

In this paper we present a new methodology for model reduction which is applicable to diagonally dominant systems [4]. The method tries to preserve in the reduced model the interaction properties of the original system, as well as the transient and steady state behavior of the diagonal elements. The method depends on examination of the root locus plots of the multivariable system [5] and elimination of the branches which are insignificant with respect to interaction and stability properties. The cancelled branches must also be chosen such that their elimination does not alter the dominance properties of the closed loop as reflected in the Bode plots of the multivariable system. The method is developed for two-by-two systems but the insight gained can be extended to systems of larger dimensions, particularly if coupled with the use of CAD package. The details of the method are illustrated by a numerical example.

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2. PROBLEM FORMULATION

Consider the two-by-two system given by:

$$T(s) = [t_{ij}(s)]/\Delta(s) \tag{1}$$

where $t_{ij}(s)$ and $\Delta(s)$ are polynomials in s. Let the system be diagonally dominant at least for the high frequency range. The general problem of model reduction is to find a reduced model Q(s),

$$Q(s) = [q_{ij}(s)]/\bar{\Delta}(s) = \hat{Q}(s)/\bar{\Delta}(s) \tag{2}$$

where $q_{ij}(s)$ and $\bar{\Delta}(s)$ are polynomials in s of lower order than $t_{ij}(s)$ and $\Delta(s)$, respectively such that the transient and steady state behavior of the reduced subsystems $q_{ij}(s)/\bar{\Delta}(s)$ resemble those of their original counterparts $t_{ij}(s)/\Delta(s)$. To this we add the following important requirement. Let the original and reduced model be incorporated in the closed loop system depicted by Fig. 1, where F is a variable gain diagonal compensator. Then it is additionally required that the transient and steady state responses of the reduced closed loop model $Q_c(s)$ approximate those of the original closed loop system $T_c(s)$ for all values of the gain F. This includes the requirement that if the original system becomes unstable at some gain F*, then the reduced system becomes unstable at approximately the same gain or another predetermined gain $\bar{F}$*. This requirement, which appears to have not received enough attention in the literature, seems to be the most important requirement if the reduced model is to be used in the analysis and design of control systems.

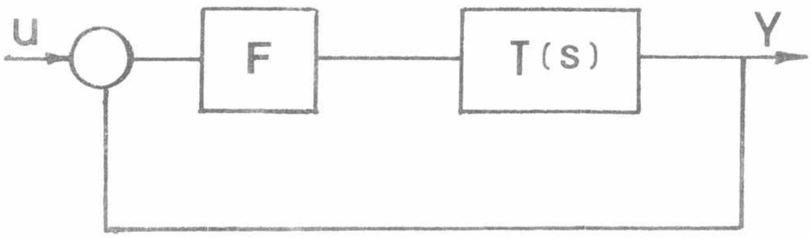

Fig. 1. Closed Loop System

3. DOMINANCE CHARACTERISTICS AND MOVEMENT OF POLES

The dominance characteristics of a closed loop system can be examined by considering the expression

$$T_c^{-1}(s) = F^{-1}\, T^{-1}(s) + I \tag{3}$$

where $T_c^{-1}(s)$ and $T^{-1}(s)$ are the inverted transfer functions of the closed loop and the open loop systems respectively. Equation (3) yields

$$F\,T_c^{-1}(s) = \frac{1}{d_o(s)} \begin{bmatrix} t_{22}(s)\,\Delta(s) + f_1\,d_o(s) & -t_{12}(s)\,\Delta(s) \\ -t_{21}(s)\,\Delta(s) & t_{11}(s)\,\Delta(s) + f_2\,d_o(s) \end{bmatrix}$$

$$= \frac{1}{d_o(s)} \begin{bmatrix} t_{c22}(s) & -t_{c12}(s) \\ -t_{c21}(s) & t_{c11}(s) \end{bmatrix} = \frac{1}{d_o(s)}\ \hat{T}_c(s) \tag{4}$$

where $d_o(s) = \det\ [t_{ij}(s)] = t_{11}(s)\ t_{22}(s) - t_{12}(s)\ t_{21}(s)$ and f_1 and f_2 are the diagonal gain elements of F. For the reduced model, similarly

$$F\,Q_c^{-1}(s) = \frac{1}{\bar{d}_o(s)} \begin{bmatrix} q_{22}(s)\,\bar{\Delta}(s) + f_1\,\bar{d}_o(s) & -q_{12}(s)\,\bar{\Delta}(s) \\ -q_{21}(s)\,\bar{\Delta}(s) & q_{11}(s)\,\bar{\Delta}(s) + f_2\,\bar{d}_o(s) \end{bmatrix}$$

$$= \frac{1}{\bar{d}_o(s)} \begin{bmatrix} q_{c22}(s) & -q_{c12}(s) \\ -q_{c21}(s) & q_{c11}(s) \end{bmatrix} = \frac{1}{\bar{d}_o(s)}\ \hat{Q}_c(s) \tag{5}$$

where $\bar{d}_o(s) = \det\ [q_{ij}(s)]$

The basic idea of this paper centers on the observation [5] that the reduced and original systems will have similar behavior if the diagonal elements in $\hat{T}_c(s)$ and $\hat{Q}_c(s)$ displayed similar characteristics for all values of F, and the diagonal dominance of the matrix $\hat{T}_c(s)$ was preserved. Therefore for model reduction we replace $t_{ij}(s)$ and $\Delta(s)$ by $q_{ij}(s)$ and $\bar{\Delta}(s)$, such that $t_{cii}(s)$ and $q_{cii}(s)$ have the same dominant roots and steady state values. Moreover, both $\hat{T}_c(s)$ and $\hat{Q}_c(s)$ should be diagonally dominant. The closed loop system is diagonally dominant if

$$|t_{cii}(j\omega)| > |t_{cij}(j\omega)| \quad \text{for } 0 \le \omega \le \infty\ ,\ i = 1,\ 2 \ \text{ and } j = 1,\ 2,\ i \ne j \tag{6}$$

The dominance characteristics of the system can be displayed by plotting the Bode diagrams of the two equations (6). These diagrams can be easily plotted if the roots of the polynomials involved are known. Therefore the change in dominance characteristics due to the change in the gains of the closed loop can be examined by examining the root locus of the polynomials t_{c11} and $t_{c22}(s)$ given in (4) for all the values of f_1 and f_2. The root locus and the Bode plots as a means of examining diagonal dominance form the basis of the design method presented in [5]. Moreover, in that paper the following stability result was shown:

Theorem:

The closed loop system given by (4) is stable if $\hat{T}_c(s)$ is diagonally dominant and its diagonal elements are Hurwitz. ■

These ideas will now be used for model reduction.

4. APPLICATION TO MODEL REDUCTION

For the reduced model, in addition to the usual transient and steady state requirements of the open loop system, in order to preserve dominance and the stability characteristics of the original system, we require the following:

1. The relative distance between the Bode plots of the diagonal and off-diagonal elements must remain unchanged between the original and reduced models.

2. The root locus of the diagonal elements of the reduced closed loop model must retain all of the critical branches of the original closed loop system.

By critical we mean the branches that give rise to unstable or oscillatory poles. The two requirements are sometimes difficult to satisfy simultaneously if the order of the reduced model is too low. In this case we require the following:

1. The reduced open loop model is high frequency dominant, while the closed loop reduced model is dominant over the entire frequency range.

2. The diagonal elements retain the critical branches of the original system.

3. The steady state requirement is satisfied only for the diagonal elements.

In such a case the reduced order closed loop system will maintain the critical pole locations of the original system and the diagonal elements will maintain the critical zeros and the correct steady state values. The off-diagonal elements will assume arbitrary transient and steady state behavior but the system will maintain its noninteracting character. The behavior of the off-diagonal elements is simply the price paid for order reduction and can be justified in light of a satisfactory behavior of the diagonal elements in all stages of the design.

5. THE DESIGN PROCEDURE

The key to the present method is the root locus behavior of equations (4) and (5). While maintaining dominance we would like to find $q_{ii}(s)$ and $\Delta(s)$ such that

$$q_{cii}(s) = q_{ii}(s)\bar{\Delta}(s) + f_j\,\bar{d}_o(s)\ ,\ i = 1,\ 2,\ j=1,\ 2,\ i\neq j \qquad (7)$$

retain the critical branches of $t_{cii}(s)$. The design procedure can thus be summarized as follows:

1. Calculate the polynomial $d_o(s)$ and plot the root loci of $t_{cii}(s)$.

2. Choose the elements of $d_o(s)$ to be retained in $\bar{d}_o(s)$. These elements represent the terminal locations (zeros) of the closed loop system root loci. Their location should reflect any oscillatory or unstable behavior of the closed loop system at high gain.

3. Choose the elements of $q_{11}(s)$, $q_{22}(s)$, and $\bar{\Delta}(s)$. These represent the departing points (poles) of the root loci. They are chosen to reflect the open loop characteristics of the original system.

4. From the equation

$$q_{12}(s)\, q_{21}(s) = q_{11}(s)\, q_{22}(s) - \bar{d}_o(s) \qquad (8)$$

find the polynomial $q_{12}(s)\ q_{21}(s)$. Note that this polynomial should retain the high frequency dominance characteristics of the original system.

5. Distribute the elements of $q_{12}(s)q_{21}(s)$ between $q_{12}(s)$ and $q_{21}(s)$ to reflect the off-diagonal subsystems open loop steady state and low frequency dominance characteristics. If possible the roots of $q_{12}(s)$ and $q_{21}(s)$ should be close to those of $t_{12}(s)$ and $t_{21}(s)$ respectively.

6. Correct for the steady state values of the diagonal element. Multiply each column in Q(s) by the same factor to avoid altering the dominance properties and the location of the poles of $q_{cii}(s)$.

7. Verify that the original system and the reduced model closed loop root loci match. Plot the Bode plots for the reduced model closed loop rows to verify dominance. If acceptable, the transient response and steady state behavior of the reduced model should match those of the original system for corresponding values of F and $\bar{F}$.

6. NUMERICAL EXAMPLE

Consider the two-by-two system given by the transfer function

$$T(s) = \begin{bmatrix} 20(s^3+8\ s^2+24\ s+32) & 11.2(s^3+17.838\ s^2+38.232\ s+21.282) \\ 5(s^3+7.162\ s^2+101.517\ s+41.592) & 35(s^3+10.6\ s^2+35\ s+17.4) \end{bmatrix} / \Delta(s) \qquad (9)$$

where $\Delta(s) = s^4 + 20\ s^3 + 151\ s^2 + 500\ s + 600$ (10)

The roots of the elements of T(s) are:

$$t_{11}(s):\ -4.0,\ -2.0\pm j2.0 \qquad t_{12}(s):\ -15.4533,\ -0.9809,\ -1.4040$$

$$t_{21}(s):\ -0.4215,\ -3.3703\pm j9.3443 \qquad t_{22}(s):\ -0.6,\ -5.0\pm j2.0 \qquad (11)$$

and $\Delta(s)$: $-3.0,\ -5.0,\ -6.0\pm j2.0$ (12)

The inverted transfer function of the system is given by

$$T^{-1}(s) = \Delta(s)\ \mathrm{adj}\ [t_{ij}(s)]/d_o(s) \qquad (13)$$

where $t_{ij}(s)$ are obtained from (9) and $d_o(s)$ is given by

$$d_o(s) = \det\,[t_{ij}(s)] = 644\,[s^6 + 18.0435\,s^5 + 133.0435\,s^4$$

$$+ 447.8261\,s^3 + 1017.8261\,s^2 + 1345.2174\,s + 528.2609] \qquad (14)$$

with roots: $(-6.9079 \pm j3.8932)$, $(-0.7353 \pm j2.4183)$, -2.1436, -0.6135

The root locus of the diagonal elements $t_{c11}(s)$ and $t_{c22}(s)$ are given in Fig. 2. The diagonal dominance plots at $f_1 = f_2 = 10$ are given in Fig. 3. The systems is diagonally dominant for all frequencies.

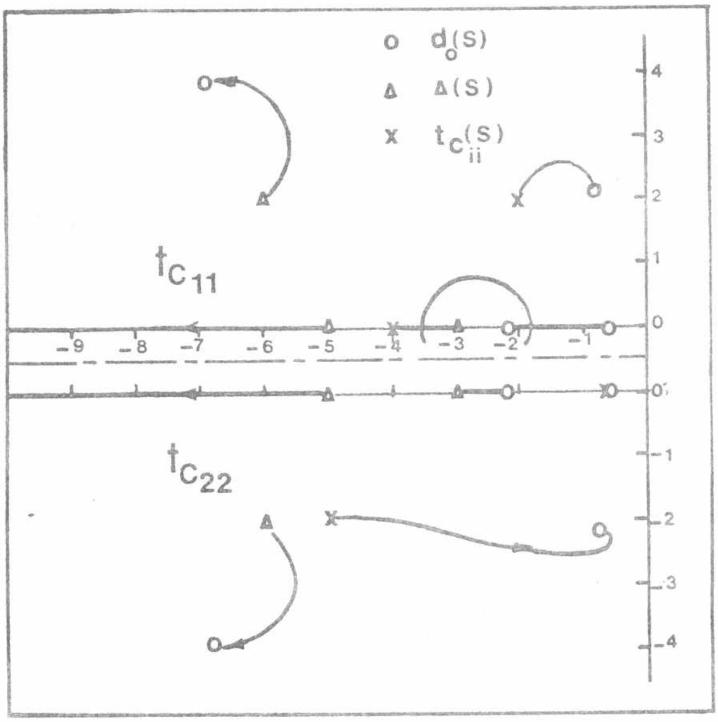

Fig. 2 Root Locus of Diagonal Elements - Original System

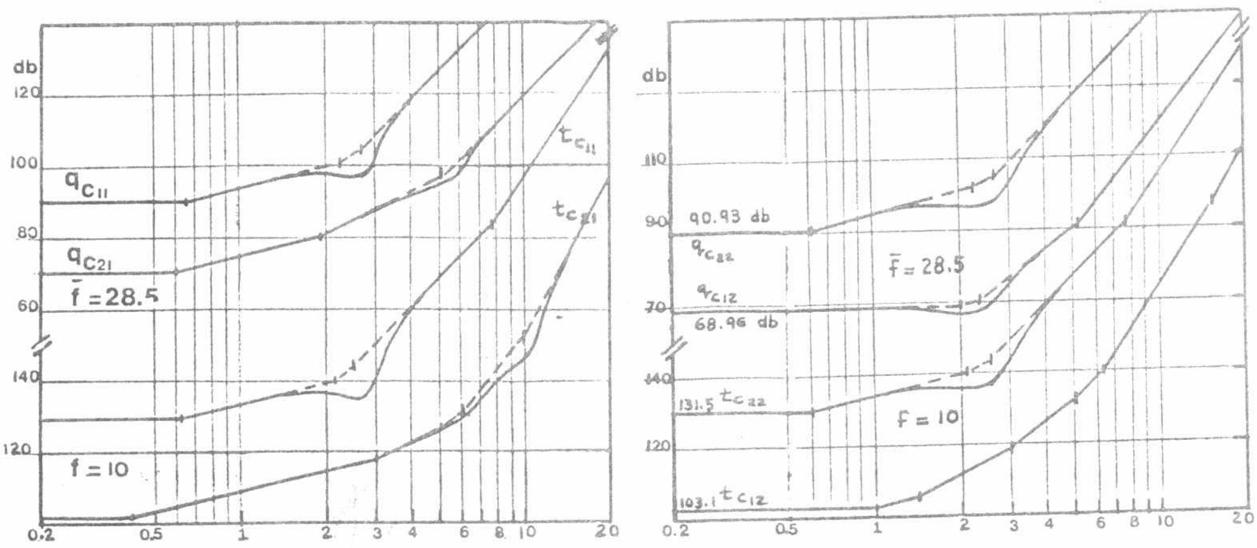

Fig. 3 Dominance Plots for Original System and Third Order Model

a. A Third Order Model:

Choosing the model roots: Using equation (13) and the root locus diagram in Fig. 2. we find that in $d_0(s)$ the following roots should be retained: $-0.7353\pm j2.4183$, -0.6135, -2.1436. In retaining these roots we notice that the first two complex roots are important to maintain the oscillatory behavior at high gain while the root at -0.6135 contributes to an almost fixed mode in $t_{c22}(s)$ and to a slowing mode as f_2 increases in $t_{c11}(s)$. Discarding these could be detrimental to the reduced model behavior. The root at -2.1436 is rather arbitrary and is chosen to maintain the general shape of the root locus. The resulting $\bar{d}_0(s)$ is given by:

$$\bar{d}_0(s) = s^4 + 4.2277\ s^3 + 11.7585\ s^2 + 19.5485\ s + 8.4019 \tag{15}$$

To obtain the reduced second order diagonal polynomials we notice that in $q_{11}(s)$ the roots at $(-2\pm j2)$ are to be maintained and in $q_{22}(s)$ the root at approximately (-0.6) should be maintained. This presents a problem since the behavior of the second diagonal element depends critically on the roots at $(-6\pm j2)$ as well as the one at (-0.6). The problem can be avoided by replacing the roots at $(-6\pm j2)$ by roots of $\bar{\Delta}(s)$ since both play the role of poles in the root locus plot. This freedom in choosing the roots is a key feature of this approach. The other root of $q_{22}(s)$ is chosen at $s = -6.0$. This yields:

$$q_{11}(s) = s^2 + 4\ s + 8 \qquad\qquad q_{22}(s) = s^2 + 6.6\ s + 3.6 \tag{16}$$

and, $$q_{11}(s)\ q_{22}(s) = s^4 + 10.6\ s^3 + 38\ s^2 + 67.2\ s + 28.8 \tag{17}$$

The denominator polynomial $\bar{\Delta}(s)$ is obtained from similar considerations as:

$$\bar{\Delta}(s) = (s + 2)(s^2 + 10\ s + 29) = s^3 + 12\ s^2 + 49\ s + 58 \tag{18}$$

Note that the reduced model root loci retain the essential features of the original system in both open loop and closed loop configurations. Therefore, it is expected that if a sufficient degree of dominance is maintained in the reduced model the resulting transient response will be similar to that of the original system for all gains. The root loci for both a third order and a second order models are shown in Figure 4 below.

From (15) and (17), maintaining high frequency dominance level we get

$$q_{12}(s)\ q_{21}(s) = q_{11}(s)\ q_{22}(s) - 0.92 \times \bar{d}_0(s) = 0.08(s^4 + 83.8815\ s^3$$
$$+ 339.7774\ s^2 + 615.1917\ s\ 263.3781) \tag{19}$$

which has the roots: $-1.7854\pm j1.5371$, -0.5953, -79.7154. Comparing these roots with those in (11) we can select the two complex roots for $q_{12}(s)$ and the two real for $q_{21}(s)$. This gives

$$q_{12}(s) = (s^2 + 3.5708\ s + 5.5503),\ q_{21}(s) = (s^2 + 80.3107\ s + 47.4546) \tag{20}$$

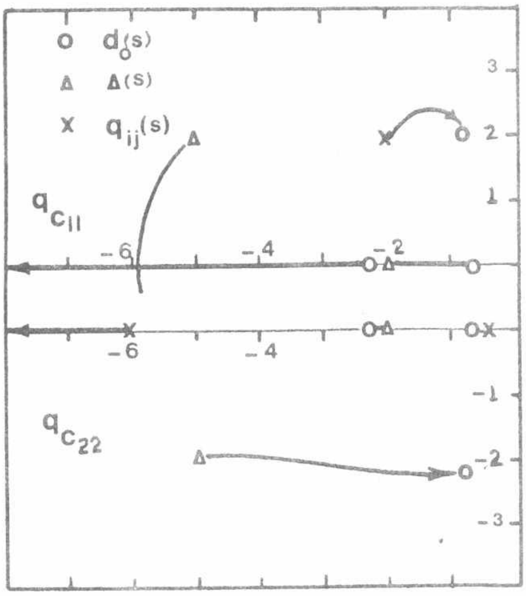

Fig. 4.a Root Locus of Diagonal Elements - Third Order System

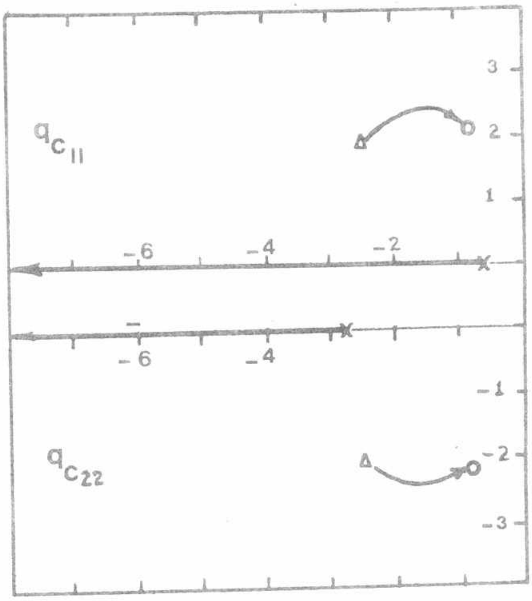

Fig. 4.b Root Locus of Diagonal Elements - Second Order System

The reduced model is, therefore

$$Q(s) = \begin{bmatrix} s^2 + 4s + 8 & 0.5333(s^2 + 3.57s + 5.55) \\ 0.15(s^2 + 80.31s + 47.45) & s^2 + 6.6s + 3.6 \end{bmatrix} / \bar{\Delta}(s) \qquad (21)$$

where $\bar{\Delta}(s)$ is given by (18).

The leading coefficients of $q_{12}(s)$ and $q_{21}(s)$ are chosen to maintain dominance at high frequency.

Steady State Correction: The next step is to correct for the steady state values of the diagonal elements. This is achieved by multiplying the first column by 7.7333 and the second column by 16.3446. This does not change dominance levels or the location of the roots of $d_o(s)$. The final reduced model is:

$$Q(s) = \begin{bmatrix} 7.7333(s^2 + 4S + 8) & 8.7166(s^2 + 3.57s + 5.55) \\ 1.1(s^2 + 80.31s + 47.45) & 16.3446(s^2 + 6.6s + 3.6) \end{bmatrix} / \bar{\Delta}(s) \qquad (22)$$

where $\bar{\Delta}(s) = s^3 + 12s^2 + 49s + 58$

Scaling the Gain F: It is important to obtain the same steady state values at equivalent gains for the diagonal elements of both the original system and reduced model. To examine gain equivalence between the original and reduced order models we consider equations (4) and (5) at $s = 0$. The steady state values of the diagonal elements of the two models at high gain (i.e., neglecting the off-diagonal elements) are:

$$t^{-1}_{cii}(o) \simeq \frac{1}{f} \frac{\Delta(o)\ t_{ii}(o)}{d_o(o)} + 1 \ , \ i = 1, 2 \tag{23}$$

$$q^{-1}_{cii}(o) \simeq \frac{1}{\bar{f}} \frac{\bar{\Delta}(o)\ q_{ii}(o)}{\bar{d}_o(o)} + 1 \ , \ i = 1, 2 \tag{24}$$

where f and $\bar{f}$ are the diagonal gains for the original and reduced model respectively, assuming equal diagonal gains for simplicity. For equal steady state values

$$\bar{f} = f \frac{\bar{\Delta}(o)\ q_{ii}(o)}{\Delta(o)\ t_{ii}(o)} \frac{d_o(o)}{\bar{d}_o(o)} \tag{25}$$

applying (25) to the present system we get $\bar{f} = 2.85$ f. Using these values in simulating the system response we get almost perfect similarity between the original system and reduced model responses for the diagonal elements at $f_1 = \bar{f}_2 = 28.5$. This is shown in Figure 5. Note that the off-diagonal responses are slightly different which is considered acceptable as long as the system is highly decoupled.

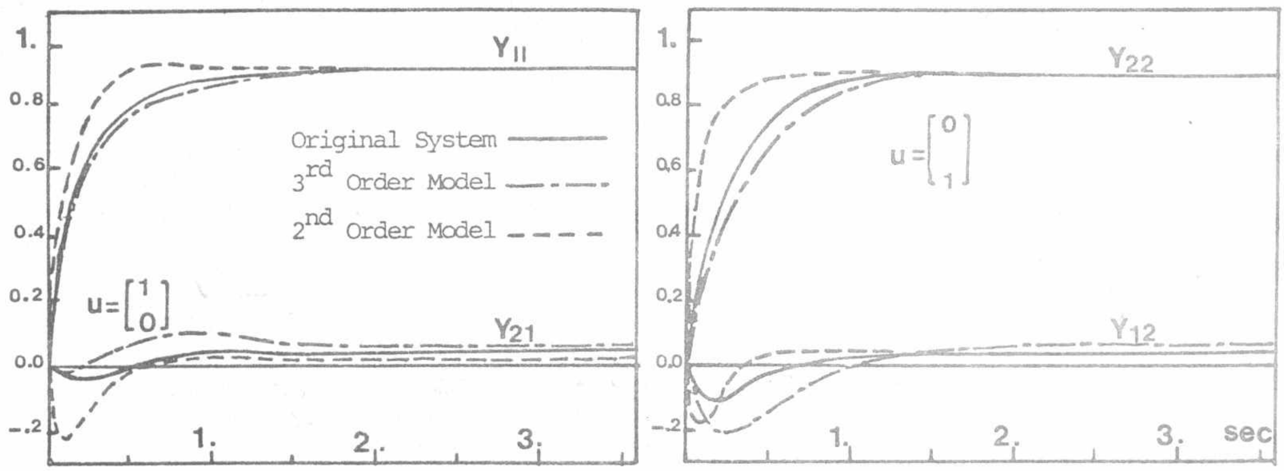

Fig. 5 Unit Step Responses of Original System and Reduced Models

b. A Second Order Model:

In this section we try to find a second order reduced model. Two oscillatory roots at -0.8 ± 0.28 are retained in $d_o(s)$. Using similar procedure to that used above we obtain the following second order system

$$Q(s) = \begin{bmatrix} 17.34\ s + 10.404 & 0.732\ s + 7.3666 \\ 4.7\ s + 1.7545 & 3.66\ s + 10.25 \end{bmatrix} /(s^2 + 5\ s + 10.25) \tag{26}$$

The root locus for the reduced model is shown in Fig. 4. Note that in selecting the poles for $q_{c11}(s)$ and $q_{c22}(s)$ the roots of $\Delta(s)$ were selected at $s = -2.5\pm j2$ to compensate for the absence of the numerator roots at $-2\pm j2$. These can no longer be used since $q_{11}(s)$ and $q_{22}(s)$ are now first order. This unique design feature allows us to get satisfactory response with orders as low as two. The response of the reduced model shown in Figure 5 shows satisfactory agreement with the original system.

Conclusion:

The method presented here for reduction of linear multivariable diagonally dominant systems provides a predictably satisfactory response for a specified range of design conditions. For diagonally dominant systems the response of the off diagonal elements can be sacrificed in order to obtain accurate response for the diagonal elements for the specified closed loop configuration. This is usually acceptable as long as the system is reasonably noninteracting. Only constant gain feedback was considered and it was shown that for all gains the reduced model and the original system diagonal elements responses agree to a considerable degree. The method depends on the use of the diagonal elements root loci for choosing the appropriate roots to be retained. The design is aided by the use of the Bode plots to display and examine dominance. With dominance and root loci conditions satisfied, the reduced model response resulting from this method is guaranteed to be satisfactory for the full range of constant gains including open loop and unstable conditions.

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