مجلة التجارة والتمويل

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كلية التجارة – جامعة طنطا

العدد : الرابع

ديسمبر 2022

Robust Estimation Methods of Exponentiated Inverted Weibull Distribution with Outliers

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الملخص

يهدف البحث الى دراسة طرق التقدير الحصينة لمعلمة الموضع والمقياس لتوزيع معكوس وايبل الاسي في حالة وجود قيم شاذة في البيانات ، حيث يؤدى وجود قيم شاذة بالبيانات الى خلل كبير في نتائج التحليل الإحصائي الخاصى بتلك البيانات ، وخصوصا عند استخدام الطرق التقليدية مثل طريقة المربعات الصغرى وطريقة الإمكان الاكبر . ولذلك فإن الهدف من الدراسة هو عمل مقارنة بين الطرق التقليدية في التقدير والطرق غير التقليدية مثل طريقة اقل الانحرافات المطلقة (LAD) وايضا M-estimation و M. Bisquare (MB) لمعرفة افضل طرق التقدير عند وجود قيم متطرفة في البيانات.

وقد تمت الدراسة على توزيع معكوس وايبل الاسي باستخدام محاكاة مونتي كارلو، كما تم استحدام بيانات حقيقية وكانت نتائج التحليل تؤكد ان طرق التقدير الغير تقليدية افضل من الطرق التقليدية حيث بينت الدراسة ان افضل طرق التقدير هي M-estimation

Abstract

This paper discussed robust estimation methods for point estimation of the shape and scale parameters for Exponentiated Inverted Weibull Distribution (EIW) using a complete dataset in the presence of various percentages of outliers. In the case of outliers, it is known that classical methods such as maximum likelihood estimation (MLE), least square (LS) in case of outliers cannot reach the best estimator. To confirm this fact, these classical methods were applied to the data of this study and compared with non-classical estimation methods. The non-classical (Robust) methods such as least absolute deviations (LAD), and M-estimation (using M. Huber (MH) weight and M. Bi-square (MB) weight) had been introduced to obtain the best estimation method for the parameters of the EIW distribution. The comparison was done numerically by using the Monte Carlo simulation study. The real dataset application confirmed that the M-estimation method is very much suitable for estimating the EIW parameters. We concluded that the M-

estimation method using Huber object function is a suitable estimation method in estimating the parameters of the EIW distribution for a complete dataset in the presence of various percentages of outliers.

Keywords: Exponentiated Inverted Weibull, Classical Estimation Methods, Robust Estimation, Least Absolute Deviations and M-estimation

1. Introduction

Gupta and Kundu (2001) discussed different methods of estimation such as MLE, LS, Moment, Weighted Least Squares methods for the parameters of a EIW distribution. Kundu and Gupta (2008) considered the Bayes estimators of the unknown parameters for EIW distribution under the assumptions of gamma priors on both the shape and scale parameters. Ahmad (2010) discussed the optimal accelerated life test designs for EIW distribution with log-linear model under periodic inspection and Type I censoring. A large number of papers covering the EIW distribution in many fields and applications such as Chen and Lio (2010); Kundu and Gupta (2011); Tahir et al. (2015); Naqash et al. (2016); Valiollahi et al. (2017); Hossain (2018); Kundu and Nekoukhou (2018).

The common estimation methods would not be appropriate in solving the parameter estimation problem if data have contained outliers or extreme observations. For more information on common estimation methods see Almetwally et al. (2018) Ahmad and Almetwally (2020), Basheer et al. (2020). Therefore, we need alternative estimation methods which can handle problems with respect to outliers or extreme observations: these methods of parameter estimation are called robust estimation methods. Almetwally and Almongy (2018) discussed six methods of estimation for regression model to reach the best parameter estimation of model. An alternative robust estimation method based on M-estimations method for the parameters of Burr III distribution have been proposed by Wang and Lee (2010, 2011 and 2014). Kantar and Yildirim (2015) considered various robust estimators for the extended Burr Type III distribution for complete data with outliers by using different methods of robust estimation. The robustness properties of the estimators are investigated by Aydın et al. (2018) for estimation of the location parameter and the scale parameter of the shifted Gompertz distribution by using least squares, maximum likelihood, and modified likelihood estimators.

The aim of this paper is to assess the effectiveness of alternative robust estimation methods in determining the parameters of the lomax distribution, where the LAD and M-estimations as Bi-square and Huber weights have been used as alternative methods of commonly estimation methods. On the other hand, MLE, LS and MPS as a more commonly estimation methods for the EIW parameters are also considered. To evaluate the performance of the estimators, a Monte Carlo simulation study is carried out. The final motivation of the paper is to develop a guideline for introducing the best estimation method for EIW distribution, where the data contains outliers or extreme observations.

The paper is organized as follows: section 2 is devoted to the EIW parameters estimation using the MLE method, the LS method and the MPS method, while in section 3 the robust estimation is considered. In section 4, we present Monte Carlo simulation study to compare the performance of the estimators of the EIW distribution parameters for all estimation methods, which are used. Moreover, application of real data is given in section 5. Finally, we show the results and the conclusion of the current study in section 6

Assume that random variable X has a standard exponentiated inverted Weibull distribution (EIW); its distribution function takes the following form:

$$F(x;\theta,\beta) = (e^{-x^{-\beta}})^{\theta} \quad ; \quad x,\theta,\beta > 0$$

Here, β and θ are the shape parameters. Therefore, the probability density function is:

$$f(x;\theta,\beta) = \theta\beta x^{-(\beta+1)}(e^{-x^{-\beta}})^{\theta} \quad ; \quad x,\theta,\beta > 0$$

The quantile function of the exponentiated inverted weibull distribution is given as:

$$x_q = \left(\frac{-1}{\theta}\ln U\right)^{\frac{-1}{\beta}} \quad , \qquad 0 < U < 1$$

$where, U_i$ uniform distribution

2. The Classical Estimation Methods

In this section, the parameter estimation by MLE, LS and MPS estimation methods will be discussed.

2.1. MLE Method

The likelihood function of the EIW distribution is

$$\boldsymbol{L(\Theta)} = (\theta\beta)^n \, (e^{-\sum_{i=1}^{n} x^{-\beta}})^{\theta} \prod_{i=1}^{n} x_i^{-(\beta+1)}$$

and the log likelihood function is given as:

$$\boldsymbol{l(\Theta)} = n\log\theta + n\log\beta - (\beta+1)\sum_{i=1}^{n}\log x_i + \theta\sum_{i=1}^{n} x_i^{-\beta} \qquad (2.1)$$

To obtain the normal equations for the unknown parameters, we differentiate (2.1) partially with respect to the vector parameter Θ = $(\beta,\ \theta)$ and equate them to zero. The estimators $\hat{\alpha}$ and $\hat{\theta}$ can be obtained as the solution of the following equations.

$$\frac{\partial \boldsymbol{l}(\boldsymbol{\Theta})}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} x_i^{-\beta}$$

$$\frac{\partial \boldsymbol{l}(\Theta)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log x_i + \theta \sum_{i=1}^{n} x_i^{-\beta} \log x_i$$

The above non-linear equations are not in closely form; hence iterative procedures such as Newton-Raphson type algorithm and others introduced by McMullen, Curtis (1987) can be used to obtain the solution.

2.1. Least Square Method

Using least square estimators introduced by van de Geer, Sara (1987), which based on the observed sample $x_1 < \cdots < x_n$ from n ordered random sample of any distribution with CDF, we get :

$$E(F(x_i) = \frac{i}{n+1}$$

where *F* (.) denotes the CDF

$$y_i = \log \log \left(\frac{i}{n+1}\right)$$

So, the straight line equation is given by:

$$y_i = -\log \theta - \beta \log x_i$$

The least squares estimators of θ and β are their value which minimizes the following equation:

$$Q(\theta, \beta) = \sum_{i=0}^{n} [y_i - (-\log \theta - \beta \log x_i)]^2 \qquad (2.2)$$

The first partial derivatives for (2.2) with respect to θ and β are given by:

$$\frac{\partial Q}{\partial \theta} = \frac{2}{\theta}\sum_{i=1}^{n}[y_i - (-\log\theta - \beta\log x_i)]$$

$$\frac{\partial Q}{\partial \beta} = 2\ \log x_i \sum_{i=1}^{n}[y_i - (-\log\theta - \beta\log x_i)]$$

$$\hat{\theta} = e^{\frac{\sum_{i=1}^{n}\log^2 x_i + \sum_{i=1}^{n}\log y_i + \sum_{i=1}^{n} y_i \log x_i \sum_{i=1}^{n}\log x_i}{(\sum_{i=1}^{n}\log x_i)^2 - n\sum_{i=1}^{n} log^2 x_i}}$$

$$\hat{\beta} = \frac{n\sum_{i=1}^{n} y_i \log x_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n}\log x_i}{(\sum_{i=1}^{n}\log x_i)^2 - n\sum_{i=1}^{n}\log^2 x_i}$$

3. Robust Estimation Method

When a dataset is contaminated with a single or few outliers, it presents a serious problem in parameter estimations. Hence, robust estimation is an important method for analyzing datasets contaminated with outliers. Robust regression estimators have a large family, such that the least absolute deviation (LAD) method, M-estimation method, the least median of squares (LMS) method, Least Quantile of Squares (LQS) method and so on. A brief description and discussion on robust estimators can be found in few existing literatures by authors such as Fang and Zhao (2006), Kantar and Yildirim (2015) and Almetwally and Almongy (2018).

The robust methods are used to estimate the parameters of the EIW distribution from a sample of n observations x_i ; $i = 1, \dots, n.$ as follows:

$$x_i = f_i(\Theta) + \varepsilon_i$$

where $f_i(\Theta)$ is the quantile function of the EIW distribution and ε_i is the error term with mean equals zero and variance equals σ^2 .For the robust regression method, it is necessary to scale the invariant error, as follows

$$e_i = \frac{\varepsilon_i}{S`}$$

where e_i denotes the i^{th}residual,

$$s = \frac{MAD(\varepsilon)}{0.6745} \; and \; MAD(\varepsilon) = Median(\varepsilon - Median(\varepsilon)).$$

3.1. Least Absolute Deviations (LAD)

Method Fang and Zhao (2006) introduced the Least Absolute Deviations (LAD) estimation method, which is a robust method in the presence of outliers and asymmetric error terms. The introduction of the easy calculus of the least-squares method by Dodge (2008), made the least-squares method much more popular than the LAD in regression analysis. Yet in recent years and with advances in statistical computing, the LAD method can be easily used. The LAD method aims to obtain the estimated regression parameters that minimizes the sum of absolute value of the residuals. Hence, it treats the outliers influence on the LS estimator sign the residuals sum of square. More so, the LAD estimator is asymptotically unbiased, normally distributed and has a lower asymptotic variance when the distribution is nonnormal.

The parameter estimates by least absolute deviation regression is given as

$$\widehat{\beta}_{LAD} = min|e_i|. \qquad (3.1)$$

where, e_i denotes the i^{th}residual

The least absolute deviation method is obtained by minimizing

$$LAD(\Theta) = min \left| \frac{x_i + ((\frac{1}{\theta}\ln U_i)^{\frac{-1}{\beta}}))}{S} \right|. \qquad (3.2)$$

where, $s = \frac{MAD(\varepsilon)}{0.6745}$, U_i uniform distribution, and β and θ are the shape parameters

After differentiating Equation (3.2) with respect to the parameters α and θ we get

$$\frac{LAD(\Theta)}{\partial\beta} = min \left| \frac{\theta\, U_i^{(1/\beta)} \ln U_i}{S\beta^2 \left(\frac{-1}{\theta} \ln U\right)^{\frac{-1}{\beta}}} \right| \tag{3.3}$$

$$\frac{LAD(\Theta)}{\partial\theta} = min \left| \frac{(\left(\frac{1}{\theta} \ln U_i\right)^{\frac{-1}{\beta}-1} \theta^2 \ln U_i}{\beta S} \right| \tag{3.4}$$

$$MAD(\varepsilon) = Median\big(\varepsilon - Median(\varepsilon)\big),$$

Equating Equations (3.3) and (3.4) to zero we obtain two nonlinear equations that cannot be solved analytically. Hence, iterative procedures such as the Newton-Raphson algorithms can be used to solve for the solution of the $\hat{\alpha}_{LAD}$ and $\hat{\theta}_{LAD}$ numerically.

3.2. M-Estimation Method

The most common general method for robust regression is the M-estimation, introduced by Huber (1964). The M-estimation method is regarded as a generalization to the maximum likelihood estimation in the context of location models. The principle of the M-estimation method is to minimize the residual function rather than minimize the sum of squared errors as the objective function. The M-estimation method for estimating the EIW distribution parameters is defined by minimizing the objective function of all invariant errors $\rho(e_i)$, as follows:

$$Minimize \sum_{i=1}^{n} \rho(ei) \qquad .(3.5)$$

To estimate of the two unknown parameters of the EIW distribution, a simple comparison between two different ρ objective functions is used. The selected objective functions are Tukey's Bi-square and Huber's weight.

Table 1: Description of the M-estimation method

	Objective Function $\rho(ei)$	Score Function $\Psi(e_i)$	a
Huber (1964)	$\begin{cases} \frac{1}{2}{e_i}^2 & if\ \lvert e_i \rvert < a \\ a\lvert e_i \rvert - \frac{1}{2}a^2 & if\ \lvert e_i \rvert \geq a \end{cases}$	$\begin{cases} e_i \\ a \text{ sign } e_i \end{cases}$	1.345
Tukey Bisquare (Huber (1981)	$\begin{cases} \frac{a^2}{6}\left(1-\left(1-\left(\frac{ei}{a}\right)^2\right)^3\right) & if\ \lvert e_i \rvert < a \\ \frac{1}{62}a^2 & if\ \lvert e_i \rvert \geq a \end{cases}$	$\begin{cases} e_i\left(1-\left(\frac{ei}{a}\right)^2\right)^2 \\ 0 \end{cases}$	4.685

e: residual, a: constant.

The estimators of the parameters can be obtained for two objective functions of the M-estimation, derived by differentiating Equation (3.5) with respect to the scale and shape parameters of EIW distribution. Then, we can obtain the simultaneous equations, which are given as follows:

$$\sum_{i=1}^{n} \Psi(e_i)\frac{\partial f_i(\Theta)}{\partial \Theta}|_{\Theta=\hat{\theta}} = 0$$

The derivative of $f_i(\Theta)$ for Θ as follows:

$$\frac{\partial f_i(\Theta)}{\partial \alpha} = \frac{-\theta\, U_i^{(1/\alpha)} \ln U_i}{\alpha^2(1-\ U_i^{(1/\alpha)})}$$

$$\frac{\partial f_i(\Theta)}{\partial \theta} = -\ln(1-\ U_i^{(1/\alpha)})$$

In order to solve the above equations, the Newton–Raphson method can be employed.

4. Simulation Study

A Monte Carlo simulation study is carried out and comparisons made between the non-robust and robust estimation methods. The non-robust methods are the maximum likelihood estimation (MLE) method, least square (LS) method, and maximum Product Spacing (MPS) method. The Robust methods are the Least Absolute Deviations (LAD)

Monte Carlo experiments were carried out using equation (1.3) to generate random samples from the EIW distribution process: obtain the error term (ε) using a normal distribution $(n, 0, \sigma^2)$. σ^2 is the variance of the normal distribution, $\sigma^2 = 0.25, 0.5$ and 0.75. Outliers are generated from a random sample from uniform distribution ~~Uniform~~ $(\bar{x} + 4S, \bar{x} + 7S)$, where $\bar{x}$ is the sample mean of $x \sim EIV(\alpha, \theta)$ and S is the sample standard deviation of x (Wang and Lee, 2010). Select different sample sizes, $n = 20, 40$ *and* 100, to investigate the robustness of the methods against outliers, we randomly generate different percentages of outliers $(P = 5\%, 10\%, 15\%, and\ 20\%)$. Setting the parameter coefficient $\Theta = (\alpha, \theta) = (2.5, 1.5)$ and (0.5, 1.8), all simulation results are based on 10000 replications. The simulation results are compared using the bias and mean square errors (MSE).

$$Bias = (\hat{\Theta}) - \Theta) \qquad (4.1)$$

$$MSE = Mean\ (\hat{\Theta} - \Theta)^2 \qquad (4.2)$$

where $\hat{\Theta}$is the estimator of Θ.

4.2. Summary and Conclusions of Simulation Results:

The simulation results are presented in Tables (2:6) and Figure (1, 2). The numerical results of the robust estimation methods and the non-robust estimation methods using different percentages of outliers (p), different standard normal error term (σ^2) and different sample sizes (n) are shown in the tables. We observe that an increase in the sample size leads to lower MSE values for robust and non-robust estimation methods. We also observe that higher percentage values of outliers' lead to higher Bias and MSE values for the robust and non-robust estimation methods. Furthermore, a higher value of the standard normal error term leads to higher Bias and MSE values for the robust estimation methods.

Table 2: The Bias and MSE for Three Different Non-Robust Methods for Data without Outliers

	Case 1						Case 1					
n	MLE		LS		MPS		MLE		LS		MPS	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	0.3011	0.7067	0.2826	0.7396	-0.2098	0.5348	0.0348	0.0098	0.019	0.0117	-0.0337	0.009
	-0.0564	0.1256	0.1259	0.2424	0.2434	0.2546	-0.0682	0.4116	0.3386	1.1112	0.5151	1.0775
40	0.2505	0.3315	0.1001	0.2556	-0.1903	0.3236	0.0224	0.0059	0.0081	0.0057	-0.0279	0.0094
	-0.0239	0.0638	0.0617	0.1043	0.1501	0.1055	-0.0308	0.2087	0.1499	0.4132	0.2997	0.4007
100	0.093	0.1895	0.0271	0.1554	-0.1182	0.1603	0.011	0.0029	0.0023	0.0015	-0.0164	0.0036
	-0.0083	0.0266	0.028	0.0387	0.0753	0.036	-0.0103	0.088	0.0245	0.144	0.1453	0.1281

Table 3: The Bias and MSE for Three Different Non-Robust Methods for Data with Different Ratio of Outliers

			Case 1						Case 1					
n			MLE		LS		MPS		MLE		LS		MPS	
	p		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	5%	α	-0.219	0.7779	-0.174	1.5051	-0.817	0.9857	0.0082	0.0116	-0.018	0.0203	-0.08	0.0155
		θ	0.4602	0.4934	0.4322	0.5502	0.9656	1.425	0.7545	1.5782	1.0288	2.9098	1.7263	5.0652
	10%	α	-0.652	0.8623	-0.529	1.6413	-1.096	1.36	-0.019	0.0125	-0.057	0.0214	-0.098	0.0172
		θ	1.0201	1.5449	0.8359	1.3142	1.659	3.5993	1.5596	4.1959	2.0323	7.8616	2.7376	10.922
	15%	α	-0.865	0.9862	-0.82	1.6711	-1.225	1.6176	-0.024	0.0134	-0.086	0.0226	-0.098	0.0179
		θ	1.5546	3.1865	1.417	3.1664	2.2801	6.4331	2.2697	7.7418	3.3527	17.717	3.5769	17.558
	20%	α	-0.987	1.135	-1.211	1.6995	-1.293	1.7593	-0.02	0.0155	-0.114	0.0298	-0.099	0.0187
		θ	2.0614	5.2824	2.3679	7.5542	2.8457	9.709	2.9156	11.921	4.9477	33.749	4.3155	24.73
40	5%	α	-0.45	0.3641	-0.264	0.3442	-0.781	0.7141	-0.027	0.0061	-0.028	0.0069	-0.076	0.01
		θ	0.4777	0.3537	0.2614	0.1837	0.7632	0.7569	0.7982	1.1228	0.5925	0.886	1.3227	2.4907
	10%	α	-0.788	0.7211	-0.555	0.5031	-1.034	1.1377	-0.045	0.0068	-0.059	0.0087	-0.089	0.0116
		θ	1.0305	1.3077	0.5782	0.5248	1.3779	2.229	1.612	3.5323	1.3983	3.0564	2.2297	6.3237
	15%	α	-0.988	1.0514	-0.874	0.9806	-1.183	1.4503	-0.053	0.0077	-0.093	0.0151	-0.093	0.0126
		θ	1.5891	2.9059	1.0991	1.5269	1.9782	4.4067	2.378	7.0782	2.7171	9.4892	3.063	11.363
	20%	α	-1.094	1.2526	-1.205	1.6113	-1.259	1.6259	-0.059	0.0081	-0.116	0.0177	-0.1	0.0137
		θ	2.1152	5.0216	2.0475	4.9318	2.5336	7.11	3.0704	11.434	4.3816	23.25	3.8057	17.205
100	5%	α	-0.466	0.3081	-0.206	0.2198	-0.63	0.4691	-0.023	0.0031	-0.012	0.0035	-0.047	0.0044
		θ	0.5158	0.3288	0.2529	0.1261	0.6462	0.4894	0.8674	0.979	0.5574	0.5777	1.0962	1.472
	10%	α	-0.806	0.7048	-0.506	0.3783	-0.925	0.8992	-0.043	0.0041	-0.043	0.0048	-0.064	0.0061
		θ	1.0713	1.2564	0.5591	0.4006	1.2249	1.623	1.6857	3.2224	1.3077	2.1565	1.9495	4.2439
	15%	α	-0.993	1.0282	-0.831	0.769	-1.085	1.2131	-0.049	0.0048	-0.079	0.0087	-0.069	0.0069
		θ	1.6254	2.82	1.0415	1.2439	1.7945	3.4182	2.4501	6.6069	2.519	7.1865	2.7397	8.1998
	20%	α	-1.098	1.2391	-1.187	1.4529	-1.176	1.411	-0.047	0.0047	-0.104	0.0129	-0.066	0.0065
		θ	2.1532	4.8883	1.935	4.0558	2.3339	5.7247	3.1482	10.741	4.0912	18.178	3.4589	12.907

Table 4: The Bias and MSE for Three Different Robust Methods for Data with Different Ratio of Outliers when $n = 20$

			Case 1						Case 1					
n			MLE		LS		MPS		MLE		LS		MPS	
	p		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	$\sigma^2 = 0.25$													
	5%	α	0.0837	0.1725	0.0261	0.0984	0.0561	0.0967	0.0207	0.0115	0.0068	0.0072	0.0052	0.007
		θ	0.0007	0.0113	0.0019	0.0072	-0.004	0.0072	0.0228	0.1168	0.0215	0.0709	0.0263	0.0521
	10%	α	0.1221	0.2147	0.0325	0.1086	0.066	0.1001	0.0222	0.0142	0.0081	0.0084	0.0054	0.0075
		θ	0.0049	0.0158	0.0012	0.0084	-0.005	0.0076	0.0918	0.3581	0.0314	0.1124	0.0421	0.1019
	15%	α	0.1547	0.3076	0.0312	0.1139	0.0758	0.1096	0.0287	0.0209	0.0073	0.0086	0.0069	0.0087
		θ	0.0254	0.0423	0.0019	0.0089	-0.004	0.0082	0.1741	0.6004	0.0355	0.1207	0.0485	0.1157
	20%	α	0.1808	0.468	0.026	0.131	0.08	0.1249	0.0432	0.0371	0.0067	0.011	0.0076	0.0097
		θ	0.0671	0.1056	0.0057	0.012	0.002	0.0101	0.214	0.6846	0.0552	0.1711	0.0809	0.1859
		$\sigma^2 = 0.5$												
	5%	α	0.2646	1.0598	0.1073	0.4899	0.1401	0.3748	0.0798	0.0589	0.0113	0.0233	0.0116	0.0211
		θ	0.0089	0.0457	0.0081	0.0287	0.003	0.0274	-0.021	0.1365	0.0773	0.1607	0.0332	0.1084
	10%	α	0.3969	1.4234	0.1185	0.5177	0.1602	0.4746	0.0861	0.061	0.0225	0.0273	0.0147	0.0238
		θ	0.0065	0.0577	0.0067	0.0312	-0.001	0.0281	0.0907	0.2101	0.0671	0.1884	0.0782	0.1402
	15%	α	0.5693	2.2385	0.124	0.5883	0.1844	0.4988	0.1349	0.1059	0.0207	0.0284	0.0305	0.0291
		θ	0.0075	0.0772	0.008	0.0327	-0.001	0.0289	-0.054	0.1632	0.0696	0.2038	0.037	0.1566
	20%	α	0.8645	9.4028	0.1125	0.6375	0.2281	0.6429	0.2497	0.3904	0.0718	0.0379	0.0536	0.0363
		θ	0.0024	0.0847	0.0154	0.0368	0.0006	0.0312	-0.124	0.1929	0.078	0.2299	0.0704	0.1905
		$\sigma^2 = 0.75$												
	5%	α	0.8219	14.302	0.2978	2.6688	0.2523	1.4318	0.1736	0.3206	0.038	0.0743	0.0403	0.0736
		θ	0.0157	0.1077	0.0257	0.0741	0.0185	0.0592	0.0243	0.4794	0.1739	0.4668	0.0703	0.2603
	10%	α	1.1754	16.623	0.28	2.7441	0.3566	2.4243	0.2366	0.5034	0.0653	0.0795	0.0598	0.0789
		θ	-0.003	0.1087	0.0334	0.0793	0.0166	0.0698	0.1927	0.6425	0.1482	0.5588	0.1093	0.349
	15%	α	1.787	62.944	0.3872	2.7729	0.4183	2.8292	0.3306	1.5774	0.0886	0.0893	0.0821	0.0878
		θ	0.0719	0.2113	0.0299	0.0946	0.0177	0.075	0.2595	0.6981	0.1684	0.6023	0.1155	0.3508
	20%	α	5.7151	81.889	0.3787	4.9432	0.5791	5.1997	0.925	3.6538	0.0623	0.0989	0.1258	0.0973
		θ	0.0546	0.2239	0.0491	0.1092	0.0193	0.0798	0.7152	0.7374	0.1975	0.685	0.1309	0.3895

Table 5: The Bias and MSE for Three Different Robust Methods for Data with Different Ratio of Outliers when $n = 40$

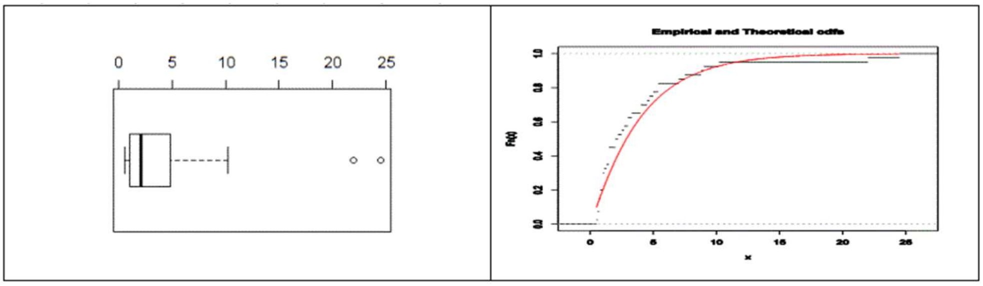
			Case 1						Case 1					
n			MLE		LS		MPS		MLE		LS		MPS	
	p		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	$\sigma^2 = 0.25$													
	5%	α	0.0481	0.0667	0.0103	0.04	0.0428	0.0435	0.0159	0.0039	0.003	0.0023	0.0013	0.0028
		θ	0.002	0.0044	0.0009	0.0027	-0.005	0.0025	-0.003	0.0192	0.0031	0.0117	0.0117	0.0134
	10%	α	0.0774	0.0796	0.0088	0.0428	0.0486	0.0437	0.0192	0.0052	0.001	0.0026	-3E-04	0.0029
		θ	0.0058	0.0053	0.0016	0.003	-0.004	0.0027	0.0262	0.0512	0.012	0.0172	0.0214	0.0157
	15%	α	0.1352	0.1099	0.0139	0.0454	0.062	0.0472	0.0384	0.0069	0.0022	0.0028	0.0039	0.003
		θ	0.004	0.0066	0.0002	0.0031	-0.005	0.003	0.005	0.0519	0.0079	0.0179	0.0151	0.0162
	20%	α	0.1952	0.1522	0.0129	0.0491	0.0732	0.052	0.0509	0.0104	0.0012	0.003	0.0065	0.0035
		θ	0.0036	0.0083	0.001	0.0034	-0.004	0.0033	0.0323	0.1268	0.0145	0.0215	0.0199	0.0186
	$\sigma^2 = 0.5$													
	5%	α	0.1199	0.2059	0.0099	0.121	0.0613	0.1212	0.0365	0.0138	0.0063	0.0074	0.0145	0.0109
		θ	-0.001	0.009	0.0064	0.0064	-8E-04	0.0068	-0.012	0.0349	0.0113	0.0258	0.0079	0.029
	10%	α	0.1812	0.3016	0.0358	0.1355	0.0819	0.1217	0.0204	0.0167	-5E-04	0.0104	0.0073	0.0106
40		θ	0.0104	0.012	0.0026	0.0071	0.0007	0.0068	0.1693	0.2872	0.0719	0.1076	0.0623	0.0696
	15%	α	0.2986	0.3921	0.0521	0.1481	0.1198	0.1285	0.0507	0.0215	-0.003	0.0105	0.0115	0.0102
		θ	0.006	0.0122	-4E-04	0.0074	-0.005	0.0069	0.1409	0.2764	0.0773	0.1097	0.0508	0.0527
	20%	α	0.4993	0.6104	0.0493	0.1647	0.1429	0.1383	0.1025	0.0352	-0.002	0.0114	0.0239	0.0133
		θ	-0.015	0.0113	0.0003	0.0076	-0.008	0.0076	0.0722	0.2253	0.0781	0.1313	0.0348	0.0495
	$\sigma^2 = 0.75$													
	5%	α	0.3089	0.8791	0.0802	0.4082	0.1421	0.3265	0.0793	0.0367	0.0206	0.0158	0.0389	0.0283
		θ	-0.006	0.0311	0.0095	0.0214	0.0024	0.0245	-0.039	0.058	0.0057	0.0458	0.0037	0.0504
	10%	α	0.4029	1.1911	0.0657	0.4362	0.1325	0.3699	0.1256	0.0634	0.0217	0.0176	0.0406	0.0295
		θ	0.013	0.0444	0.0178	0.0283	0.0086	0.0237	-0.063	0.065	0.0091	0.0449	0.0004	0.0543
	15%	α	0.6716	1.8218	0.1005	0.4841	0.1892	0.3771	0.1995	0.1208	0.0154	0.0184	0.0507	0.0328
		θ	-0.005	0.0474	0.0118	0.0294	-0.001	0.021	-0.108	0.0755	0.0135	0.0475	-0.013	0.0594
	20%	α	0.8972	2.7996	0.0753	0.5031	0.2201	0.4268	0.2684	0.2166	0.0074	0.021	0.0542	0.0353
		θ	-0.005	0.0488	0.0213	0.0326	0.0049	0.0222	-0.087	0.1418	0.0576	0.0883	0.0064	0.0591

Table 5: The Bias and MSE for Three Different Robust Methods for Data with Different Ratio of Outliers when n = 100

n	p		Case 1						Case 1					
			MLE		LS		MPS		MLE		LS		MPS	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	$\sigma^2 = 0.25$													
100	5%	α	0.0445	0.024	0.0042	0.0138	0.0372	0.0197	0.0128	0.0012	0.001	0.0007	0.0038	0.0005
		θ	-0.001	0.0013	0.0001	0.0008	-0.005	0.0011	-0.009	0.004	0.0004	0.0025	-0.001	0.0017
	10%	α	0.0953	0.0342	0.0023	0.0142	0.0462	0.0198	0.0273	0.002	0.0009	0.0007	0.0067	0.0005
		θ	-0.005	0.0014	0.0007	0.0009	-0.006	0.0011	-0.022	0.0046	0.001	0.0027	-0.003	0.0017
		α	0.1418	0.0526	0.0053	0.0164	0.0585	0.023	0.0456	0.0038	0.0017	0.0008	0.0109	0.0007
	15%	θ	-0.006	0.0019	0.0002	0.001	-0.007	0.0012	-0.041	0.0068	-1E-03	0.0032	-0.009	0.0018
	20%	α	0.2151	0.0868	0.0047	0.0171	0.0716	0.0256	0.0608	0.0057	0.0011	0.0008	0.0128	0.0008
		θ	-0.011	0.0024	7E-05	0.0011	-0.008	0.0012	-0.055	0.0086	0.0002	0.0033	-0.01	0.0019
	$\sigma^2 = 0.5$													
	5%	α	0.1055	0.1076	0.0172	0.057	0.0619	0.0525	0.0257	0.0053	0.0029	0.0028	0.0103	0.0024
		θ	-0.003	0.0053	0.0004	0.0034	-0.005	0.0022	-0.009	0.0166	0.0046	0.0114	-0.002	0.0063
	10%	α	0.2033	0.1536	0.0163	0.0597	0.0852	0.0594	0.0551	0.0089	0.0031	0.0029	0.0157	0.0027
		θ	-0.007	0.0055	0.0013	0.0038	-0.007	0.0032	-0.033	0.0179	0.005	0.0112	-0.007	0.0077
		α	0.3052	0.244	0.0182	0.0688	0.1002	0.0658	0.0889	0.0164	0.0049	0.0035	0.0238	0.0034
	15%	θ	-0.01	0.007	0.0008	0.004	-0.008	0.0035	-0.054	0.0269	0.0038	0.0149	-0.015	0.0086
	20%	α	0.5009	0.4665	0.017	0.0716	0.1285	0.0788	0.1271	0.0279	0.0036	0.0036	0.031	0.0042
		θ	-0.028	0.0093	0.0014	0.0043	-0.01	0.0039	-0.076	0.0348	0.007	0.0158	-0.019	0.0092
	$\sigma^2 = 0.75$													
	5%	α	0.1784	0.2705	0.0303	0.1342	0.0935	0.1171	0.0419	0.0137	0.0098	0.0067	0.0233	0.009
		θ	-0.004	0.0117	0.003	0.0078	-0.001	0.0061	-0.011	0.038	0.0055	0.0256	-0.004	0.0186
	10%	α	0.3415	0.413	0.0234	0.1367	0.1127	0.122	0.0873	0.0237	0.0066	0.007	0.0314	0.0098
		θ	-0.012	0.0123	0.0055	0.008	-0.004	0.007	-0.043	0.0392	0.0134	0.0269	-0.012	0.0193
		α	0.5096	0.671	0.0282	0.1639	0.1413	0.132	0.1464	0.0429	0.0087	0.0075	0.047	0.011
	15%	θ	-0.014	0.0151	0.0072	0.0097	-0.005	0.0074	-0.09	0.0447	0.0092	0.0273	-0.031	0.0205
	20%	α	0.7737	1.1957	0.0282	0.1702	0.1753	0.1497	0.2184	0.0777	0.0061	0.0079	0.0508	0.0113
		θ	-0.029	0.0185	0.0072	0.0104	-0.005	0.0077	-0.14	0.0574	0.0171	0.0311	-0.032	0.0271

5. Application of Real Data Analysis

In this section, we present the results of the EIW distribution parameter estimation using the robust and non-robust estimation methods on real data set. Data set: The data set on the active repair times (hours) for an airborne communication transceiver. This data set was analyzed by Jorgensen (2012). The data observations are: 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, and 24.50.



	MLE	LS	LAD	MB	MH
$\hat{\alpha}$ $(\hat{\sigma}_\alpha)$	1.1137 (0.2445)	1.0452 (1.0600)	1.1437 (0.1993)	0.9241 (0.1629)	0.9591 (0.0561)
$\hat{\theta}$ $(\hat{\sigma}_\theta)$	3.7349 (0.7820)	3.2954 (3.5999)	3.1981 (0.6175)	3.9808 (0.3546)	4.0626 (0.3764)
D $(P-Value)$	0.1583 (0.2684)	0.1288 (0.5201)	0.1249 (0.5605)	0.1388 (0.4239)	0.1264 (0.5443)

By examining the numerical results in Table (7), we observe the best result is from the MH method, since it has the least standard deviation, followed by the MB method and the LAD method, and so on as shown in Table 7. The goodness of fit test for the models; the Kolmogorov-Smirnov test shows that the efficiency of the test increases with the robust methods for the overall fit of the models with outliers as shown in Figure (3).

6. Conclusion

In this paper, we present various parameter estimation methods for the generalized exponential distribution using a complete dataset in the presence of outliers. We assessed the ability of the classical and robust estimation methods in determining the parameter estimates of the EIW distribution using complete dataset in the presence of various percentage of outliers. The classical estimation methods are the MLE, LS, and MPS methods and the non-classical estimation methods are the LAD and M-estimation. The M-estimation method is used to minimize the invariant errors in the Bi-square objective function and Huber objective function. The Monte Carlo simulation results showed that the M-estimation method using the Huber objective function outperformed the other methods in terms of Bias and MSE values. The simulation results also showed that the MLE method, which is a classical method is more suitable than the LS and MPS methods for estimating the EIW parameters. The real datasets application confirmed that the M estimation method is very much suitable for estimating the EIW parameters. We concluded that the M-estimation method using Huber object function is the robust method in estimating the parameters of the EIW distribution for a complete dataset in the presence of various percentage of outliers. This study concluded that, before performing the study and analyzing data of it, researchers must examine data through (Boxplot, the goodness of fit, etc.) and verify the presence or absence of outliers to determine the optimal way for estimation of the model's parameters.

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