Extended Exponentiated Inverse Lindley Distribution:

Model, Properties and Applications

Mahmoud.A.Eltehiwy

Faculty of the Politics and Economics, Department of Statistics, Beni-suef University, Egypt.

Email: Mahmoud.mohamed111@com.svu.edu.com

Abstract

In this article, a four-parameter generalization of inverse Lindley distribution is obtained, with the purpose of obtaining a more flexible model relative to the behavior of hazard rate functions. Various statistical properties such as density, hazard rate functions, moments, moment generating functions, stochastic ordering, Renyi entropy, distribution of k^{th} order statistics has been derived. The method of maximum likelihood estimation has been used to estimate parameters. Further confidence intervals are also obtained. Finally, applicability of the proposed model to the real data is analyzed. A comparison has also been made with some existing distributions.

***Keywords and Phrases*:** Lambert function, maximum likelihood estimation, order statistics, Extended inverse Lindley distribution, Stochastic ordering.

1. Introduction

Survival and reliability analysis is a very important branch of statistics. It has many applications in many applied sciences, such as engineering, public health, actuarial science, biomedical studies, demography, and industrial reliability. The failure behavior of any system can be considered as a random variable due to the variations from one system to another resulting from the nature of the system. Therefore, it seems logical to find a statistical model for the failure of the system. In other applications, survival data are categorized by their hazard rate, e.g., the number of deaths per unit in a period of time. The modeling of survival data depends on the behavior of the hazard rate. The hazard rate may belong to the monotone (non-increasing and non-decreasing hazard rate) or non-monotone (bathtub and upside-down bathtub [UBT] or unimodal hazard rate). Several lifetime models have been suggested in statistics literature to model survival data. The Weibull distribution is one of the most popular and widely used models in life testing and reliability theory. Lindley (1958) suggested a one-parameter distribution as an alternative model for survival data. This model is known as Lindley distribution. However, the Weibull and Lindley distributions are restricted when data shows non-monotone hazard rate shapes, such as the unimodal hazard rate function (Almalki and Nadarajah 2014; Almalki and Yuan 2013).

There are several real applications where the data show the non-monotone shape for their hazard rate. For example, Langlands et al. (1997) studied the data of 3878 cases of breast carcinoma seen in Edinburgh from 1954 to 1964 and noticed that mortality was initially low in the first year, reaching a peak in the subsequent years, and then declining slowly. Another real problem was analyzed by Efron (1988) who, using head and neck cancer data, found the hazard rate initially increased, reached a maximum, and decreased before it finally stabilized due to therapy. The inverse versions of some existing probability distributions, such as inverse Weibull, inverse Gaussian, inverse gamma, and inverse Lindley, show non-monotone shapes for their hazard rates; hence, we were able to model a non-monotone shape data. Erto and Rapone (1984) showed that the inverse Weibull distribution is a good fit for survival data, such as the time to breakdown of an insulating fluid subjected to the action of constant tension. The use of Inverse Weibull was comprehensively described by Murthy et al. (2004). Glen (2011) proposed the inverse gamma distribution as a lifetime model in the context of reliability and survival studies. Recently, a new upside-down bathtub-shaped hazard rate model for survival data analysis was proposed by Sharma et al. (2014) by using transmuted Rayleigh distribution. Sharma et al. (2015) introduced the inverse Lindley distribution as a one-parameter model for a stress-strength reliability model. Sharma et al. (2016) generalized the inverse Lindley into a two-parameter model called "the generalized inverse Lindley distribution. Finally, a new reliability model of inverse gamma distribution referred to as "the generalized inverse gamma distribution" was proposed by Mead (2015), which includes the inverse exponential, inverse Rayleigh, inverse Weibull, inverse gamma, inverse Chi square, and other inverse distributions.

Lindley (1958) proposed the Lindley distribution in the context of the Bayes theorem as a counter example of fiducial statistics with the probability density function (pdf)

$$f(y;\beta) = \frac{\beta^2}{1+\beta}(1+y)e^{-\beta y}\ ;\ y,\beta > 0. \qquad (1)$$

Shanker et al. (2013) proposed two parameter extensions of the Lindley distribution with the pdf

$$f(z;\beta) = \frac{\beta^2}{\beta+\gamma}(1+\gamma z)e^{-\beta z}\ ;\ \gamma,\beta,z > 0. \qquad (2)$$

Ghitany et al. (2008) discussed the Lindley distribution and its applications extensively and showed that the Lindley distribution is a better fit than the exponential distribution based on the waiting time at the bank for service. The lindley distribution has been extended by different researchers including Zakerzadeh and Dolati (2009), Nadarajah et al. (2011), Bakouch et al. (2012), Shanker and Mishra (2013), Ghitany et al. (2013), Ashour and Eltehiwy (2015). The inverse Lindley distribution was proposed by Sharma et al. (2015) using the transformation $X = \frac{1}{Y}$ with the pdf

$$f(x;\beta) = \frac{\beta^2}{1+\beta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\beta}{x}}; \qquad \beta, x > 0,$$

where Y is a random variable having pdf (1).

Another two parameters inverse Lindley distribution introduced by Sharma et al. (2016), called "the generalized inverse Lindley distribution," is a new statistical inverse model for upside-down bathtub survival data that uses the transformation $X = Y^{-\frac{1}{\alpha}}$ with the pdf

$$f(x;\beta,\alpha) = \frac{\alpha\beta^2}{1+\beta}\left(\frac{1+x^\alpha}{x^{2\alpha+1}}\right)e^{-\frac{\beta}{x^\alpha}}; \qquad \beta, \alpha, x > 0,$$

with Y being a random variable having pdf (1). Note that Barco et al. (2017) also obtained the generalized inverse Lindley distribution by taking the transformation $X = Y^{-\frac{1}{\alpha}}$ where Y follows inverse Lindley distribution known as inverse power Lindley distribution with the same pdf.

Using the transformation $X = Z^{-\frac{1}{\alpha}}$, Alkarni (2015) introduced a more flexible distribution with three parameters called "extended inverse Lindley distribution", (EIL) with the pdf

$$f(x;\beta,\gamma,\alpha) = \frac{\alpha\beta^2}{\beta+\gamma}\left(\frac{\gamma+x^\alpha}{x^{2\alpha+1}}\right)e^{-\frac{\beta}{x^\alpha}}; \qquad \gamma, \beta, \alpha, x > 0. \quad (3)$$

The pdf (3) can be shown as a mixture of two distributions as follows:

$$f(x;\beta,\gamma,\alpha) = pf_1(x) + (1-p)f_2(x)$$

Where

$$p = \frac{\beta}{\beta+\gamma},\ f_1 = \frac{\alpha\beta}{x^{\alpha+1}}e^{-\frac{\beta}{x^\alpha}},\ x > 0 \text{ and } f_2 = \frac{\alpha\beta^2}{x^{2\alpha+1}}e^{-\frac{\beta}{x^\alpha}},\quad x > 0.$$

We see that, EIL is a two-component mixture of inverse Weibull distribution (shape α and scale β) and generalized inverse gamma distribution (with shape parameters 2, α and scale θ), with mixing proportion $p = \beta/(\beta+\gamma)$.

Using the transformation $F(x) = [G(x)]^\theta$, where $G(x)$ is the cumulative distribution function of EIL and θ is a positive real number, we introduce a more flexible distribution with four parameter called "Extended Exponentiated Inverse Lindley distribution", (EEILD) and this gives us a better fit for upside-down bathtub data.

The aim of this paper is to introduce a new extension of inverse Lindley distribution with its mathematical properties. These include the shapes of the density and hazard rate functions, the moments, moment generating function and some associated measures, the quantile function, and stochastic orderings. Maximum likelihood estimation of the model parameters and their asymptotic standard distribution and confidence interval are derived. Rényi entropy as a measure of the uncertainty in the model is derived. Application of the model to a real data set is finally presented and compared to the fit attained by some other well-known distributions.

2. The extended exponentiated inverse Lindley distribution

The new extension of the generalized inverse Lindley distribution is most conveniently specified in terms of the cumulative distribution function:

$$F(x;\alpha,\beta,\theta,\gamma)=\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta}, \qquad (4)$$

for $x>0,\theta,\beta,\alpha>0$ and the corresponding pdf is given by

$$f(x;\alpha,\beta,\gamma,\theta)=\frac{\alpha\theta\beta^{2}}{\beta+\gamma}\left(\frac{\gamma+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^{\alpha}}}\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}\right)\right]^{\theta-1}, \qquad (5)$$

The corresponding hazard rate function is

$$h(x)=\frac{\alpha\theta\beta^{2}}{\beta+\gamma}\left(\frac{\gamma+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^{\alpha}}}\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}\right)\right]^{\theta-1}\{S(x)\}^{-1}, \qquad (6)$$

where

$$S(x)=1-\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta}. \qquad (7)$$

Note that Equation (5) has closed form survival functions and hazard rate functions.

For $(\theta=1)$ and $(\alpha=\theta=1)$ we have the pdfs of extended inverse Lindley and inverse Lindley distributions respectively.

The cdf of X, Eq. 4 can also be represented in an extended form

$$F(x)=\sum_{i=0}^{\infty}\binom{\theta}{i}p^{\theta-i}(1-p)^{i}F_{IW(\alpha,\beta)}^{\theta-i}(x)\,F_{GIG(2,\alpha,\beta)}^{i}(x),$$

where $F_{IW(\alpha,\beta)}(x)$ is the cumulative distribution of inverse Weibull with shape parameter α and scale θ and $F_{GIG(2,\alpha,\beta)}(x)$ is the cumulative distribution of generalized inverse gamma distribution (with shape parameters 2, α and scale θ).

The following propositions discuss the limiting behavior and other characteristics of EEIL distribution.

Proposition 1. The Extended inverse Lindley is a limiting distribution of the EEIL distribution when $\theta \to 1$.

Proof. Using Eq.4

$$\lim_{\theta\to 1} F(x;\alpha,\beta,\theta,\gamma) = \lim_{\theta\to 1}\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta}$$
$$= \left(1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}$$

Clearly, for $\theta \to 1$; the proposed model (EEIL distribution) given in Equation (5) reduces to the Extended inverse Lindley distribution. Therefore, the EEIL distribution can be viewed as an extension of the base model (which is asymptotically related to the usual one-parameter inverse Lindley distribution).

Proposition 2. For the pdf of the EEIL distribution, we have

$$\lim_{x\to 0} f(x;\alpha,\beta,\gamma,\theta) = 0$$

and

$$\lim_{x\to\infty^{+}} f(x;\alpha,\beta,\gamma,\theta) = 0$$

The EEIL distribution is always unimodal. Figures 1 illustrates some of the possible shapes of the pdf of the EEIL distribution for different values of the parameters α,β,θ and γ.

Plots of the pdf are shown in Figure 1. The pdfs appear always unimodal. The mode moves more to the right and the pdf becomes less peaked with increasing values of β. The mode moves more to the right and the pdf becomes less peaked with increasing values of θ. The pdf becomes more peaked with increasing values of γ.

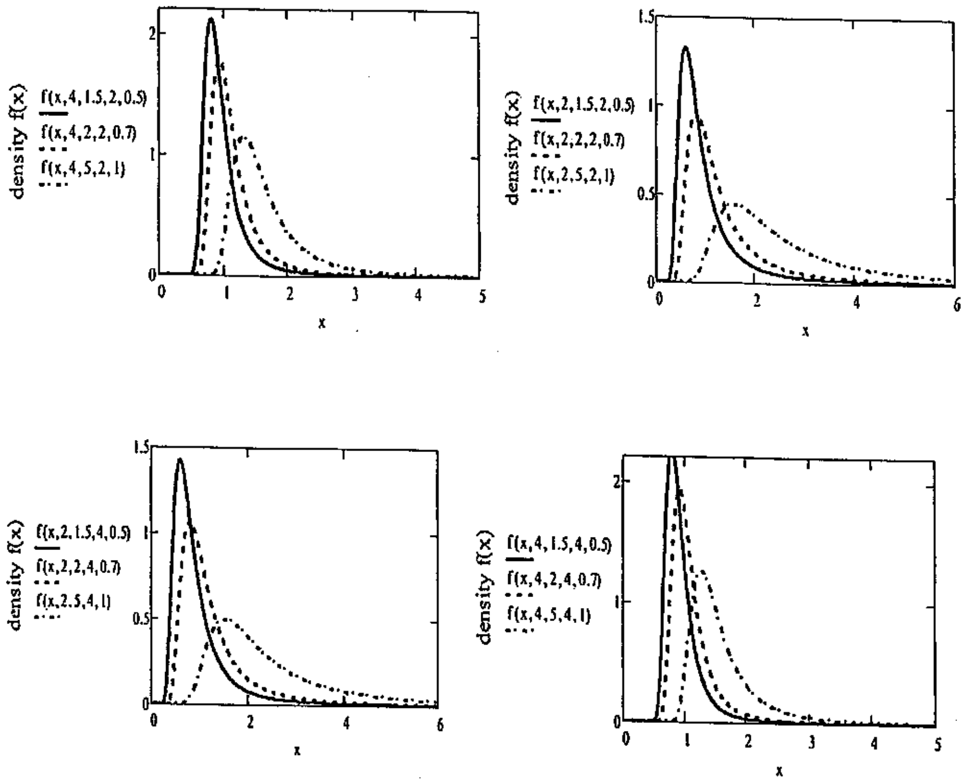

Figure1: Behavior of the density function of the Extended exponentiated inverse Lindley distribution for different values of $\beta = (1.5, 2, 5), \theta = (0.5, 0.7, 1)$, and $\alpha, \gamma = (2,2), (2,4), (4,2), (4,4)$.

The behavior of $h(x)$ in (6) of the EEILD for different values of the parameters α, β, θ and γ are showed graphically in figure 2.

By taking the limit of Equation (6) when $x \to 0$ and when $x \to \infty$ as follows

$$\lim_{x \to 0} h(x) = 0,$$

and

$$\lim_{x \to \infty} h(x) = 0$$

Because the hazard rate function of extended inverse Lindley distribution is always unimodel function in x, the new distribution is also a unimodal.

Figure 2 illustrates the behavior of the hazard rate function of the EEIL distribution at different values of the parameters involved. Concerning the hazard rate function of the extended exponentiated inverse Lindley distribution, which is shown in Figure 2, it notably has the shape of an upside-down bathtub, therefore being unimodal in x.

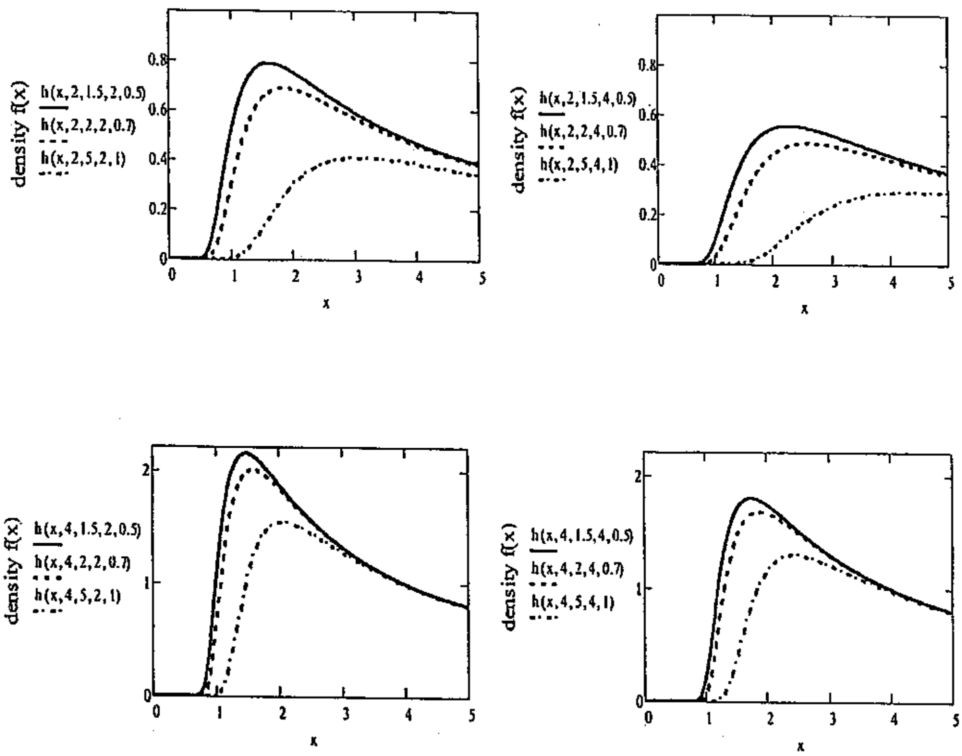

Figure 2: Behavior of the hazard rate function of the Extended exponentiated inverse Lindley distribution for different values of $\beta = (1.5, 2, 5), \theta = (0.5, 0.7, 1)$, and $\alpha, \gamma = (2,2), (2,4), (4,2), (4,4)$.

This attractive flexibility makes the EEIL hazard rate function useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

Special cases of the EEIL distribution

The EEIL distribution contains some well-known distributions as sub-models, described below in brief.

- When $\theta = 1$, we obtain the extended inverse Lindley distribution.
- When $\theta = \alpha = \gamma = 1$, we obtain the inverse Lindley distribution.
- When $\theta = \gamma = 1$, we obtain the generalized inverse Lindley distribution.
- When $\theta = 1$ and $\gamma = 0$, we obtain the inverse Weibull distibution.

2. Moments

Theorem 1. If X be a random variable following EEILD given as in (5), then, the r-th moments $E(X^r)$ about origin is given by

$$E(X^r) = (\theta\beta)^{\frac{r}{\alpha}} \sum_{i=0}^{\infty} \binom{\theta-1}{i} \frac{\left[\gamma^2\left(i+1-\frac{r}{\alpha}\right)+\gamma\theta\beta\right]}{[\theta(\gamma+\beta)]^{i+1}} \Gamma\left(1+i-\frac{r}{\alpha}\right).$$

Proof.

$$E(X^r) = \int_0^{\infty} x^r f(x) dx$$

$$E(X^r) = \frac{\alpha\theta\beta^2}{\beta+\gamma} \int_0^{\infty} x^r \left(\frac{\gamma+x^\alpha}{x^{2\alpha+1}}\right) e^{-\frac{\beta\theta}{x^\alpha}} \left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^\alpha}\right)\right]^{\theta-1} dx \quad (8)$$

Using the following binomial series expansion of $\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^\alpha}\right)\right]^{\theta-1}$ given by

$$\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^\alpha}\right)\right]^{\theta-1} = \sum_{i=0}^{\infty} \binom{\theta-1}{i} \left(\frac{\gamma\beta}{(\gamma+\beta)x^\alpha}\right)^i$$

Equation (8) takes the following form

$$E(X^r) = \frac{\alpha\theta\beta^2}{\beta+\gamma} \sum_{i=0}^{\infty} \binom{\theta-1}{i} \left(\frac{\gamma\beta}{(\gamma+\beta)}\right)^i \left[\int_0^{\infty} \frac{\gamma}{x^{\alpha(2+i)-r+1}} e^{-\frac{\beta\theta}{x^\alpha}} dx + \int_0^{\infty} \frac{1}{x^{\alpha(i+1)-r+1}} e^{-\frac{\beta\theta}{x^\alpha}} dx\right] (9)$$

Let $t = x^\beta$ and using the definition of inverse gamma distribution (9) reduces to

$$E(X^r) = \frac{\theta\beta^2}{\beta+\gamma} \sum_{i=0}^{\infty} \binom{\theta-1}{i} \left(\frac{\gamma\beta}{(\gamma+\beta)}\right)^i \left[\frac{\gamma\Gamma\left(2+i-\frac{r}{\alpha}\right)}{(\theta\beta)^{2+i-\frac{r}{\alpha}}} + \frac{\Gamma\left(1+i-\frac{r}{\alpha}\right)}{(\theta\beta)^{1+i-\frac{r}{\alpha}}}\right]$$

$$\mu_r' = E(X^r) = (\theta\beta)^{\frac{r}{\alpha}} \sum_{i=0}^{\infty} \binom{\theta-1}{i} \frac{\left[\gamma^2\left(i+1-\frac{r}{\alpha}\right)+\gamma\theta\beta\right]}{[\theta(\gamma+\beta)]^{i+1}} \Gamma\left(1+i-\frac{r}{\alpha}\right) \quad (10)$$

For r^{th} moment to exist, the constraint $\alpha > r$ must be satisfied. Note that for $\theta = 1$, (10) reduces to the r^{th} moment of extended inverse Lindley distribution.

3. Moment generating function

Theorem 2. Let X be a random variable of EEILD having pdf (5), then the moment generating function of X, $M_X(t)$, is

$$M_X(t) = \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\theta\beta)^{\frac{n}{\alpha}} \binom{\theta-1}{i} \frac{\left[\gamma^{i+1}\left(i+1-\frac{n}{\alpha}\right)+\gamma^i\theta\beta\right]}{[\theta(\gamma+\beta)]^{i+1}} \Gamma\left(1+i-\frac{n}{\alpha}\right),$$

Proof. $M_X(t) = \int_0^{\infty} e^{tx} f(x) dx$

$$M_X(t) = \frac{\alpha\theta\beta^2}{\beta+\gamma}\int_0^\infty e^{tx}\left(\frac{\gamma+x^\alpha}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^\alpha}}\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^\alpha}\right)\right]^{\theta-1}dx \quad (11)$$

Using the following binomial series expansion of $\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^\alpha}\right)\right]^{\theta-1}$ given by

$$\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^\alpha}\right)\right]^{\theta-1} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}\left(\frac{\gamma\beta}{(\gamma+\beta)x^\alpha}\right)^i$$

Equation (11) takes the following form

$$M_X(t) = \frac{\alpha\theta\beta^2}{\beta+\gamma}\sum_{i=0}^{\infty}\binom{\theta-1}{i}\left(\frac{\gamma\beta}{(\gamma+\beta)}\right)^i\left[\int_0^\infty e^{tx}\left(\frac{\gamma e^{-\frac{\beta\theta}{x^\alpha}}}{x^{\alpha(2+i)-r+1}}+\frac{e^{-\frac{\beta\theta}{x^\alpha}}}{x^{\alpha(i+1)-r+1}}\right)dx\right]$$
(12)

Using the following expansion of e^{tx} given by

$$e^{tx} = \sum_{n=0}^{\infty}\frac{t^n x^n}{n!}$$

Equation (12) can be rewritten as follow:

$$M_X(t) = \frac{\alpha\theta\beta^2}{\beta+\gamma}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\binom{\theta-1}{i}\left(\frac{\gamma\beta}{(\gamma+\beta)}\right)^i\frac{t^n}{n!}\int_0^\infty\left(\frac{\gamma e^{-\frac{\beta\theta}{x^\alpha}}}{x^{\alpha(2+i)-n+1}}+\frac{e^{-\frac{\beta\theta}{x^\alpha}}}{x^{\alpha(i+1)-n+1}}\right)dx$$

Let $v = x^\alpha$ and using the definition of inverse gamma distribution the above equation becomes

$$M_X(t) = \frac{\theta\beta^2}{\beta+\gamma}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\binom{\theta-1}{i}\left(\frac{\gamma\beta}{(\gamma+\beta)}\right)^i\frac{t^n}{n!}\left[\frac{\gamma\Gamma\left(2+i-\frac{n}{\alpha}\right)}{(\theta\beta)^{2+i-\frac{n}{\alpha}}}+\frac{\Gamma\left(1+i-\frac{n}{\alpha}\right)}{(\theta\beta)^{1+i-\frac{n}{\alpha}}}\right]$$

Thus, the moment generating function of EEILD is given by:

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^n}{n!}(\theta\beta)^{\frac{n}{\alpha}}\binom{\theta-1}{i}\frac{\left[\gamma^{i+1}\left(i+1-\frac{n}{\alpha}\right)+\gamma^i\theta\beta\right]}{[\theta(\gamma+\beta)]^{i+1}}\Gamma\left(1+i-\frac{n}{\alpha}\right)$$

4. Quantile Function

Let X denotes a random variable with the probability density function (Eq. 5). The quantile function, say $Q(p)$, defined by $F\big(Q(p)\big) = p$ is the root of the equation

$$\left(1+\frac{\gamma\beta}{(\gamma+\beta)Q(p)^\alpha}\right)e^{-\frac{\beta}{Q(p)^\alpha}} = p^{1/\theta} \quad (13)$$

for $0 < p < 1$. Multiplying (13) both sides by $e^{-\gamma-\beta}$ we get,

$$-\left(\gamma+\beta+\frac{\gamma\beta}{Q(p)^\alpha}\right)e^{-\left(\gamma+\beta+\frac{\beta}{Q(p)^\alpha}\right)} = -(\gamma+\beta)p^{1/\theta}e^{-(\gamma+\beta)}$$

Using the Lambert W function which is the solution of the equation $W(z)e^{W(z)}$, where z is a complex number, we have

$$W\left(-p^{1/\theta}(\gamma+\beta)e^{-(\gamma+\beta)}\right) = -\left(\gamma+\beta+\frac{\gamma\beta}{Q(p)^{\alpha}}\right)$$

The negative Lambert W function of the real argument $-p(\gamma+\beta)e^{(\gamma+\beta)}$ is

$$W_{-1}\left(-p^{1/\theta}(\gamma+\beta)e^{-(\gamma+\beta)}\right) = -\left(\gamma+\beta+\frac{\gamma\beta}{Q(p)^{\alpha}}\right)$$

Which upon solving for $Q(p)$ results in

$$Q(p) = \left[-\gamma-\frac{1}{\beta}-\frac{1}{\gamma\beta}W_{-1}\left(-p^{1/\theta}(\gamma+\beta)e^{-(\gamma+\beta)}\right)\right]^{-\frac{1}{\alpha}}$$

Using above equation, the quartiles of the extended exponentiated inverse Lindley distribution can be determined. Median of extended exponentiated inverse Lindley distribution is given by

$$Q(0.5) = \left[-\gamma-\frac{1}{\beta}-\frac{1}{\gamma\beta}W_{-1}\left(-(0.5)^{\frac{1}{\theta}}(\gamma+\beta)e^{-(\gamma+\beta)}\right)\right]^{-\frac{1}{\alpha}}$$

5. Rényi entropy

Entropy is a measure of variation of the uncertainty in the distribution of any random variable. It provides important tools to indicate variety in distributions at moments in time and to analyze evolutionary processes over time. For a given probability distribution, Rényi (1961) gave an expression of the entropy function, so called Rényi entropy, defined by

$$Re(\delta) = \frac{1}{1-\delta}\log\left[\int f^{\delta}(x)dx\right]$$

Where $\delta > 0$ and $\delta \neq 1$

$$Re(\delta) = \frac{1}{1-\delta}\log\left[\int_0^{\infty}\left[\frac{\alpha\theta\beta^2}{\beta+\gamma}\left(\frac{\gamma+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^{\alpha}}}\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}\right)\right]^{\theta-1}\right]^{\delta}\right]$$

Using the binomial series expansion $(1+x)^n = \sum_{i=0}^{\infty}\binom{n}{i}z^i$ for $\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}\right)^{\theta-1}\right]^{\delta}$ and $\left[\gamma\left(1+\frac{1}{x^{\alpha}}\right)\right]^{\delta}$, we have

$Re(\delta) =$

$$\frac{1}{1-\delta}\log\left(\frac{\alpha\theta\gamma\beta^2}{\beta+\gamma}\right)^{\delta}\sum_{i=0}^{\infty}\binom{\delta\theta-\delta}{i}\left(\frac{\gamma\beta}{(\gamma+\beta)}\right)^{i}\sum_{j=0}^{\infty}\binom{\delta}{j}\int_0^{\infty}\frac{e^{-\frac{\delta\beta\theta}{x^{\alpha}}}}{x^{\alpha(\delta+i+j)+\delta}}dx \quad (14)$$

Letting $x^{\alpha} = t$ and using the definition of inverse gamma distribution (14) reduces to

$Re(\delta) =$

$$\frac{1}{1-\delta}\log\left[\frac{\alpha^{\delta-1}\theta^{\delta}\gamma^{\delta}\beta^{2\delta}}{(\beta+\gamma)^{\delta}}\right]\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\delta\theta-\delta}{i}\left(\frac{\gamma\beta}{(\gamma+\beta)}\right)^{i}\binom{\delta}{j}\frac{\Gamma\left(\delta+i+j+\frac{\delta-1}{\alpha}\right)}{(\delta\beta\theta)^{\delta+i+j+\frac{\delta-1}{\alpha}}}.$$

6. Distribution of order statistics

The pdf of k^{th} order statistic is given by:

$$g_k(x) = \frac{n!\,g(x)}{(k-1)!\,(n-k)!}[G(x)]^{k-1}[1-G(x)]^{n-k}$$

Where $g(x)$ and $G(x)$ denotes the pdf and cdf respectively.
Using the binomial series expansion $(1-x)^n = \sum_{i=0}^{\infty}\binom{n}{i}(-1)^i(x)^i$, we get

$$g_k(x) = \frac{n!\,g(x)}{(k-1)!\,(n-k)!}\sum_{i=0}^{\infty}\binom{n-k}{i}(-1)^i\,[G(x)]^{i+k-1}$$

$$g_k(x) = \frac{n!}{(k-1)!(n-k)!}\sum_{i=0}^{\infty}\binom{n-k}{i}(-1)^i\frac{\alpha\theta\beta^2}{\beta+\gamma}\left(\frac{\gamma+x^\alpha}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^\alpha}}\left[\left(1+\frac{\gamma\beta}{(\gamma+\beta)x^\alpha}\right)\right]^{\theta i+\theta k-1}$$

$$= \frac{\alpha\theta\beta^2}{\beta+\gamma}\frac{n!}{(k-1)!\,(n-k)!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^i\binom{n-k}{i}\binom{\theta i+\theta k-1}{j}\left(\frac{\gamma\beta}{(\gamma+\beta)}\right)^j\left(\frac{1}{(x^\alpha)}\right)^j\left(\frac{\gamma+x^\alpha}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^\alpha}}$$

$$= \frac{\alpha\theta\gamma^j\beta^{2+j}}{(\beta+\gamma)^{j+1}}\frac{n!}{(k-1)!(n-k)!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^i\binom{n-k}{i}\binom{\theta i+\theta k-1}{j}\left(\frac{\gamma+x^\alpha}{x^{(j+2)\alpha+1}}\right)e^{-\frac{\beta\theta}{x^\alpha}}.$$

7. Stochastic Orders

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. Suppose X_i is distributed according to (Eqs. 4 and 5) with common parameter β and parameters θ_i and α_i for $i = 1,2$. Let F_i denote the cumulative distribution of X_i and let f_i denote the probability density function of X_i. A random variable X_1 is said to be smaller than a random variable X_2 in the

I. Stochastic order ($X_1 \le_{st} X_2$) if $F_1(x) \ge F_2(x)$ for all x.
II. Hazard rate order ($X_1 \le_{hr} X_2$)if $h_1(x) \ge h_2(x)$ for all x.
III. Likelihood ratio order ($X_1 \le_{Lr} X_2$) if $\frac{f_1(x)}{f_2(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$X_1 \le_{Lr} X_2 \Rightarrow X_1 \le_{hr} X_2 \Rightarrow X_1 \le_{st} X_2$$

The EEILD is ordered with respect to the strongest "likelihood ratio" ordering as shown in the following theorem:

Theorem3.
Let $X_1 \sim EEILD(\theta_1, \gamma, \beta_1, \alpha_1)$ and $X_2 \sim EEILD(\theta_2, \gamma, \beta_2, \alpha_2)$. If $\beta_1 = \beta_2$, and $\theta_2 \geq \theta_1$ (or if $\beta_2 \geq \beta_1$and $\theta_1 = \theta_2$), then $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$.
Proof.

$$\frac{f_2(x)}{f_1(x)} = \frac{\alpha_2\beta_2^2\theta_2(\gamma_1+\beta_1)}{\alpha_1\beta_1^2\theta_1(\gamma_2+\beta_2)}\left(\frac{1+x^{\alpha_2}}{1+x^{\alpha_1}}\right)\left(\frac{x^{2\alpha_1+1}}{x^{2\alpha_2+1}}\right) exp - \left(\frac{\theta_2\beta_2}{x^{\alpha_2}} - \frac{\theta_1\beta_1}{x^{\alpha_1}}\right)$$

$$\left[\frac{\left(1+\frac{\gamma_2\beta_2}{(\gamma_2+\beta_2)x^{\alpha_2}}\right)^{\theta_2-1}}{\left(1+\frac{\gamma_1\beta_1}{(\gamma_1+\beta_1)x^{\alpha_1}}\right)^{\theta_1-1}}\right]$$

Setting $\alpha_1 = \alpha_2$ and $\gamma_1 = \gamma_2$.

Case 1: $\beta_1 = \beta_2$ and $\theta_2 \geq \theta_1$ we obtained $\frac{d}{dx}\left(\frac{f_2(x)}{f_1(x)}\right)$ as an increasing function of x.

Case 2: $\beta_1 \geq \beta_2$ and $\theta_2 = \theta_1$ we obtained $\frac{d}{dx}\left(\frac{f_2(x)}{f_1(x)}\right)$ as an increasing function of x.

This implies $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$.

8. Maximum Likelihood Estimation of Parameters

Let $x_1, \ldots, x_n$ be a random sample of size n from EEILD. Then, the log-likelihood function is given by

$$\mathcal{L}(\alpha,\beta,\gamma,\theta) = \sum_{i=1}^n \ln f(x_i),$$

$$= n[\ln(\alpha) + 2\ln(\beta) + \ln(\theta) - \ln(\beta+\gamma)] + \sum_{i=1}^n \ln(\gamma + x_i^\alpha) -$$

$$(2\alpha+1)\sum_{i=1}^n \ln(x_i) - \theta\beta\sum_{i=1}^n x_i^{-\alpha} + (\theta-1)\sum_{i=1}^n \ln\left[1 + \frac{\gamma\beta}{(\gamma+\beta)x_i^\alpha}\right] \quad (15)$$

The MLEs $\hat{\theta}, \hat{\beta}, \hat{\gamma}, \hat{\alpha}$ of $\theta, \beta, \gamma, \alpha$ are then the solutions of the following non-linear equations:

$$\frac{\partial}{\partial\alpha}\mathcal{L}(\alpha,\beta,\gamma,\theta) = \frac{n}{\alpha} + \sum_{i=1}^n \frac{x_i^\alpha \ln(x_i)}{x_i^\alpha + \gamma} - 2\sum_{i=1}^n \ln(x_i)$$

$$+\theta\beta\sum_{i=1}^n x_i^{-\alpha} . \ln(x_i) - \frac{(\theta-1)\gamma\beta}{\gamma+\beta}\sum_{i=1}^n \left[\frac{x_i^{-\alpha}\ln(x_i)}{\left[1+\frac{\gamma\beta}{(\gamma+\beta)x_i^\alpha}\right]}\right] = 0,$$

$$\frac{\partial}{\partial\beta}\mathcal{L}(\alpha,\beta,\gamma,\theta) = \frac{n(\beta+2)}{\beta(\beta+\gamma)} - \theta\sum_{i=1}^n x_i^{-\alpha} + (\theta-1)$$

$$\sum_{i=1}^n \left[\frac{1}{1+\frac{\gamma\beta}{(\gamma+\beta)x_i^\alpha}}\right]\left[\frac{\gamma^2}{x_i^\alpha(\gamma+\beta)^2}\right] = 0$$

$$\frac{\partial}{\partial\theta}\mathcal{L}(\alpha,\beta,\gamma,\theta) = \frac{n}{\theta} - \beta\sum_{i=1}^n x^{-\alpha} + \sum_{i=1}^n \ln\left[1 + \frac{\gamma\beta}{(\gamma+\beta)x_i^\alpha}\right] = 0$$

$$\frac{\partial}{\partial\gamma}\mathcal{L}(\alpha,\beta,\gamma,\theta) = \frac{n}{\beta+\gamma} + \sum_{i=1}^n \frac{1}{(\gamma+x_i^\alpha)} + (\theta-1)\sum_{i=1}^n \left[\frac{1}{1+\frac{\gamma\beta}{(\gamma+\beta)x_i^\alpha}}\right]\left[\frac{\beta^2}{x_i^\alpha(\gamma+\beta)^2}\right] = 0$$

The above non-linear system of equations are solved by numerical iteration technique and maximum likelihood estimates are obtained.

For the four parameters extended exponentiated inverse Lindley distribution $EEILD(\theta,\beta,\alpha)$, all the second order derivatives exist. Thus we have the inverse dispersion matrix is

$$\begin{pmatrix}\hat{\theta}\\ \hat{\beta}\\ \hat{\alpha}\\ \hat{\gamma}\end{pmatrix}\sim N\left[\begin{pmatrix}\theta\\ \beta\\ \alpha\\ \gamma\end{pmatrix},\begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34}\\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44}\end{pmatrix}\right]$$

$$V^{-1}=-E\left[\begin{pmatrix}V_{11} & \dots & V_{14}\\ \dots & \dots & \dots\\ V_{41} & \dots & V_{44}\end{pmatrix}\right]=-E\begin{pmatrix}\frac{\partial^2\mathcal{L}}{\partial\theta^2} & \dots & \frac{\partial^2\mathcal{L}}{\partial\theta\partial\gamma}\\ \dots & \dots & \dots\\ \frac{\partial^2\mathcal{L}}{\partial\theta\partial\gamma} & \dots & \frac{\partial^2\mathcal{L}}{\partial\gamma^2}\end{pmatrix} \qquad (15)$$

Equation 15 is the variance covariance matrix of the $EEILD(\theta,\beta,\alpha,\gamma)$

$$V_{11}=\frac{\partial^2\mathcal{L}}{\partial\theta^2}\quad V_{12}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\beta}\quad V_{13}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha}\quad V_{14}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\gamma}\quad V_{22}=\frac{\partial^2\mathcal{L}}{\partial\beta^2}$$

$$V_{23}=\frac{\partial^2\mathcal{L}}{\partial\beta\partial\alpha}\quad V_{24}=\frac{\partial^2\mathcal{L}}{\partial\beta\partial\gamma}\quad V_{33}=\frac{\partial^2\mathcal{L}}{\partial\alpha^2}\quad V_{34}=\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\gamma}\quad V_{44}=\frac{\partial^2\mathcal{L}}{\partial\gamma^2}$$

The second derivatives of $\mathcal{L}$ can be derived as follows:

$$\frac{\partial^2\mathcal{L}}{\partial\theta^2}=\frac{-n}{\theta^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha^2}=-\frac{n}{\alpha^2}+\sum_{i=1}^{n}\frac{x_i^{\alpha}\ln(x_i)^2}{\left(x_i^{\alpha}+\gamma\right)^2}-\theta\beta\sum_{i=1}^{n}\frac{\ln x_i}{x_i^{\alpha}}+(\theta-1)\left(\frac{\gamma\beta}{\gamma+\beta}\right)\sum_{i=1}^{n}\frac{x_i^{-\alpha}\ln(x_i)^2}{\left(1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right)^2}$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta^2}=\left[\frac{-2n}{\beta^2}+\frac{n}{(\beta+1)^2}\right]-\frac{(\theta-1)\gamma^2}{(\gamma+\beta)^3}\sum_{i=1}^{n}\left(\frac{1}{x_i^{\alpha}\left(1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right)}\right)\left[\frac{\frac{\gamma(2\beta+\gamma)}{(\gamma+\beta)x_i^{\alpha}}+2}{\left(1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right)}\right]$$

$$\frac{\partial^2\mathcal{L}}{\partial\gamma^2}=\frac{-n}{(\gamma+\beta)^2}-\sum_{i=1}^{n}\frac{1}{\left(\gamma+x_i^{\alpha}\right)^2}-\frac{(\theta-1)\beta^2}{(\gamma+\beta)^3}\sum_{i=1}^{n}\left(\frac{1}{x_i^{\alpha}\left(1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right)}\right)\left[\frac{\frac{\beta(2\gamma+\beta)}{(\gamma+\beta)x_i^{\alpha}}+2}{\left(1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right)}\right]$$

$$\frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha}=\beta\sum_{i=1}^{n}\frac{\ln x_i}{x_i^{\alpha}}-\frac{\gamma\beta}{(\gamma+\beta)}\sum_{i=1}^{n}\left[\frac{x_i^{-\alpha}\ln(x_i)}{\left[1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right]}\right],$$

$$\frac{\partial^2\mathcal{L}}{\partial\theta\partial\beta}=-\sum_{i=1}^{n}x_i^{-\alpha}+\sum_{i=1}^{n}\left[\frac{1}{1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}}\right]\left[\frac{\gamma^2}{x^{\alpha}(\gamma+\beta)^2}\right],$$

$$\frac{\partial^2\mathcal{L}}{\partial\theta\partial\gamma}=\sum_{i=1}^{n}\left[\frac{1}{1+\frac{\gamma\beta}{(\gamma+\beta)x^{\alpha}}}\right]\left[\frac{\beta^2}{x^{\alpha}(\gamma+\beta)^2}\right]$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta}=\theta\sum_{i=1}^{n}\frac{\ln x_i}{x_i^{\alpha}}-\frac{\gamma^2(\theta-1)}{(\gamma+\beta)^2}\sum_{i=1}^{n}\frac{x_i^{-\alpha}\ln(x_i)}{\left[1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right]}\left[1-\frac{\gamma\beta}{x_i^{\alpha}(\gamma+\beta)+\gamma\beta}\right]$$

$$\frac{\partial^2 \mathcal{L}}{\partial\alpha\partial\gamma} = \sum_{i=1}^{n} \frac{-x_i^{\alpha}\ln x_i}{(\gamma+x_i^{\alpha})^2} + \frac{\beta^2(\theta-1)}{(\gamma+\beta)^2}\sum_{i=1}^{n} \frac{x_i^{-\alpha}\ln(x_i)}{\left[1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right]}\left[1-\frac{\gamma\beta}{x_i^{\alpha}(\gamma+\beta)+\gamma\beta}\right]$$

$$\frac{\partial^2 \mathcal{L}}{\partial\beta\partial\gamma} = \frac{-n}{(\beta+\gamma)^2} + \frac{2\gamma(\theta-1)}{(\gamma+\beta)^2}\sum_{i=1}^{n} \frac{x_i^{-\alpha}}{\left[1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right]}\left[1-\frac{\gamma}{(\gamma+\beta)}-\frac{2\gamma\beta^2 x_i^{-\alpha}}{\left(1+\frac{\gamma\beta}{(\gamma+\beta)x_i^{\alpha}}\right)(\gamma+\beta)^2}\right]$$

By solving this inverse dispersion matrix, these solution will yield the asymptotic variance and co-variances of these ML estimators for $\hat{\theta}$, $\hat{\beta}$, $\hat{\alpha}$ and $\hat{\gamma}$. By using (Eq.15), approximately $100(1-\alpha)\%$ confidence intervals for θ, β, α and γ can be determined as

$$\hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}} \qquad \hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}} \qquad \hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{33}} \qquad \hat{\gamma} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{44}}$$

where $Z_{\frac{\alpha}{2}}$ is the upper α-th percentile of the standand normal distribution.

9. Data Analysis

In this section, real data analysis is performed to illustrate the applicability of Extended Inverse Lindley distribution. The data set given in Table 1 represents the active repair times (hr) for an airborne communication transceiver. This data has been widely used by various authors and were initially used by Jorjensen (1982).

Table (1): Active repair times

0.50	0.60	0.60	0.70	0.70	0.70	0.80	0.80
1.00	1.00	1.00	1.00	1.10	1.30	1.50	1.50
1.50	1.50	2.00	2.00	2.20	2.50	2.70	3.00
3.00	3.30	4.00	4.00	4.50	4.70	5.00	5.40
5.40	7.00	7.50	8.80	9.00	10.20	22.00	24.50

The applicability of EEILD is demonstrated by using some statistical tools suchs as Kolmogrov-Smirnov statistic, Akaike information criterion (AIC) defined by $-2\log L + 2q$, Bayesian information criterion(BIC) defined by$-2\log L + q\log(n)$, where q is the number of estimated parameters and n is the sample size, and are compared with other distributions. AIC and BIC values estimates the quality of each model relative to each of the other models. The MLEs of the parameters are given in Table 3 and the statistical values mentioned above are computed and are given in Table 2. These values indicate that the proposed distribution fits well to the data compared to other tested distributions.

Table 2:Comparison criterion

Model	AIC	BIC	$-\log L$	K-S statistic	p-value
(EEILD)	182.3388	189.0943	87.16939	0.09597	0.8365
Exponentiated power Lindley (EPLD)	186.5721	191.6387	90.2861	0.10714	0.7239
Power Lindley (PL)	195.8854	199.2631	95.9427	0.1596	0.4361
Generalized Lindley (GL)	280.1635	283.54126	138.08175	0.5332	0.0012
Lindley distribution (LD)	199.8218	203.1995	97.9109	0.1907	0.2959
Exponentiated exponential (EED)	194.9158	198.2936	95.4579	0.1334	0.6472
Weibull distribution (WD)	195.0227	198.4005	95.5114	0.1540	0.4753

The EEILD takes the smallest K-S test statistic value and the largest value of its corresponding p-value. In addition, it takes the largest log likelihood.

Table (3) Parameters MLES

Model	α	β	θ	γ
EEILD	1.18001	0.73294	2.97043	0.33838
EPLD	0.23716	4.54017	82.8491	----------
PLD	0.79884	0.58672	---------	----------
GLD	---------	0.35884	0.74604	----------
LD	---------	0.42421	---------	----------
EE	---------	0.2678	1.1137	----------
WD	0.9604	0.2688	---------	----------

10. Generation Algorithms and Monte Carlo Simulation Study

In this section, the algorithms for generating random data from EEIL distribution are given. A simulation study was also conducted to check the performance and accuracy of maximum likelihood estimates of the EEIL model parameters.

10.1 Generation algorithms

In this subsection, different algorithms that can be used to generate random data from EEIL distribution are presented. The scheme for simulating from the exponentiated power Lindley distribution given in Ashour and Eltehiwy (2015) is used for simulating ELIP distribution. The following scheme is obtained:

Algorithm I. (mixture form of the inverse Lindley distribution)

1. Generate $U_i \sim uniform\ (0,1), i = 1, \dots, n;$
2. Generate $V_i \sim\ inverse\ Exponential\ (\beta), i = 1, \dots, n;$
3. Generate $G_i \sim inverse\ Gamma\ (2, \beta), i = 1, \dots, n.$
4. if $U_i^{1/\theta} \le \frac{\beta}{\gamma+\beta}$, then set $X_i = V_i^{1/\alpha}$, otherwise, set $X_i = G_i^{1/\alpha}, i = 1, \dots, n.$

Algorithm II. (mixture form of the Extended inverse Lindley distribution)

1. Generate $U_i \sim uniform\ (0,1), i = 1, \dots, n;$
2. Generate $Y_i \sim\ inverse\ Weibull\ (\alpha, \beta), i = 1, \dots, n;$
3. Generate$S_i \sim Generalized\ inverse\ Gamma\ (2, \alpha, \beta), i = 1, \dots, n.$
4. if $U_i^{1/\theta} \le \frac{\beta}{\gamma+\beta}$, then set $X_i = Y_i$, otherwise, set $X_i = S_i, i = 1, \dots, n.$

Algorithm III: (inverse CDF)

1. Generate $U_i \sim uniform(0,1), i = 1, \dots, n;$
2. Set

$$X_i = \left[-\gamma - \frac{1}{\beta} - \frac{1}{\gamma\beta} W_{-1}\left(-U_i^{1/\theta}(\gamma+\beta)e^{-(\gamma+\beta)}\right)\right]^{-\frac{1}{\alpha}}$$

10.2 Monte Carlo simulation study

In this subsection, we study the performance and accuracy of maximum likelihood estimates of the EEIL model parameters by conducting various simulations for different combinations of 6 sample sizes with two sets of parameter values. Algorithm II was used to generate random data from the EEIL distribution. The simulation study was repeated N = 10,000 times each with samples of size n = 25, 50, 100, 200, 400, 600 combined with parameter values (*I*): $\theta = 0.7, \beta = 1.5, \alpha = 2, \gamma = 2$ and (*II*): $\theta = 1.5, \beta = 0.7, \alpha = 4, \gamma = 4$. Four quantities were computed in this simulation study: (i) Average bias of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \alpha, \beta, \theta, \gamma$: $\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta)$; (ii) Root mean squared error (RMSE) of the MLE $\hat{\vartheta}$ of the parameter $\vartheta =$

$\alpha,\beta,\theta,\gamma$: $\left[\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta}-\vartheta)^2\right]^{0.5}$; (iii) Coverage probability (CP) of 95% confidence intervals of the parameter $\vartheta=\alpha,\beta,\theta,\gamma$; (iv) Average width (AW) of 95% confidence intervals of the parameter $\vartheta=\alpha,\beta,\theta,\gamma$. Table 4 presents the Average Bias, RMSE, CP and AW values of the parameters α,β,θ and γ for different sample sizes. According to the results, it can be concluded that as the sample size n increases, the RMSEs decrease toward zero. We also observe that for all the parameters, the biases decrease as the sample size n increases. The results show that the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and that the average confidence widths decrease as the sample size increases. Consequently, the MLE's and their asymptotic results can be used for estimating and constructing confidence intervals even for reasonably small sample sizes.

11. Concluding Remarks

In this paper, we have proposed a new family of distributions called extended exponentiated inverse Lindley distribution. We get the probability density functions for generalized inverse Lindley and inverse Lindley distributions as special cases from EEILD. Some mathematical properties along with estimation issues are addressed. The hazard rate function behavior of the extended exponentiated inverse Lindley distribution shows that the subject distribution can be used to model reliability data. The estimation of parameters is approached by the method of maximum likelihood. We present a simulation study to exhibit the performance and accuracy of maximum likelihood estimates of the EEIL model parameters. Real data application was also presented to illustrate the usefulness and applicability of the EEIL distribution.

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