Bayesian Estimation of $R = P[Y < X]$ for Burr Type XII Distribution Using Extreme Ranked Set Sampling

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Abstract

In this paper, a comparison of Bayesian estimators using non-informative priors under different loss functions, assuming that both the stress and the strength are independently identically Burr XII random variables. Bayes estimates of R based on extreme ranked set sampling are developed using Jeffery prior under symmetric and asymmetric loss functions and compared with the known estimators using simple random sampling technique. The Bayes estimator cannot be obtained in explicit form, and therefore extensive numerical investigation will be carried out to compare the Bayesian estimators under simple random sampling and extreme ranked set sampling techniques.

Keywords: *Burr XII distribution, ranked set sampling, extreme ranked set sampling, Bayesian estimation.*

Introduction

The two-parameter Burr XII distribution was derived by Burr (1942), Burr type XII distribution denoted by $Burr\ (c, b)$ is received more attention by the researchers due to its applications in many practical situations. The probability density function (pdf) and the cumulative density function (cdf) are as follows:

$$f(x; c, b) = bcx^{c-1}(1 + x^c)^{-(b+1)}, x > 0, c > 0, b > 0. \quad (1)$$

and

$$F(x; c, b) = 1 - (1 + x^c)^{-b}, x > 0, c > 0, b > 0. \quad (2)$$

The methodology of ranked set sampling (RSS) was introduced by McIntyre (1952) in order to estimate mean pasture and forage yields. The notation of RSS provides an efficient way to achieve observational economy under certain particular condition. The ranked set sampling process consists of drawing n simple random sample (SRS) from the population, each of size n, and ranking each of the n samples by a visual inspection or by some other inexpensive method, then the smallest observation from the first sample is chosen for measurement, as is the second smallest observation from the second sample, the first cycle is completed when the unit with ranked n is selected from the n^{th} set. The cycle can be repeated r times if needed to get a sample of size nr units from RSS data. Takahasi and Wakimoto (1968) established a rigorous statistical foundation of the theory of ranked set sampling. Dell and Clutter (1972) showed that the mean of the RSS is an unbiased estimator of the population mean, whether or not there are errors in ranking.

To reduce the errors in ranking in estimating the population mean, the extreme ranked set sampling (ERSS) procedure was introduced by Samawi et al. (1996). In the ERSS procedure, select n random samples of size n units from the population and rank the units within each sample with respect to a variable of interest by visual inspection. If the sample size n is even, select from $n/2$ samples the smallest unit and from the other $\frac{n}{2}$ samples the largest unit for actual measurement. If the sample size is odd, select from $(n-1)/2$ samples the smallest unit, from the other $(n-1)/2$ the largest unit and from one sample the median of the sample for actual measurement. The cycle may be repeated r times to get nr units. These nr units form the ERSS data.

Estimation of reliability of a single component stress-strength model has been discussed in the literature extensively. In the stress strength model P[Y < X], the stress Y and the strength X are treated as random variables and the reliability of a component during a given period is taken to be the probability that its strength exceeds the stress during the entire interval. The problems of estimating R based on SRS technique have been widely used in the statistical literature. See, for example Downtown (1973), Kelley et al. (1976), Beg (1980), McCool (1991), Abd-Elfattah and Mandouh (2004), Ali et al. (2005), Ali and Woo (2005), Raqab and Kundu (2005), Raqab et al. (2008), Ali et al. (2010), Rezaei et (2010) and Ali et al. (2012).

Hassan et al. (2014) introduced the estimation of $R = P[Y < X]$ when Y and X are two independent Burr type XII distribution with common known shape parameter c using maximum likelihood method of estimation based on ranked set sampling data. Hassan et al. (2015a) obtained Maximum likelihood estimator of R based on median ranked set sampling, ERSS and percentile ranked set sampling data when Y and X follow Burr XII distribution. These estimators are compared with known estimators based on SRS and RSS data in terms of their mean square errors (MSEs) and efficiencies. Reliability estimators for multicomponent stress strength model based on different types of ranked set sampling when samples drawn from strength and stress distributions were developed by Hassan et al. (2015b).

Bayesian estimators of R based on median ranked set sampling, ranked set sampling and simple random sampling were obtained by Hassan et al. (2015c), the main objective in this article is to obtain Bayesian estimators of R under squared error and linear exponential loss functions for $R = P[Y < X]$ when X and Y are two independent Burr type XII distributions

This study seeks to focus on Bayesian estimation for R based on SRS and ERSS techniques. using non-informative priors under symmetric and asymmetric loss functions. Numerical study will be performed to compare the different estimates. The rest of the study is organized as follows. In Section 2, Bayesian estimation of R based on SRS will be reviewed. In Section 3, Bayesian estimation of R based on ERSS in case of even and odd set sizes will be derived respectively. Numerical study is presented in Section 4. Finally conclusions are presented in Section 5.

2. Bayesian Estimator for R based on SRS data

Hassan et al. (2016) introduced Bayesian estimator of R using non-informative priors under the assumptions that the shape parameter c is known and the shape parameters b and a have independent non-informative priors with the following probability density functions:

$$\pi(b) \propto \frac{1}{b}; \qquad b > 0, \tag{3}$$

and

$$\pi(a) \propto \frac{1}{a}; \qquad a > 0. \tag{4}$$

Let $\underline{X} = \{X_1, \dots, X_p\}$ be a SRS from Burr (c, b) and $\underline{Y} = \{Y_1, \dots, Y_q\}$ be a simple random sample from Burr (c, a), Hassan et al. (2016) obtained Bayesian estimator of R under squared error (SE) loss function denoted by $\hat{R}_{SE}$ and linear exponential (LINEX) loss function denoted by $\hat{R}_{LIN}$ as follows:

$$\hat{R}_{SELF} = \frac{\lambda_1^{\,p}\,\lambda_2^{\,q}}{B(p,q)} \int_0^1 \frac{(1-r_1)^{p-1}\, r_1^{\,q}}{[(1-r_1)\lambda_1 + r_1\lambda_2]^{p+q}}\, dr_1, \tag{5}$$

and

$$\hat{R}_{LINEX} = \frac{-1}{\vartheta} \ln \left\{ \frac{\lambda_1^{\,p}\,\lambda_2^{\,q}}{B(p,q)} \int_0^1 \frac{e^{-\vartheta r_1}(1-r_1)^{p-1}\, r_1^{\,q-1}}{[(1-r_1)\lambda_1 + r_1\lambda_2]^{p+q}}\, dr_1 \right\}. \tag{6}$$

3. Bayesian Estimator for R based on ERSS data

In this section, Bayesian estimator for R based on ERSS with odd and even set sizes will be derived using non-informative priors under the assumptions that the shape parameter c is known and the shape parameters b and a have independent non-informative priors.

3.1. Bayesian Estimator for R with odd set size based on ERSS data

Let $\{X^*_{i(1)s}, i = 1, \dots, g-1; s = 1,2,\dots,r\}$, $\{X^*_{i(n)s}, i = g, \dots, n-1; s = 1,2,\dots,r\}$ are the smallest and largest order statistics from Burr (c, b), where n is the set size, r is the number of cycles. The PDFs of $X^*_{i(1)s}$ and $X^*_{i(n)s}$, will be as follows

$$f_1(x^*_{i(1)s}) = nbc\, x^{*c-1}_{i(1)s}\left(1 + x^{*c}_{i(1)s}\right)^{-(bn+1)}, \qquad x^*_{i(1)s} > 0,$$

and

$$f_n(x^*_{i(n)s}) = nbc\, x^{*c-1}_{i(n)s}\left(1 + x^{*c}_{i(n)s}\right)^{-(b+1)}\left[1 - \left(1 + x^{*c}_{i(n)s}\right)^{-b}\right]^{n-1}, x^*_{i(n)s} > 0.$$

Let $\{x^*_{1(g)s}, \dots, x^*_{n(g)s}\}$ is a MRSS from Burr (c, b) with sample size $p = nr$, where n is the set size, r is the number of cycles. Then, the PDF of $X^*_{i(g)s}$ will be as follows:

$$f_g(x^*_{i(g)s}) = \frac{n!}{[(g-1)!]^2} bc x^{*c-1}_{i(g)s}\left[1 + x^{*c}_{i(g)s}\right]^{-(bg+1)}\left[1 - \left(1 + x^{*c}_{i(g)s}\right)^{-b}\right]^{g-1}, \quad x^*_{i(g)s} > 0.$$

The likelihood function denoted by $L(\underline{x}^*|b)$ will be as follows

$$L(\underline{x}^*|b) \propto b^p e^{-b\left(n\sum_{s=1}^{r}\sum_{i=1}^{g-1} ln\left(1+x^{*c}_{i(1)s}\right)+\sum_{s=1}^{r}\sum_{i=g}^{n-1} ln\left(1+x^{*c}_{i(n)s}\right)+g\sum_{s=1}^{r} ln\left(1+x^{*c}_{n(n)s}\right)\right)}$$
$$\prod_{s=1}^{r}\prod_{i=g}^{n-1}\left[1 - \left(1 + x^{*c}_{i(n)s}\right)^{-b}\right]^{n-1}\prod_{s=1}^{r}\left[1 - \left(1 + x^{*c}_{n(g)s}\right)^{-b}\right]^{g-1}$$

by using binomial expansion,

$$L(\underline{x}^*|b) \propto b^p e^{-b\left(n\sum_{s=1}^{r}\sum_{i=1}^{g-1} ln\left(1+x^{*c}_{i(1)s}\right)+\sum_{s=1}^{r}\sum_{i=g}^{n-1} ln\left(1+x^{*c}_{i(n)s}\right)+g\sum_{s=1}^{r} ln\left(1+x^{*c}_{n(g)s}\right)\right)}\prod_{s=1}^{r}\prod_{i=g}^{n-1}$$
$$\left[\sum_{L^*=0}^{n-1}(-1)^{L^*}\binom{n-1}{L^*}\left(1 + x^{*c}_{i(n)s}\right)^{-bL^*}\right]\prod_{s=1}^{r}\left[\sum_{L^*=0}^{g-1}(-1)^{L^*}\binom{g-1}{L^*}\left(1 + x^{*c}_{n(g)s}\right)^{-bL^*}\right]$$

$L(\underline{x}^*|b)$

$$\propto b^p \sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1}\dots\sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_g^{2*}=0}^{n-1}\sum_{L_{g+1}^{2*}=0}^{n-1}\dots\sum_{L_{2(g-1)}^{2*}=0}^{n-1}\dots\sum_{L_g^{r*}=0}^{n-1}\sum_{L_{g+1}^{r*}=0}^{n-1}\dots\sum_{L_{2(g-1)}^{r*}=0}^{n-1}\left[\prod_{s=1}^{r}\prod_{i=g}^{n-1}\Phi_{L_i^{s*}}(i)\right]$$
$$\left[\prod_{s=1}^{r}\Phi(\mathrm{L}_g^{s*})\right]e^{-b\sum_{s=1}^{r}\left[n\sum_{i=1}^{g-1} ln\left(1+x^{*c}_{i(1)s}\right)+\sum_{i=g}^{n-1}(1+L_i^{s*})ln\left(1+x^{*c}_{i(n)s}\right)+(g+L_g^{s*})ln(1+x^{*c}_{n(g)s})\right]}, \tag{7}$$

where $\Phi_{L_i^{s*}}(i) = (-1)^{L_i^{s*}}\binom{n-1}{L_i^{s*}}$ and $\Phi(\mathrm{L}_g^{s*}) = (-1)^{L_g^{s*}}\binom{g-1}{L_g^{s*}}$

Combine the prior density in Equation (3) and the likelihood function in Equation (7), the posterior density of b denoted by $\pi(\underline{x}^*|b)$ based on ERSS in case of odd set size will be obtained as follows

$$\pi(\underline{x}^*|b) \propto b^{p-1}\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1}\dots\sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_g^{2*}=0}^{n-1}\sum_{L_{g+1}^{2*}=0}^{n-1}\dots\sum_{L_{2(g-1)}^{2*}=0}^{n-1}\dots\sum_{L_g^{r*}=0}^{n-1}\sum_{L_{g+1}^{r*}=0}^{n-1}\dots\sum_{L_{2(g-1)}^{r*}=0}^{n-1}$$
$$\sum_{L_{2(g-1)}^{r*}=0}^{n-1}\left[\Phi_{L_i^{s*}}(i)\right]\left[\Phi_{\mathrm{L}_g^{s*}}(\mathrm{g})\right]e^{-b\sum_{s=1}^{r}\left[n\sum_{i=1}^{g-1} ln\left(1+x^{*c}_{i(1)s}\right)+\sum_{i=g}^{n-1}T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]} \tag{8}$$

where $T_{L_i^{s*}}(i) = (1 + \mathrm{L}_i^{s*})ln\left(1 + x^{*c}_{i(n)s}\right)$ and $T_{L_g^{s*}}(g) = (g + \mathrm{L}_g^{s*})ln(1 + x^{*c}_{n(g)s})$

Similarly, let $\{Y^*_{j(1)s}, j = 1, \dots, h-1; s = 1,2,\dots,r\}$, $\{Y^*_{j(m)s}, j = h, \dots, m-1; s = 1,2,\dots,r\}$ are the smallest and largest order statistics from Burr (c, a), where m is the set size, r is the number of cycles. The PDFs of $Y^*_{j(1)s}$ and $Y^*_{j(m)s}$ respectively will be as follows:

$$f_1\left(y^*_{j(1)s}\right) = mac\, y^{*c-1}_{j(1)s}\left(1 + y^{*c}_{j(1)s}\right)^{-(am+1)}, y^*_{j(1)s} > 0,$$

and

$$f_m\left(y^*_{j(m)s}\right) = mac\, y^{*c-1}_{j(m)s}\left(1 + y^{*c}_{j(m)s}\right)^{-(a+1)}\left[1 - \left(1 + y^{*c}_{j(m)s}\right)^{-a}\right]^{m-1}, y^*_{j(m)s} > 0.$$

let $\{Y^*_{j(h)s}, s = 1,2,\dots,r\}$ is the h^{th} order statistics from Burr (c, a). Then, the PDF of h^{th} order statistics is obtained as follows

$$f_h\left(y^*_{j(h)s}\right) = \frac{m!}{[(h-1)!]^2} ac y^{c-1}_{j(h)s}\left[1 + y^{*c}_{j(h)s}\right]^{-(ah+1)}\left[1 - \left(1 + y^{*c}_{j(h)s}\right)^{-a}\right]^{h-1}, y_{j(h)s} > 0.$$

The likelihood function denoted by $L\left(\underline{y}^*|a\right)$ will be as follows

$$L\left(\underline{y}^*|a\right) \propto a^q \sum_{L_h^{1*}=0}^{m-1}\sum_{L_{h+1}^{1*}=0}^{m-1}\dots\sum_{L_{2(h-1)}^{1*}=0}^{m-1}\sum_{L_h^{2*}=0}^{m-1}\sum_{L_{h+1}^{2*}=0}^{m-1}\dots\sum_{L_{2(h-1)}^{2*}=0}^{m-1}\dots\sum_{L_h^{r*}=0}^{m-1}\sum_{L_{h+1}^{r*}=0}^{m-1}\sum_{L_{2(h-1)}^{r*}=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{j=h}^{m-1}\Phi_{L_j^{s*}}(j)\right]$$

$$\left[\prod_{s=1}^{r}\Phi_{L_h^{s*}}(g)\right] e^{-a\sum_{s=1}^{r}\left[m\sum_{j=1}^{h-1} ln\left(1+y^{*c}_{j(1)s}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]} \quad (9)$$

where $\Phi_{L_j^{s*}}(j) = (-1)^{L_j^{s*}}\binom{m-1}{L_j^{s*}}$, $\Phi_{L_h^{s*}}(g) = (-1)^{L_h^{s*}}\binom{h-1}{L_h^{s*}}$, $T_{L_j^{s*}}(j) = \left(1 + L_j^{s}\right)ln\left(1 + y^{*c}_{j(m)s}\right)$ and $T_{L_h^{s*}}(h) = (h + L_h^{s})ln(1 + y^{*c}_{m(h)s})$

Combine the prior density in Equation (4) and the likelihood function in Equation (9), then the posterior density of a denoted by $\pi\left(\underline{y}^*|a\right)$ based on ERSS is obtained as follows

$$\pi\left(\underline{y}^*|a\right)$$
$$\propto a^{q-1}\sum_{L_h^{1*}=0}^{m-1}\sum_{L_{h+1}^{1*}=0}^{m-1}\dots\sum_{L_{2(h-1)}^{1*}=0}^{m-1}\sum_{L_h^{2*}=0}^{m-1}\sum_{L_{h+1}^{2*}=0}^{m-1}\dots\sum_{L_{2(h-1)}^{2*}=0}^{m-1}\dots\sum_{L_h^{r*}=0}^{m-1}\sum_{L_{h+1}^{r*}=0}^{m-1}\sum_{L_{2(h-1)}^{r*}=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{j=h}^{m-1}\Phi_{L_j^{s*}}(j)\right]$$

$$\left[\prod_{s=1}^{r}\Phi_{L_h^{s*}}(h)\right] e^{-a\sum_{s=1}^{r}\left[m\sum_{j=1}^{h-1} ln\left(1+y^{*c}_{j(1)s}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]} \quad (10)$$

Assume that b and a are independent, from posteriors densities in Equations (6) and (8), the joint bivariate posterior density of b and a denoted by $\pi\left(b, a|\underline{x}^*, \underline{y}^*\right)$ based on ERSS data will be

$$\pi(b, a|\underline{x}^*, \underline{y}^*) \propto b^{p-1}a^{q-1}\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1}\dots\sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_g^{2*}=0}^{n-1}\sum_{L_{g+1}^{2*}=0}^{n-1}\dots\sum_{L_{2(g-1)}^{2*}=0}^{n-1}\dots\sum_{L_g^{r*}=0}^{n-1}\sum_{L_{g+1}^{r*}=0}^{n-1}\sum_{L_{2(g-1)}^{r*}=0}^{n-1}$$

$$\sum_{L_h^{1*}=0}^{m-1}\sum_{L_{h+1}^{1*}=0}^{m-1}\dots\sum_{L_{2(h-1)}^{1*}=0}^{m-1}\sum_{L_h^{2*}=0}^{m-1}\sum_{L_{h+1}^{2*}=0}^{m-1}\dots\sum_{L_{2(h-1)}^{2*}=0}^{m-1}\dots\sum_{L_h^{r*}=0}^{m-1}\sum_{L_{h+1}^{r*}=0}^{m-1}\dots\sum_{L_{2(h-1)}^{r*}=0}^{m-1}\left[\Phi_{L_i^{s*}}(i)\right]\left[\Phi_{L_g^{s*}}(g)\right]\left[\Phi_{L_j^{s*}}(j)\right]\left[\Phi_{L_h^{s*}}(h)\right]$$

$$e^{-b\sum_{s=1}^{r}\left[n\sum_{i=1}^{g-1} ln\left(1+x^{*c}_{i(1)s}\right)+\sum_{i=g}^{n-1} T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]-a\sum_{s=1}^{r}\left[m\sum_{j=1}^{h-1} ln\left(1+y^{*c}_{j(1)s}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]} \quad (11)$$

Bayesian estimator of R can be obtained using the following traditional transformation technique as follows: Let

$$r_1 = \frac{a}{a+b} \text{ and } A_1 = a+b, \text{ then } a = r_1 A_1 \text{ and } b = A_1(1-r_1), \quad 0 < r_1 < 1, \quad A_1 > 0, \tag{12}$$

then, the posterior density function in Equation (11) can be written as follows

$$\pi\left(r_1, A_1 | \underline{x}^*, \underline{y}^*\right) \propto A_1^{p+q-1}(1-r_1)^{p-1} r_1^{q-1} \sum_{L_g^{1*}=0}^{n-1} \sum_{L_{g+1}^{1*}=0}^{n-1} \dots \sum_{L_{2(g-1)}^{1*}=0}^{n-1} \sum_{L_g^{2*}=0}^{n-1}$$

$$\sum_{L_{g+1}^{2*}=0}^{n-1} \dots \sum_{L_{2(g-1)}^{2*}=0}^{n-1} \sum_{L_g^{r*}=0}^{n-1} \dots \sum_{L_{g+1}^{r*}=0}^{n-1} \dots \sum_{L_{2(g-1)}^{r*}=0}^{n-1} \sum_{L_h^{1*}=0}^{m-1} \sum_{L_{h+1}^{1*}=0}^{m-1} \dots \sum_{L_{2(h-1)}^{1*}=0}^{m-1}$$

$$\sum_{L_h^{2*}=0}^{m-1} \sum_{L_{h+1}^{2*}=0}^{m-1} \dots \sum_{L_{2(h-1)}^{2*}=0}^{m-1} \dots \sum_{L_h^{r*}=0}^{m-1} \sum_{L_{h+1}^{r*}=0}^{m-1} \sum_{L_{2(h-1)}^{r*}=0}^{m-1} \left[\Phi_{L_i^{s*}}(\mathrm{i})\right]\left[\Phi_{L_g^{s*}}(\mathrm{g})\right]\left[\Phi_{L_j^{s*}}(\mathrm{j})\right]\left[\Phi_{L_h^{s*}}(\mathrm{h})\right]$$

$$e^{-A_1\left[(1-r_1)\sum_{s=1}^{r}\left[n\sum_{i=1}^{g-1} ln\left(1+x_{i(1)s}^{*c}\right)+\sum_{i=g}^{n-1} T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]+r_1\sum_{s=1}^{r}\left[m\sum_{j=1}^{h-1} ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]\right]}$$

Integrate out of A_1, the posterior density function of r_1 using non-informative prior for $0 < r_1 < 1$ is given by

$$\pi\left(r_1 | \underline{x}^*, \underline{y}^*\right) = \Psi \sum_{L_g^{1*}=0}^{n-1} \sum_{L_{g+1}^{1*}=0}^{n-1} \dots \sum_{L_{2(g-1)}^{1*}=0}^{n-1} \sum_{L_g^{2*}=0}^{n-1} \sum_{L_{g+1}^{2*}=0}^{n-1} \dots \sum_{L_{2(g-1)}^{2*}=0}^{n-1} \dots \sum_{L_g^{r*}=0}^{n-1} \sum_{L_{g+1}^{r*}=0}^{n-1} \dots \sum_{L_{2(g-1)}^{r*}=0}^{n-1}$$

$$\sum_{L_h^{1*}=0}^{m-1} \sum_{L_{h+1}^{1*}=0}^{m-1} \dots \sum_{L_{2(h-1)}^{1*}=0}^{m-1} \sum_{L_h^{2*}=0}^{m-1} \sum_{L_{h+1}^{2*}=0}^{m-1} \dots \sum_{L_{2(h-1)}^{2*}=0}^{m-1} \dots \sum_{L_h^{r*}=0}^{m-1} \sum_{L_{h+1}^{r*}=0}^{m-1} \sum_{L_{2(h-1)}^{r*}=0}^{m-1} \dots \left[\Phi_{L_i^{s*}}(\mathrm{i})\right]\left[\Phi_{L_g^{s*}}(\mathrm{g})\right]\left[\Phi_{L_j^{s*}}(\mathrm{j})\right]$$

$$\left[\Phi_{L_h^{s*}}(\mathrm{h})\right] r_1^{q-1}(1-r_1)^{p-1}\left[\left[(1-r_1)\sum_{s=1}^{r}\left[n\sum_{i=1}^{g-1} ln\left(1+x_{i(1)s}^{*c}\right)+\sum_{i=g}^{n-1} T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]\right.\right.$$

$$\left.+r_1\sum_{s=1}^{r}\left[m\sum_{j=1}^{h-1} ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]\right]^{-(p+q)} \tag{13}$$

where Ψ is a constant and it can be obtained as follows

$$\frac{1}{\Psi} = \int_0^1 \pi\left(r_1 | \underline{x}^*, \underline{y}^*\right) dr_1$$

$$\frac{1}{\Psi} = \sum_{L_g^{1*}=0}^{n-1} \sum_{L_{g+1}^{1*}=0}^{n-1} \dots \sum_{L_{2(g-1)}^{1*}=0}^{n-1} \sum_{L_g^{2*}=0}^{n-1} \sum_{L_{g+1}^{2*}=0}^{n-1} \dots \sum_{L_{2(g-1)}^{2*}=0}^{n-1} \dots \sum_{L_g^{r*}=0}^{n-1} \sum_{L_{g+1}^{r*}=0}^{n-1} \dots \sum_{L_{2(g-1)}^{2*}=0}^{n-1} \dots \sum_{L_h^{1*}=0}^{m-1}$$

$$\sum_{L_{h+1}^{1*}=0}^{m-1} \cdots \sum_{L_{2(h-1)}^{1*}=0}^{m-1} \cdots \sum_{L_h^{2*}=0}^{m-1} \sum_{L_{h+1}^{2*}=0}^{m-1} \cdots \sum_{L_{2(h-1)}^{2*}=0}^{m-1} \cdots \sum_{L_h^{r*}=0}^{m-1} \sum_{L_{h+1}^{r*}=0}^{m-1} \cdots \sum_{L_{2(h-1)}^{r*}=0}^{m-1} \left[\Phi_{L_i^{s*}}(i)\right]\left[\Phi_{L_g^{s*}}(g)\right]\left[\Phi_{L_j^{s*}}(j)\right]$$

$$\left[\Phi_{L_h^{s*}}(h)\right]\left[\sum_{s=1}^{r}[m\sum_{j=1}^{h-1} ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]^{-(p+q)} \int_0^1 r_1^{-p-1}(1-r_1)^{p-1}$$

$$\left[\frac{[(1-r_1)\sum_{s=1}^{r}[n\sum_{i=1}^{g-1} ln\left(1+x_{i(1)s}^{*c}\right)+\sum_{i=g}^{n-1} T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)]}{r_1\sum_{s=1}^{r}[m\sum_{j=1}^{h-1} ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_j^{s*}}(j)]}+1\right]^{-(p+q)} dr_1$$

Let $\eta_1=\frac{(1-r_1)}{r_1}$, $r_1=\frac{1}{1+\eta_1}$ and $1-r_1=\frac{\eta_1}{1+\eta_1}$, (14)

$$\frac{1}{\Psi}=\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1} \cdots \sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_g^{2*}=0}^{n-1}\sum_{L_{g+1}^{2*}=0}^{n-1} \cdots \sum_{L_{2(g-1)}^{2*}=0}^{n-1} \cdots \sum_{L_g^{r*}=0}^{n-1}\sum_{L_{g+1}^{r*}=0}^{n-1} \cdots \sum_{L_{2(g-1)}^{r*}=0}^{n-1} \cdots \sum_{L_h^{1*}=0}^{m-1}$$

$$\sum_{L_{h+1}^{1*}=0}^{m-1} \cdots \sum_{L_{2(h-1)}^{1*}=0}^{m-1} \sum_{L_h^{2*}=0}^{m-1}\sum_{L_{h+1}^{2*}=0}^{m-1} \cdots \sum_{L_{2(h-1)}^{2*}=0}^{m-1} \cdots \sum_{L_h^{r*}=0}^{m-1}\sum_{L_{h+1}^{r*}=0}^{m-1} \cdots \sum_{L_{2(h-1)}^{r*}=0}^{m-1} \left[\Phi_{L_i^{s*}}(i)\right]\left[\Phi_{L_g^{s*}}(g)\right]$$

$$\left[\Phi_{L_j^{s*}}(j)\right]\left[\Phi_{L_h^{s*}}(h)\right]\left[\sum_{s=1}^{r}[m\sum_{j=1}^{h-1} ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]^{-(p+q)}$$

$$\int_0^{\infty} \eta_1^{p-1}\left[\eta_1\frac{\sum_{s=1}^{r}[n\sum_{i=1}^{g-1} ln\left(1+x_{i(1)s}^{*c}\right)+\sum_{i=g}^{n-1} T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)]}{\sum_{s=1}^{r}[m\sum_{j=1}^{h-1} ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)]}+1\right]^{-(p+q)} d\eta_1 \qquad (15)$$

Let $\eta_2=\eta_1\frac{\sum_{s=1}^{r}\left[n\sum_{i=1}^{g-1} ln\left(1+x_{i(1)s}^{*c}\right)+\sum_{i=g}^{n-1} T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]}{\sum_{s=1}^{r}\left[m\sum_{j=1}^{h-1} ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1} T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]}$, then Equation (15) will be as follows

$$\frac{1}{\Psi}=\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1} \cdots \sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_g^{2*}=0}^{n-1}\sum_{L_{g+1}^{2*}=0}^{n-1} \cdots \sum_{L_{2(g-1)}^{2*}=0}^{n-1} \cdots \sum_{L_g^{r*}=0}^{n-1}\sum_{L_{g+1}^{r*}=0}^{n-1} \cdots \sum_{L_{2(g-1)}^{r*}=0}^{n-1} \cdots$$

$$\sum_{L_h^{1*}=0}^{m-1} \sum_{L_{h+1}^{1*}=0}^{m-1} \cdots \sum_{L_{2(h-1)}^{1*}=0}^{m-1} \sum_{L_h^{2*}=0}^{m-1}\sum_{L_{h+1}^{2*}=0}^{m-1} \cdots \sum_{L_{2(h-1)}^{2*}=0}^{m-1} \cdots \sum_{L_h^{r*}=0}^{m-1}\sum_{L_{h+1}^{r*}=0}^{m-1} \cdots \sum_{L_{2(h-1)}^{r*}=0}^{m-1} \left[\Phi_{L_i^{s*}}(i)\right]\left[\Phi_{L_g^{s*}}(g)\right]$$

$$\left[\Phi_{L_j^{s*}}(j)\right]\left[\Phi_{L_h^{s*}}(h)\right]\left(\sum_{s=1}^{r}\left[n\sum_{i=1}^{g-1} ln\left(1+x_{i(1)s}^{*c}\right)+\sum_{i=g}^{n-1} T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]\right)^{-p}$$

$$\left(\sum_{s=1}^{r}\left[m\sum_{j=1}^{h-1} ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1}T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]\right)^{-q} B(p,q)$$

Therefore, the posterior pdf of R based on ERSS in case of odd set size, for $0 < r < 1$ is given by

$$\pi\left(r_1|\underline{x}^*,\underline{y}^*\right)=\left[\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_g^{2*}=0}^{n-1}\sum_{L_{g+1}^{2*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{2*}=0}^{n-1}\cdots\sum_{L_g^{r*}=0}^{n-1}\sum_{L_{g+1}^{r*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{r*}=0}^{n-1}\right.$$

$$\sum_{L_h^{1*}=0}^{m-1}\sum_{L_{h+1}^{1*}=0}^{m-1}\cdots\sum_{L_{2(h-1)}^{1*}=0}^{m-1}\sum_{L_h^{2*}=0}^{m-1}\sum_{L_{h+1}^{2*}=0}^{m-1}\cdots\sum_{L_{2(h-1)}^{2*}=0}^{m-1}\cdots\sum_{L_h^{r*}=0}^{m-1}\sum_{L_{h+1}^{r*}=0}^{m-1}\sum_{L_{2(h-1)}^{r*}=0}^{m-1}\left[\Phi_{L_i^{s*}}(\mathrm{i})\right]$$

$$\left[\Phi_{L_g^{s*}}(\mathrm{g})\right]\left[\Phi_{L_j^{s*}}(\mathrm{j})\right]\left[\Phi_{L_h^{s*}}(\mathrm{h})\right]\left(\sum_{s=1}^{r}\left[n\sum_{i=1}^{g-1}ln\left(1+x_{i(1)s}^{*c}\right)+\sum_{i=g}^{n-1}T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]\right)^{-p}$$

$$\left.\left(\sum_{s=1}^{r}\left[m\sum_{j=1}^{h-1} ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1}T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]\right)^{-q} B(p,q)\right]^{-1}\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{1*}=0}^{n-1}$$

$$\sum_{L_g^{2*}=0}^{n-1}\sum_{L_{g+1}^{2*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{2*}=0}^{n-1}\cdots\sum_{L_g^{r*}=0}^{n-1}\sum_{L_{g+1}^{r*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{r*}=0}^{n-1}\cdots\sum_{L_h^{1*}=0}^{m-1}\sum_{L_{h+1}^{1*}=0}^{m-1}\cdots\sum_{L_{2(h-1)}^{1*}=0}^{m-1}\sum_{L_h^{2*}=0}^{m-1}\sum_{L_{h+1}^{2*}=0}^{m-1}\cdots\sum_{L_{2(h-1)}^{2*}=0}^{m-1}\cdots\sum_{L_h^{r*}=0}^{m-1}\sum_{L_{h+1}^{r*}=0}^{m-1}\cdots$$

$$\sum_{L_{2(h-1)}^{r*}=0}^{m-1}\left[\Phi_{L_i^{s*}}(\mathrm{i})\right]\left[\Phi_{L_g^{s*}}(\mathrm{g})\right]\left[\Phi_{L_j^{s*}}(\mathrm{j})\right]\left[\Phi_{L_h^{s*}}(\mathrm{h})\right]r_1^{q-1}(1-r_1)^{p-1}\left[\left[(1-r_1)\sum_{s=1}^{r}\left[n\sum_{i=1}^{g-1}ln\left(1+x_{i(1)s}^{*c}\right)+\right.\right.\right.$$

$$\left.\left.\sum_{i=g}^{n-1}T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]+r_1\sum_{s=1}^{r}\left[m\sum_{j=1}^{h-1}ln\left(1+y_{j(1)s}^{*c}\right)+\sum_{j=h}^{m-1}T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]\right]^{-(p+q)} \tag{16}$$

From Equation (16), Bayesian estimator of R based on ERSS in case of odd set size using non-informative prior under squared error loss function (SELF) and LINEX loss functions respectively, can be obtained as follows

$$R_{SELF}^{*}=\int_0^1 r_1\,\pi\left(r_1|\underline{x}^*,\underline{y}^*\right)dr_1, \tag{17}$$

and

$$R_{LINEX}^{*}=\frac{-1}{\vartheta}\ln\int_0^1 e^{-\vartheta r_1}\,\pi\left(r_1|\underline{x}^*,\underline{y}^*\right)dr_1, \tag{18}$$

Bayesian estimators R_{SELF}^{*} and R_{LINEX}^{*} of R under SELF and LINEX loss functions using Jeffrey priors based on ERSS data for the simple case of a single cycle, can be obtained by substituting $(r=1)$ in Equations (17) and (18) as follows

$$R^*_{SELF} = \left[\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_h^{1*}=0}^{m-1}\sum_{L_{h+1}^{1*}=0}^{m-1}\cdots\sum_{L_{2(h-1)}^{1*}=0}^{m-1}\left[\Phi_{L_i^{s*}}(i)\right]\left[\Phi_{L_g^{s*}}(g)\right]\left[\Phi_{L_j^{s*}}(j)\right]\left[\Phi_{L_h^{s*}}(h)\right]B(p,q)\right.$$

$$\left.\left(n\sum_{i=1}^{g-1} ln\left(1+x^{*c}_{i(1)s}\right)+\sum_{i=g}^{n-1}T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]\right)^{-p}\left(\left[m\sum_{j=1}^{h-1} ln\left(1+y^{*c}_{j(1)s}\right)+\sum_{j=h}^{m-1}T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]\right)^{-q}\right]^{-1}$$

$$\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_h^{1*}=0}^{m-1}\sum_{L_{h+1}^{1*}=0}^{m-1}\cdots\sum_{L_{2(h-1)}^{1*}=0}^{m-1}\left[\Phi_{L_i^{s*}}(i)\right]\left[\Phi_{L_g^{s*}}(g)\right]\left[\Phi_{L_j^{s*}}(j)\right]\left[\Phi_{L_h^{s*}}(h)\right]$$

$$\int_0^1 r_1^q\,(1-r_1)^{p-1}\left[\left[(1-r_1)\left[n\sum_{i=1}^{g-1} ln\left(1+x^{*c}_{i(1)s}\right)+\sum_{i=g}^{n-1}T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]+\right.\right.$$

$$\left. r_1\left[m\sum_{j=1}^{h-1} ln\left(1+y^{*c}_{j(1)s}\right)+\sum_{j=h}^{m-1}T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]\right]^{-(p+q)} dr_1. \qquad (19)$$

Bayes estimator of R under LINEX loss function, based on ERSS in case of odd set size when $(r = 1)$is given by

$$R^*_{LINEX} = \frac{-1}{\vartheta}\ln\left[\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_h^{1*}=0}^{m-1}\sum_{L_{h+1}^{1*}=0}^{m-1}\cdots\sum_{L_{2(h-1)}^{1*}=0}^{m-1}\left[\Phi_{L_i^{s*}}(i)\right]\left[\Phi_{L_g^{s*}}(g)\right]\left[\Phi_{L_j^{s*}}(j)\right]\left[\Phi_{L_h^{s*}}(h)\right]B(p,q)\right.$$

$$\left.\left(\left[n\sum_{i=1}^{g-1} ln\left(1+x^{*c}_{i(1)s}\right)+\sum_{i=g}^{n-1}T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]\right)^{-p}\left(\left[m\sum_{j=1}^{h-1} ln\left(1+y^{*c}_{j(1)s}\right)+\sum_{j=h}^{m-1}T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]\right)^{-q}\right]^{-1}$$

$$\sum_{L_g^{1*}=0}^{n-1}\sum_{L_{g+1}^{1*}=0}^{n-1}\cdots\sum_{L_{2(g-1)}^{1*}=0}^{n-1}\sum_{L_h^{1*}=0}^{m-1}\sum_{L_{h+1}^{1*}=0}^{m-1}\cdots\sum_{L_{2(h-1)}^{1*}=0}^{m-1}\left[\Phi_{L_i^{s*}}(i)\right]\left[\Phi_{L_g^{s*}}(g)\right]\left[\Phi_{L_j^{s*}}(j)\right]\left[\Phi_{L_h^{s*}}(h)\right]$$

$$\int_0^1 e^{-\vartheta r_1} r_1^q\,(1-r_1)^{p-1}\left[\left[(1-r_1)\left[n\sum_{i=1}^{g-1} ln\left(1+x^{*c}_{i(1)s}\right)+\sum_{i=g}^{n-1}T_{L_i^{s*}}(i)+T_{L_g^{s*}}(g)\right]\right.\right.$$

$$\left. +r_1\left[m\sum_{j=1}^{h-1} ln\left(1+y^{*c}_{j(1)s}\right)+\sum_{j=h}^{m-1}T_{L_j^{s*}}(j)+T_{L_h^{s*}}(h)\right]\right]^{-(p+q)} dr_1. \qquad (20)$$

Cleary, it is not easy to obtain a closed form solution of (19) and (20). Therefore, an iterative technique must be applied to solve these equations numerically and then obtain an estimate of R.

3.2. Bayesian Estimator for R with Even set size based on ERSS data

In this section, Bayesian estimator for R based on ERSS with even set size will be obtained. To obtain Bayes estimators of R based ERSS in case of even set size, let $\{\acute{X}_{i(1)s}, i = 1,2,\dots,u; s = 1,2,\dots,r\}$ and $\{\acute{X}_{i(n)s}, i = u+1,\dots,n; s = 1,2,\dots,r\}$ are the smallest and largest order statistics from Burr (c, b) with the following PDFs:

$$f_1(\acute{x}_{i(1)s}) = nbc\, \acute{x}_{i(1)s}^{c-1}\left(1+\acute{x}_{i(1)s}^{c}\right)^{-(bn+1)}, \qquad \acute{x}_{i(1)s} > 0,$$

and

$$f_n(\acute{x}_{i(n)s}) = nbc\, \acute{x}_{i(n)s}^{c-1}\left(1+\acute{x}_{i(n)s}^{c}\right)^{-(b+1)}\left[1-\left(1+\acute{x}_{i(n)s}^{c}\right)^{-b}\right]^{n-1}, \acute{x}_{i(n)s} > 0,$$

respectively.

The likelihood function denoted by $L(\underline{\acute{x}}|b)$ will be as follows

$$L(\underline{\acute{x}}|b) \propto b^{p} \sum_{L_{u+1}^{1}=0}^{n-1}\sum_{L_{u+2}^{1}=0}^{n-1}\dots\sum_{L_{n}^{1}=0}^{n-1}\dots\sum_{L_{u+1}^{2}=0}^{n-1}\sum_{L_{u+2}^{2}=0}^{n-1}\dots\sum_{L_{n}^{2}=0}^{n-1}\dots\sum_{L_{u+1}^{r}=0}^{n-1}\sum_{L_{u+2}^{r}=0}^{n-1}\dots\sum_{L_{n}^{r}=0}^{n-1}\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}(-1)^{L_i^s}\binom{n-1}{L_i^s}\right] e^{-b\sum_{s=1}^{r}[n\sum_{i=1}^{u}\acute{x}_{i(1)s}^{c})+\sum_{i=u+1}^{n}(L_i^s+1)\,ln(1+\acute{x}_{i(n)s}^{c})}. \qquad (21)$$

Combine the prior density in Equation (3) and the likelihood function in Equation (21), the posterior density of b denoted by $\pi(\underline{\acute{x}}|b)$ based on ERSS in case of even set size will be as follows

$$\pi(\underline{\acute{x}}|b) \propto b^{p-1} \sum_{L_{u+1}^{1}=0}^{n-1}\sum_{L_{u+2}^{1}=0}^{n-1}\sum_{L_{n}^{1}=0}^{n-1}\dots\sum_{L_{u+1}^{2}=0}^{n-1}\sum_{L_{u+1}^{2}=0}^{n-1}\sum_{L_{n}^{2}=0}^{n-1}\dots\sum_{L_{u+1}^{r}=0}^{n-1}\sum_{L_{u+2}^{r}=0}^{n-1}\sum_{L_{n}^{r}=0}^{n-1}\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}(-1)^{L_i^s}\binom{n-1}{L_i^s}\right] e^{-b\sum_{s=1}^{r}[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^{c})+\sum_{i=u+1}^{n}(L_i^s+1)\ln(1+\acute{x}_{i(n)s}^{c})}. \qquad (22)$$

Similarly, let $\{\acute{Y}_{j(1)s}, j = 1,\dots,h-1; s = 1,2,\dots,r\}$, $\{\acute{Y}_{j(m)s}, j = h,\dots,m-1; s = 1,2,\dots,r\}$ are the smallest and largest order statistics from Burr (c, a), where m is the set size, r is the number of cycles. The PDFs of $\acute{Y}_{j(1)s}$ and $\acute{Y}_{j(m)s}$, respectively, will be as follows:

$$f_1(\acute{y}_{j(1)s}) = mac\, \acute{y}_{j(1)s}^{c-1}\left(1+\acute{y}_{j(1)s}^{c}\right)^{-(am+1)}, \acute{y}_{j(1)s} > 0,$$

and

$$f_m(\acute{y}_{j(m)s}) = mac\, \acute{y}_{j(m)s}^{c-1}\left(1+\acute{y}_{j(m)s}^{c}\right)^{-(a+1)}\left[1-\left(1+\acute{y}_{j(m)s}^{c}\right)^{-a}\right]^{m-1}, \acute{y}_{j(m)s} > 0.$$

The likelihood function denoted by $L(\underline{\acute{y}}; a)$ is given by

$$L(\underline{\acute{y}}; a) \propto a^{q} \sum_{l_{j=v+1}^{1}=0}^{m-1}\sum_{l_{j=v+2}^{1}=0}^{m-1}\sum_{l_{j=m}^{1}=0}^{m-1}\dots\sum_{l_{j=v+1}^{2}=0}^{m-1}\sum_{l_{j=v+2}^{2}=0}^{m-1}\sum_{l_{j=m}^{2}=0}^{m-1}\dots\sum_{l_{j=v+1}^{r}=0}^{m-1}\sum_{l_{j=v+2}^{r}=0}^{m-1}\sum_{l_{j=m}^{r}=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}(-1)^{l_j^m}\binom{m-1}{l_j^m}\right] e^{-a\sum_{s=1}^{r}[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(m)s}^{c})+\sum_{j=v+1}^{m}(l_j^s+1)\ln\left(1+\left(1+\acute{y}_{j(m)s}^{c}\right)\right)]}. \qquad (23)$$

Combine the prior density in Equation (4) and the likelihood function in Equation (23), the posterior density function of a denoted by $\pi(\underline{\acute{y}}|a)$ based on ERSS is obtained as follows .

$$\pi(\underline{\acute{y}}|a) \propto a^{q-1} \sum_{l_{j=v+1}^1=0}^{m-1} \sum_{l_{j=v+2}^1=0}^{m-1} \sum_{l_{j=m}^1=0}^{m-1} \cdots \sum_{l_{j=v+1}^2=0}^{m-1} \sum_{l_{j=v+2}^2=0}^{m-1} \sum_{l_{j=m}^2=0}^{m-1} \cdots \sum_{l_{j=v+1}^r=0}^{m-1} \sum_{l_{j=v+2}^r=0}^{m-1} \sum_{l_{j=m}^r=0}^{m-1}$$

$$\left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} (-1)^{l_j^m} \binom{m-1}{l_j^m}\right] e^{-a\sum_{s=1}^{r}\left[m\sum_{j=1}^{v} \ln(1+\acute{y}_{j(1)s}^c)+\sum_{j=v+1}^{m}(l_j^s+1)\ln\left(1+\left(1+\acute{y}_{j(m)s}^c\right)\right)\right]} . \quad (24)$$

Assume that b and a are independent, from posteriors densities in Equations (22) and (24), the joint bivariate posterior density of b and a denoted by $\pi(b,a\,|\underline{\acute{x}},\underline{\acute{y}})$ based on even ERSS data will be

$$\pi(b,a\,|\underline{\acute{x}},\underline{\acute{y}}) \propto b^{p-1}a^{q-1} \sum_{L_{u+1}^1=0}^{n-1} \sum_{L_{u+2}^1=0}^{n-1} \sum_{L_n^1=0}^{n-1} \cdots \sum_{L_{u+1}^2=0}^{n-1} \sum_{L_{u+2}^2=0}^{n-1} \sum_{L_n^2=0}^{n-1} \cdots \sum_{L_{u+1}^r=0}^{n-1} \sum_{L_{u+2}^r=0}^{n-1} \sum_{L_n^r=0}^{n-1}$$

$$\left[\prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{L_i^s}(i)\right] \sum_{l_{j=v+1}^1=0}^{m-1} \sum_{l_{j=v+2}^1=0}^{m-1} \sum_{l_{j=m}^1=0}^{m-1} \cdots \sum_{l_{j=v+1}^2=0}^{m-1} \sum_{l_{j=v+2}^2=0}^{m-1} \sum_{l_{j=m}^2=0}^{m-1} \cdots \sum_{l_{j=v+1}^r=0}^{m-1} \sum_{l_{j=v+2}^r=0}^{m-1} \sum_{l_{j=m}^r=0}^{m-1}$$

$$\left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} \Phi_{l_j^s}(j)\right] e^{-b\sum_{s=1}^{r}\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s})+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]-a\sum_{s=1}^{r}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^c)+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]} \quad (25)$$

where

$\Phi_{L_i^s}(i) = (-1)^{L_i^s}\binom{n-1}{L_i^s}$, $\Phi_{l_j^s}(j) = (-1)^{l_j^s}\binom{m-1}{l_j^s}$, $T_{L_i^s}(i) = \left(L_i^s+1\right)\ln(1+\acute{x}_{i(n)s})$ *and* $T_{l_j^s}(j) = (l_j^s+1)\ln\left(1+\acute{y}_{j(m)s}^c\right)$.

Bayes estimate of R can be obtained by using the traditional transformation technique in (12) , then the posterior density function in Equation (25) will be as follows:

$$\pi(r_1,A_1\,|\underline{\acute{x}},\underline{\acute{y}}) \propto A_1^{p+q-1}r_1^{q-1}(1-r_1)^{p-1} \sum_{L_{u+1}^1=0}^{n-1} \sum_{L_{u+2}^1=0}^{n-1} \sum_{L_n^1=0}^{n-1} \cdots \sum_{L_{u+1}^2=0}^{n-1} \sum_{L_{u+2}^2=0}^{n-1} \sum_{L_n^2=0}^{n-1} \cdots \sum_{L_{u+1}^r=0}^{n-1}$$

$$\sum_{L_{u+2}^r=0}^{n-1} \cdots \sum_{L_n^r=0}^{n-1} \cdots \sum_{L_n^r}^{n-1} \prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{L_i^s}(i)\Bigg] \sum_{l_{j=v+1}^1=0}^{m-1} \sum_{l_{j=v+2}^1=0}^{m-1} \sum_{l_{j=m}^1=0}^{m-1} \cdots \sum_{l_{j=v+1}^2=0}^{m-1} \sum_{l_{j=v+2}^2=0}^{m-1} \sum_{l_{j=m}^2=0}^{m-1} \cdots \sum_{l_{j=v+1}^r=0}^{m-1} \sum_{l_{j=v+2}^r=0}^{m-1} \sum_{l_{j=m}^r=0}^{m-1}$$

$$\left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} \Phi_{l_j^s}(j)\right] e^{-A_1\sum_{s=1}^{r}[(1-r_1)\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^c)+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]+r_1\sum_{s=1}^{r}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^c)+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]} .$$

Integrate out of A_1, the posterior density function of r_1 using non-informative prior for $0 < r_1 < 1$ is given by

$$\pi(r_1,A_1\,|\underline{\acute{x}},\underline{\acute{y}}) \propto \Psi \sum_{L_{u+1}^1=0}^{n-1} \sum_{L_{u+2}^1=0}^{n-1} \sum_{L_n^1=0}^{n-1} \cdots \sum_{L_{u+1}^2=0}^{n-1} \sum_{L_{u+2}^2=0}^{n-1} \sum_{L_n^2=0}^{n-1} \cdots \sum_{L_{u+1}^r=0}^{n-1} \sum_{L_{u+2}^r=0}^{n-1} \sum_{L_n^r=0}^{n-1} \left[\prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{L_i^s}(i)\right]$$

$$\sum_{l_{j=v+1}^1=0}^{m-1} \sum_{l_{j=v+2}^1=0}^{m-1} \sum_{l_{j=m}^1=0}^{m-1} \cdots \sum_{l_{j=v+1}^2=0}^{m-1} \sum_{l_{j=v+2}^2=0}^{m-1} \sum_{l_{j=m}^2=0}^{m-1} \cdots \sum_{l_{j=v+1}^r=0}^{m-1} \sum_{l_{j=v+2}^r=0}^{m-1} \sum_{l_{j=m}^r=0}^{m-1} \left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} \Phi_{l_j^s}(j)\right] r_1^{q-1}(1-r_1)^{p-1}$$

$$\left[\sum_{s=1}^{r}[(1-r_1)\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^c)+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]+r_1\sum_{s=1}^{r}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^c)+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]\right]^{-(p+q)}$$

where Ψ is a constant and it can be obtained as follows

$$\frac{1}{\Psi} = \sum_{L_{u+1}^{1}=0}^{n-1} \sum_{L_{u+2}^{1}=0}^{n-1} \sum_{L_{n}^{1}=0}^{n-1} \cdots \sum_{L_{u+1}^{2}=0}^{n-1} \sum_{L_{u+2}^{2}=0}^{n-1} \sum_{L_{n}^{2}=0}^{n-1} \cdots \sum_{L_{u+1}^{r}=0}^{n-1} \sum_{L_{u+2}^{r}=0}^{n-1} \sum_{L_{n}^{r}=0}^{n-1} \left[\prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{L_i^s}(i)\right] \sum_{l_{j=v+1}^{1}=0}^{m-1} \sum_{l_{j=v+2}^{1}=0}^{m-1} \sum_{l_{j=m}^{1}=0}^{m-1} \cdots$$

$$\sum_{l_{j=v+1}^{2}=0}^{m-1} \sum_{l_{j=v+2}^{2}=0}^{m-1} \sum_{l_{j=m}^{2}=0}^{m-1} \cdots \sum_{l_{j=v+1}^{r}=0}^{m-1} \sum_{l_{j=v+2}^{r}=0}^{m-1} \sum_{l_{j=m}^{r}=0}^{m-1} \left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} \Phi_{l_j^s}(j)\right] \int_{0}^{1} r_1^{q-1}(1-r_1)^{p-1} \left[\sum_{s=1}^{r} [(1-r_1)\right.$$

$$\left.\left[n\sum_{i=1}^{u} \ln(1+\acute{x}_{i(1)s}^{c}) + \sum_{i=u+1}^{n} T_{L_i^s}(i)\right] + r_1 \sum_{s=1}^{r}\left[m\sum_{j=1}^{v} \ln(1+\acute{y}_{j(1)s}^{c}) + \sum_{j=v+1}^{m} T_{l_j^s}(j)\right]\right]^{-(p+q)} dr_1$$

$$\frac{1}{\Psi} = \sum_{L_{u+1}^{1}=0}^{n-1} \sum_{L_{u+2}^{1}=0}^{n-1} \sum_{L_{n}^{1}=0}^{n-1} \cdots \sum_{L_{u+1}^{2}=0}^{n-1} \sum_{L_{u+2}^{2}=0}^{n-1} \sum_{L_{n}^{2}=0}^{n-1} \cdots \sum_{L_{u+1}^{r}=0}^{n-1} \sum_{L_{u+2}^{r}=0}^{n-1} \sum_{L_{n}^{r}=0}^{n-1} \left[\prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{L_i^s}(i)\right] \sum_{l_{j=v+1}^{1}=0}^{m-1}$$

$$\sum_{l_{j=v+2}^{1}=0}^{m-1} \sum_{l_{j=m}^{1}=0}^{m-1} \cdots \sum_{l_{j=v+1}^{2}=0}^{m-1} \sum_{l_{j=v+2}^{2}=0}^{m-1} \sum_{l_{j=m}^{2}=0}^{m-1} \cdots \sum_{l_{j=v+1}^{r}=0}^{m-1} \sum_{l_{j=v+2}^{r}=0}^{m-1} \sum_{l_{j=m}^{r}=0}^{m-1} \left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} \Phi_{l_j^s}(j)\right]$$

$$\int_{0}^{1} r_1^{q-1}(1-r_1)^{p-1} \left[r_1 \sum_{s=1}^{r}\left[m\sum_{j=1}^{v} \ln(1+\acute{y}_{j(1)s}^{c}) + \sum_{j=v+1}^{m} T_{l_j^s}(j)\right]\right]^{-(p+q)}$$

$$\left[\frac{[\sum_{s=1}^{r}[(1-r_1)\left[n\sum_{i=1}^{u} \ln(1+\acute{x}_{i(1)s}^{c}) + \sum_{i=u+1}^{n} T_{L_i^s}(i)\right]}{r_1 \sum_{s=1}^{r}\left[m\sum_{j=1}^{v} \ln(1+\acute{y}_{j(1)s}^{c}) + \sum_{j=v+1}^{m} T_{l_j^s}(j)\right]\right]} + 1\right]^{-(p+q)} dr_1. \qquad (26)$$

Using Equation (14), Equation (26) will be as follows

$$\frac{1}{\Psi} = \sum_{L_{u+1}^{1}=0}^{n-1} \sum_{L_{u+2}^{1}=0}^{n-1} \sum_{L_{n}^{1}=0}^{n-1} \cdots \sum_{L_{u+1}^{2}=0}^{n-1} \sum_{L_{u+2}^{2}=0}^{n-1} \sum_{L_{n}^{2}=0}^{n-1} \cdots \sum_{L_{u+1}^{r}=0}^{n-1} \sum_{L_{u+2}^{r}=0}^{n-1} \sum_{L_{n}^{r}=0}^{n-1} \left[\prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{L_i^s}(i)\right]$$

$$\sum_{l_{j=v+1}^{1}=0}^{m-1} \sum_{l_{j=v+2}^{1}=0}^{m-1} \sum_{l_{j=m}^{1}=0}^{m-1} \cdots \sum_{l_{j=v+1}^{2}=0}^{m-1} \sum_{l_{j=v+2}^{2}=0}^{m-1} \sum_{l_{j=m}^{2}=0}^{m-1} \cdots \sum_{l_{j=v+1}^{r}=0}^{m-1} \sum_{l_{j=v+2}^{r}=0}^{m-1} \sum_{l_{j=m}^{r}=0}^{m-1} \left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} \Phi_{l_j^s}(j)\right]$$

$$\int_{0}^{\infty} \eta_1^{p-1} \left[\sum_{s=1}^{r}\left[m\sum_{j=1}^{v} \ln(1+\acute{y}_{j(1)s}^{c}) + \sum_{j=v+1}^{m} T_{l_j^s}(j)\right]\right]^{-(p+q)}$$

$$\left[\eta_1 \frac{\sum_{s=1}^{r}\left[n\sum_{i=1}^{u} \ln(1+\acute{x}_{i(1)s}^{c}) + \sum_{i=u+1}^{n} T_{L_i^s}(i)\right]}{\sum_{s=1}^{r}\left[m\sum_{j=1}^{v} \ln(1+\acute{y}_{j(1)s}^{c}) + \sum_{j=v+1}^{m} T_{l_j^s}(j)\right]} + 1\right]^{-(p+q)} d\eta_1. \qquad (27)$$

Let $\eta_2 = \eta_1 \frac{\sum_{s=1}^{r}\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^{c})+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]}{\sum_{s=1}^{r}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^{c})+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]}$, *and* $d\eta_1 = \frac{\sum_{s=1}^{r}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^{c})+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]}{\sum_{s=1}^{r}\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^{c})+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]}d\eta_2,$

then Equation (27) will be as follows

$$\frac{1}{\Psi} = \sum_{L_{u+1}^{1}=0}^{n-1}\sum_{L_{u+2}^{1}=0}^{n-1}\sum_{L_{n}^{1}=0}^{n-1}\cdots\sum_{L_{u+1}^{2}=0}^{n-1}\sum_{L_{u+2}^{2}=0}^{n-1}\sum_{L_{n}^{2}=0}^{n-1}\cdots\sum_{L_{u+1}^{r}=0}^{n-1}\sum_{L_{u+2}^{r}=0}^{n-1}\sum_{L_{n}^{r}=0}^{n-1}\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}\Phi_{L_i^s}(i)\right]$$

$$\sum_{l_{j=v+1}^{1}=0}^{m-1}\sum_{l_{j=v+2}^{1}=0}^{m-1}\sum_{l_{j=m}^{1}=0}^{m-1}\cdots\sum_{l_{j=v+1}^{2}=0}^{m-1}\sum_{l_{j=v+2}^{2}=0}^{m-1}\sum_{l_{j=m}^{2}=0}^{m-1}\cdots\sum_{l_{j=v+1}^{r}=0}^{m-1}\sum_{l_{j=v+2}^{r}=0}^{m-1}\sum_{l_{j=m}^{r}=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi_{l_j^s}(j)\right]$$

$$\left[\sum_{s=1}^{r}\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^{c})+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]^{-p}\right]\left[\sum_{s=1}^{r}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^{c})+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]\right]^{-q}$$

$$\int_{0}^{\infty}\eta_2^{p-1}\,[\eta_2+1]^{-(p+q)}d\eta_2$$

$$\frac{1}{\Psi} = \sum_{L_{u+1}^{1}=0}^{n-1}\sum_{L_{u+2}^{1}=0}^{n-1}\sum_{L_{n}^{1}=0}^{n-1}\cdots\sum_{L_{u+1}^{2}=0}^{n-1}\sum_{L_{u+2}^{2}=0}^{n-1}\sum_{L_{n}^{2}=0}^{n-1}\cdots\sum_{L_{u+1}^{r}=0}^{n-1}\sum_{L_{u+2}^{r}=0}^{n-1}\sum_{L_{n}^{r}=0}^{n-1}\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}\Phi_{L_i^s}(i)\right]$$

$$\sum_{l_{j=v+1}^{1}=0}^{m-1}\sum_{l_{j=v+2}^{1}=0}^{m-1}\sum_{l_{j=m}^{1}=0}^{m-1}\cdots\sum_{l_{j=v+1}^{2}=0}^{m-1}\sum_{l_{j=v+2}^{2}=0}^{m-1}\sum_{l_{j=m}^{2}=0}^{m-1}\cdots\sum_{l_{j=v+1}^{r}=0}^{m-1}\sum_{l_{j=v+2}^{r}=0}^{m-1}\sum_{l_{j=m}^{r}=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi_{l_j^s}(j)\right]$$

$$\left[\sum_{s=1}^{r}\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^{c})+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]^{-p}\right]\left[\sum_{s=1}^{r}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^{c})+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]\right]^{-q}B(P,q)$$

$$\pi(r_1, A_1 \,|\, \underline{x}, \underline{y})$$
$$= \left[\sum_{L_{u+1}^{1}=0}^{n-1}\sum_{L_{u+2}^{1}=0}^{n-1}\sum_{L_{n}^{1}=0}^{n-1}\cdots\sum_{L_{u+1}^{2}=0}^{n-1}\sum_{L_{u+2}^{2}=0}^{n-1}\sum_{L_{n}^{2}=0}^{n-1}\cdots\sum_{L_{u+1}^{r}=0}^{n-1}\sum_{L_{u+2}^{r}=0}^{n-1}\sum_{L_{n}^{r}=0}^{n-1}\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}\Phi_{L_i^s}(i)\right]\sum_{l_{j=v+1}^{1}=0}^{m-1}\right.$$

$$\sum_{l_{j=v+2}^{1}=0}^{m-1}\sum_{l_{j=m}^{1}=0}^{m-1}\cdots\sum_{l_{j=v+1}^{2}=0}^{m-1}\sum_{l_{j=v+2}^{2}=0}^{m-1}\sum_{l_{j=m}^{2}=0}^{m-1}\cdots\sum_{l_{j=v+1}^{r}=0}^{m-1}\sum_{l_{j=v+2}^{r}=0}^{m-1}\sum_{l_{j=m}^{r}=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi_{l_j^s}(j)\right]\left[\sum_{s=1}^{r}\left[n\sum_{i=1}^{u}\right.\right.$$

$$\left.\left.\left.\ln(1+\acute{x}_{i(1)s}^{c}+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]\right]^{-p}\left[\sum_{s=1}^{r}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^{c})+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]\right]^{-q}B(P,q)\right]^{-1}$$

$$\sum_{L_{u+1}^{1}=0}^{n-1}\sum_{L_{u+2}^{1}=0}^{n-1}\sum_{L_{n}^{1}=0}^{n-1}\cdots\sum_{L_{u+1}^{2}=0}^{n-1}\sum_{L_{u+2}^{2}=0}^{n-1}\sum_{L_{n}^{2}=0}^{n-1}\cdots\sum_{L_{u+1}^{r}=0}^{n-1}\sum_{L_{u+2}^{r}=0}^{n-1}\sum_{L_{n}^{r}=0}^{n-1}\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}\Phi_{L_i^s}(i)\right]\sum_{l_{j=v+1}^{1}=0}^{m-1}\sum_{l_{j=v+2}^{1}=0}^{m-1}\sum_{l_{j=m}^{1}=0}^{m-1}\cdots$$

$$\sum_{l_{j=v+1}^{2}=0}^{m-1}\sum_{l_{j=v+2}^{2}=0}^{m-1}\sum_{l_{j=m}^{2}=0}^{m-1}\cdots\sum_{l_{j=v+1}^{r}=0}^{m-1}\sum_{l_{j=v+2}^{r}=0}^{m-1}\sum_{l_{j=m}^{r}=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi(l_j^s)\right]r_1^{q-1}(1-r_1)^{p-1}\left[\sum_{s=1}^{r}[(1-r_1)\right.$$

$$\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^{c})+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]+r_1\sum_{s=1}^{r}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^{c})+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]^{-(p+q)} \quad (28)$$

Thereby, the posterior density function of R denoted by $\pi(r_1, A_1 | \underline{\acute{x}}, \underline{\acute{y}})$, can be obtained by using Equation (28), for $0 < r < 1$. Then Bayes estimator of r denoted by $\grave{R}_{SELF}$ under SELF is as follows:

$$\grave{R}_{SELF} = \int_0^1 r_1\, \pi\left(r_1 | \underline{\acute{x}}, \underline{\acute{y}}\right) dr_1. \quad (29)$$

Additionally, the Bayes estimator of R under LINEX loss function, denoted by $\grave{R}_{LINEX}$, based on ERSS in case of even set size is given as follows:

$$\grave{R}_{LINEX} = \frac{-1}{\vartheta}\ln\int_0^1 e^{-\vartheta r_1}\, \pi\left(r_1 | \underline{\acute{x}}, \underline{\acute{y}}\right) dr_1. \quad (30)$$

By using Equation (29), Bayes estimator of R under SELF in the simple case of a single cycle when $(r = 1)$ as follows:

$$\grave{R}_{SELF} = \left[\sum_{L_{u+1}^{1}=0}^{n-1}\sum_{L_{u+2}^{1}=0}^{n-1}\sum_{L_{n}^{1}=0}^{n-1}\cdots\left[\prod_{i=u+1}^{n}\Phi_{L_i^s}(i)\right]\sum_{l_{j=v+1}^{1}=0}^{m-1}\sum_{l_{j=v+2}^{1}=0}^{m-1}\sum_{l_{j=m}^{1}=0}^{m-1}\cdots\left[\prod_{j=v+1}^{m}\Phi_{l_j^s}(j)\right]\right.$$

$$\left.\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^{c})+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]^{-p}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^{c})+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]^{-q} B(P,q)\right]^{-1}$$

$$\sum_{L_{u+1}^{1}=0}^{n-1}\sum_{L_{u+2}^{1}=0}^{n-1}\sum_{L_{n}^{1}=0}^{n-1}\cdots\left[\prod_{i=u+1}^{n}\Phi_{L_i^s}(i)\right]\sum_{l_{j=v+1}^{1}=0}^{m-1}\sum_{l_{j=v+2}^{1}=0}^{m-1}\sum_{l_{j=m}^{1}=0}^{m-1}\cdots\left[\prod_{j=v+1}^{m}\Phi(l_j^s)\right]\int_0^1 r_1^{q}(1-r_1)^{p-1}$$

$$\left[(1-r_1)\left[n\sum_{i=1}^{u}\ln(1+\acute{x}_{i(1)s}^{c})+\sum_{i=u+1}^{n}T_{L_i^s}(i)\right]+r_1\left[m\sum_{j=1}^{v}\ln(1+\acute{y}_{j(1)s}^{c})+\sum_{j=v+1}^{m}T_{l_j^s}(j)\right]\right]^{-(p+q)} dr_1. \quad (31)$$

Bayes estimator of R under LINEX loss function, based on ERSS in case of even set size when $(r = 1)$ can be obtained by using Equation (30) as follows:

$$\grave{R}_{LINEX} = \frac{-1}{\vartheta}ln\left[\sum_{L_{u+1}^{1}=0}^{n-1}\sum_{L_{u+2}^{1}=0}^{n-1}\sum_{L_{n}^{1}=0}^{n-1}\cdots\left[\prod_{i=u+1}^{n}\Phi_{L_i^s}(i)\right]\sum_{l_{j=v+1}^{1}=0}^{m-1}\sum_{l_{j=v+2}^{1}=0}^{m-1}\sum_{l_{j=m}^{1}=0}^{m-1}\cdots\left[\prod_{j=v+1}^{m}\Phi_{l_j^s}(j)\right]\right.$$

$$\left[n\sum_{i=1}^{u}\ln(1+\acute{x}^{c}_{i(1)s}+\sum_{i=u+1}^{n}T_{l^s_i}(i)\right]^{-p}\left[m\sum_{j=1}^{v}\ln(1+\acute{y}^{c}_{j(1)s})+\sum_{j=v+1}^{m}T_{l^s_j}(j)\right]^{-q}B(P,q)\Bigg]^{-1}$$

$$\sum_{L^1_{u+1}=0}^{n-1}\sum_{L^1_{u+2}=0}^{n-1}\sum_{L^1_{n}=0}^{n-1}\cdots\left[\prod_{i=u+1}^{n}\Phi_{L^s_i}(i)\right]\sum_{l^1_{j=v+1}=0}^{m-1}\sum_{l^1_{j=v+2}=0}^{m-1}\sum_{l^1_{j=m}=0}^{m-1}\cdots\left[\prod_{j=v+1}^{m}\Phi(l^s_j)\right]\int_0^1 e^{\vartheta r_1}r_1{}^{q}(1-r_1)^{p-1}$$

$$\left[(1-r_1)\left[n\sum_{i=1}^{u}\ln(1+\acute{x}^{c}_{i(1)s})+\sum_{i=u+1}^{n}T_{l^s_i}(i)\right]+r_1\left[m\sum_{j=1}^{v}\ln(1+\acute{y}^{c}_{j(1)s})+\sum_{j=v+1}^{m}T_{l^s_j}(j)\right]\right]^{-(p+q)} dr_1(32)$$

Cleary, it is not easy to obtain a closed form solution for Bayes estimators under Jeffery prior in (31) and (32)using SELF and LINEX loss functions respectively, so an iterative procedure can be used to evaluate these equations.

4. Simulation Study

A simulation study is carried out to compare Bayes estimates of R based on SRS and ERSS techniques. The ratio $\rho=\frac{b}{a}$ is selected as 0.25, 0.33, 0.50, 1, 2, 3, 4, 5, 6. Sample sizes will be considered to be $(n,m)=(2{,}2),(3{,}3),(4{,}4)$. Without loss of generality the number of cycles $(r=1)$ and the shape parameter c will be considered to be one in all cases. For a given generated sample, Bayes estimates of R for SRS and ERSS in case of odd and even set sizes under SELF and LINEX loss functions will be computed and the process will be replicated 1000 times. The biases, mean square errors (MSEs) and efficiencies of the estimators, namely $\{\hat{R}_{SELF}, \hat{R}_{LINEX}, R^*_{SELF}, R^*_{LINEX}, \grave{R}_{SELF}\ and\ \grave{R}_{LINEX}\}$ based on SRS and ERSS with odd and even set sizes respectively will be obtained. The efficiencies of, $\hat{R}_{LINEX}, R^*_{SELF}, R^*_{LINEX}, \grave{R}_{SELF}$ $and\ \grave{R}_{LINEX}$ estimates will be performed with respect to $\hat{R}_{SELF}$. The results of biases and MSEs are summarizes in Tables (1 – 6), the efficiencies of the different estimators will be reported in Tables (7- 9). From the tables and figures the following conclusions can be represented as follows:

1. The biases of all estimates $\hat{R}_{SELF}, \hat{R}_{LINEX}, R^*_{SELF}, R^*_{LINEX}, \grave{R}_{SELF}\ and\ \grave{R}_{LINEX}$ are small. (See Tables (1 - 6).
2. In all cases, MSEs of all estimates $R^*_{SELF}, R^*_{LINEX}, \grave{R}_{SELF}\ and\ \grave{R}_{LINEX}$ based on ERSS with odd and even set sizes are smaller than $\hat{R}_{SELF}$, $\hat{R}_{LINEX}$ based on SRS. (See Tables (1- 6) and Figure (1)).
3. In all cases, MSEs of all estimates $\hat{R}_{SELF}, \hat{R}_{LINEX}, R^*_{SELF}, R^*_{LINEX}, \grave{R}_{SELF}\ and\ \grave{R}_{LINEX}$ are decreasing as the set size increasing at the same value of ρ. (See Tables (1- 6)and Figure (1)).).

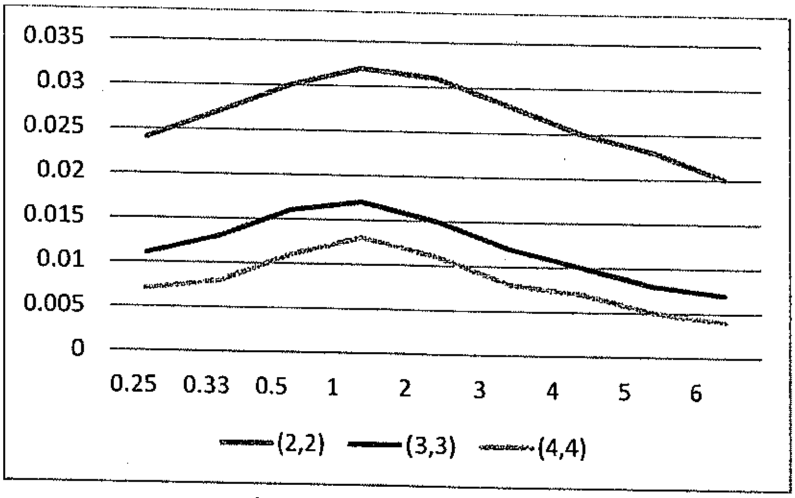

Figure 1: MSEs of the R^*_{SELF} and $\grave{R}_{SELF}$ estimates based on ERSS with odd and even set sizes at different values of ρ

4. In almost all cases, for $\rho < 1$, MSEs of $\hat{R}_{LINEX}, R^*_{LINEX}, and\ \grave{R}_{LINEX}$ estimates under LINEX loss function at $\theta = -2$, have the smallest MSEs compared to the corresponding estimates based on SELF loss function and LINEX loss function at $\theta = 2$. On the other hand, MSEs of $\hat{R}_{LINEX}, R^*_{LINEX}$, and $\grave{R}_{LINEX}$ estimates based on LINEX loss function at $\theta = 2$, have the largest value of MSEs comparing with the other estimators at the same value of ρ. (See tables (1 – 6)).
5. In all cases, the efficiencies of all $R^*_{SELF}, R^*_{LINEX}, \grave{R}_{SELF}\ and\ \grave{R}_{LINEX}$ estimates based on ERSS with odd and even set sizes are greater than the corresponding estimates $\hat{R}_{LINEX}$ and $\hat{R}_{LINEX}$ based on SRS. (See Tables (7 – 9) and Figure 2).

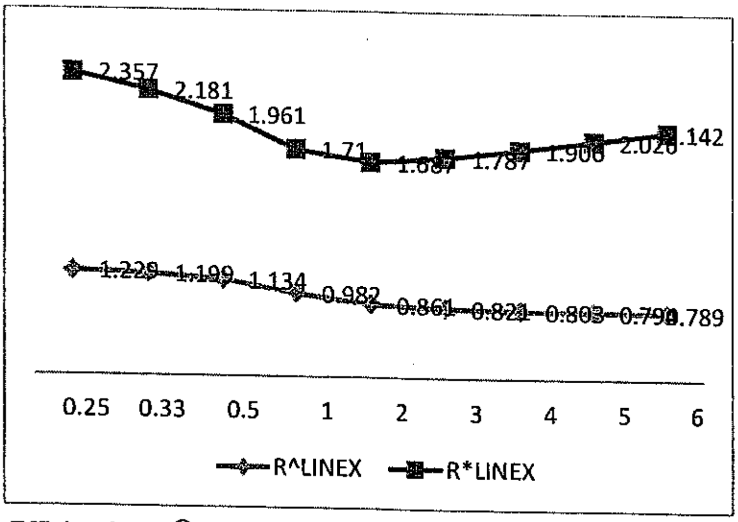

Figure 2: Efficiencies of $\hat{R}_{LINEX}$ and R^*_{LINEX} estimates when $\theta = -2$ at the set size (3,3)

6. In almost all cases, the efficiencies of all estimators increase as the set size increases at the same value of ρ. (See tables (7 – 9) and Figure 3).

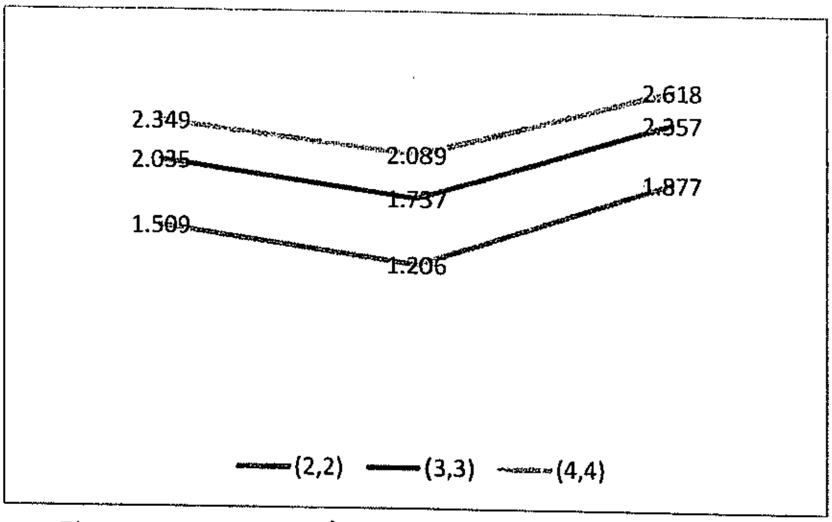

Figure 3: Efficiencies of $\grave{R}_{SELF}$, R^*_{LINEX} and $\grave{R}_{LINEX}$ estimates.

7. In almost all cases, for $\rho < 1$, the efficiencies of the estimates $\widehat{R}_{LINEX}$, R^*_{LINEX}, and $\grave{R}_{LINEX}$ under LINEX loss function at $\theta = -2$ take the largest comparing with the other estimates $\widehat{R}_{SELF}$, R^*_{SELF}, $\grave{R}_{SELF}$ based on SELF and $\widehat{R}_{LINEX}$, R^*_{LINEX}, and $\grave{R}_{LINEX}$ based on LINEX loss function at $\theta = 2$. On the other hand, the efficiencies of the estimates $\widehat{R}_{LINEX}$, R^*_{LINEX}, and $\grave{R}_{LINEX}$ based on LINEX loss function at $\theta = 2$, take the smallest values comparing with the other estimates at the same value of ρ. (See Tables (7 – 9) and Figure 4).

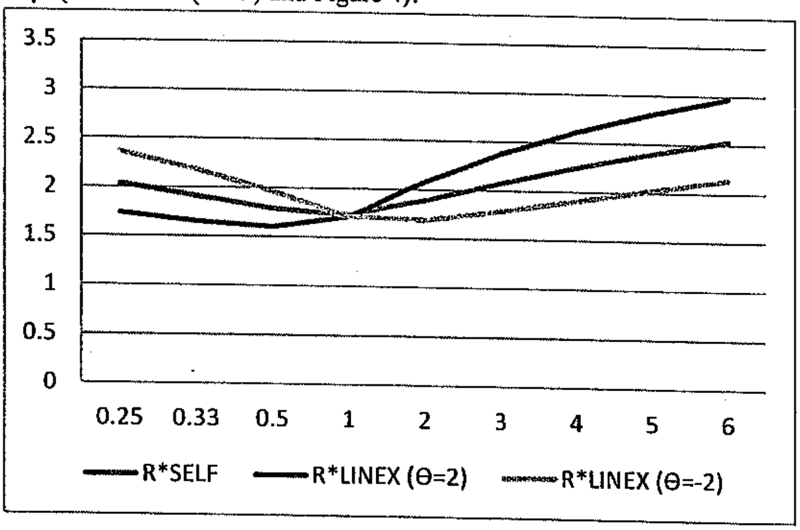

Figure 4: Efficiencies of R^*_{SELF} and R^*_{LINEX} estimates at $\theta = 2$ and $\theta = -2$ at the set size (3,3)

8. In almost all cases, for $\rho > 1$, the efficiencies of the estimates $\widehat{R}_{LINEX}$, R^*_{LINEX}, and $\grave{R}_{LINEX}$ under LINEX loss function at $\theta = 2$ are the largest comparing with the other estimates $\widehat{R}_{SELF}$, R^*_{SELF}, $\grave{R}_{SELF}$ based on SELF and $\widehat{R}_{LINEX}$, R^*_{LINEX}, and $\grave{R}_{LINEX}$ based on LINEX loss function at $\theta = -2$. On the other hand, the efficiencies of the estimators $\widehat{R}_{LINEX}$, R^*_{LINEX}, and $\grave{R}_{LINEX}$ based on LINEX loss function when $\theta = -2$, are the smallest comparing with the other estimates at the same value of ρ. (See tables (7 – 9) and Figure 4).

Table 1: Biases and MSEs for $\widehat{R}_{SELF}$ and $\widehat{R}_{LINEX}$ estimates based on SRS at $(n,m) = (2,2)$.

ρ	Biases			MSEs		
	$\widehat{R}_{SELF}$	$\widehat{R}_{LINEX}$ $\vartheta = 2$	$\widehat{R}_{LINEX}$ $\vartheta = -2$	$\widehat{R}_{SELF}$	$\widehat{R}_{LINEX}$ $\vartheta = 2$	$\widehat{R}_{LINEX}$ $\vartheta = -2$
0.25	-0.087	-0.119	-0.059	0.036	0.048	0.028
0.33	-0.081	-0.117	-0.048	0.038	0.048	0.031
0.50	-0.062	-0.102	-0.026	0.039	0.047	0.033
1	-0.014	-0.054	0.026	0.038	0.039	0.038
2	0.036	0.0009	0.075	0.035	0.030	0.041
3	0.060	0.029	0.096	0.034	0.027	0.042
4	0.064	0.037	0.101	0.030	0.023	0.040
5	0.068	0.043	0.097	0.027	0.021	0.036
6	0.068	0.046	0.096	0.025	0.019	0.033

Table 2: Biases and MSEs for $\widehat{R}_{SELF}$ and $\widehat{R}_{LINEX}$ estimates based on SRS at $(n,m) = (3,3)$.

ρ	Biases			MSEs		
	$\widehat{R}_{SELF}$	$\widehat{R}_{LINEX}$ $\vartheta = 2$	$\widehat{R}_{LINEX}$ $\vartheta = -2$	$\widehat{R}_{SELF}$	$\widehat{R}_{LINEX}$ $\vartheta = 2$	$\widehat{R}_{LINEX}$ $\vartheta = -2$
0.25	-0.053	-0.075	-0.034	0.022	0.027	0.018
0.33	-0.048	-0.073	-0.026	0.025	0.030	0.021
0.50	-0.035	-0.063	-0.009	0.028	0.032	0.024
1	0.003	-0.027	0.032	0.030	0.030	0.030
2	0.039	0.013	0.068	0.028	0.025	0.033
3	0.053	0.030	0.078	0.025	0.021	0.031
4	0.057	0.038	0.080	0.023	0.018	0.028
5	0.058	0.041	0.078	0.020	0.016	0.025
6	0.058	0.042	0.075	0.018	0.014	0.023

Table 3: Biases and MSEs for $\widehat{R}_{SELF}$ and $\widehat{R}_{LINEX}$ estimates based on SRS data at $(n,m) = (4,4)$.

ρ	Biases			MSEs		
	$\widehat{R}_{SELF}$	$\widehat{R}_{LINEX}$ $\vartheta = 2$	$\widehat{R}_{LINEX}$ $\vartheta = -2$	$\widehat{R}_{SELF}$	$\widehat{R}_{LINEX}$ $\vartheta = 2$	$\widehat{R}_{LINEX}$ $\vartheta = -2$
0.25	-0.043	-0.059	-0.028	0.017	0.020	0.014
0.33	-0.038	-0.057	-0.021	0.019	0.023	0.017
0.50	-0.028	-0.049	-0.007301	0.023	0.025	0.020
1	0.002561	-0.020	0.026	0.025	0.025	0.025
2	0.033	0.013	0.056	0.023	0.021	0.026
3	0.043	0.026	0.062	0.020	0.017	0.024
4	0.046	0.031	0.062	0.018	0.015	0.021
5	0.046	0.034	0.061	0.015	0.013	0.018
6	0.045	0.035	0.058	0.013	0.011	0.016

Table 4: Biases and MSEs for R^*_{SELF} and R^*_{LINEX} estimates based on ERSS data at $(n,m) = (2,2)$.

ρ	Biases			MSEs		
	R^*_{SELF}	R^*_{LINEX}		R^*_{SELF}	R^*_{LINEX}	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.058	-0.081	-0.038	0.024	0.030	0.019
0.33	-0.053	-0.079	-0.030	0.027	0.033	0.022
0.50	-0.039	-0.068	-0.012	0.030	0.035	0.026
1	-0.00005	-0.030	0.030	0.032	0.033	0.032
2	0.039	0.012	0.068	0.031	0.027	0.035
3	0.054	0.031	0.079	0.028	0.023	0.034
4	0.059	0.039	0.082	0.025	0.021	0.031
5	0.060	0.043	0.081	0.023	0.018	0.028
6	0.060	0.044	0.078	0.020	0.016	0.026

Table 5: Biases and MSEs for R^*_{SELF} and R^*_{LINEX} estimates based on ERSS data at $(n,m) = (3,3)$.

ρ	Biases			MSEs		
	R^*_{SELF}	R^*_{LINEX}		R^*_{SELF}	R^*_{LINEX}	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.032	-0.044	-0.022	0.011	0.013	0.009
0.33	-0.030	-0.044	-0.018	0.013	0.015	0.011
0.50	-0.023	-0.040	-0.008	0.016	0.017	0.014
1	0.001	-0.020	0.017	0.017	0.017	0.017
2	0.021	0.005	0.037	0.015	0.014	0.017
3	0.028	0.015	0.042	0.012	0.011	0.014
4	0.030	0.020	0.041	0.010	0.009	0.012
5	0.030	0.021	0.040	0.008	0.007	0009
6	0.029	0.022	0.037	0.007	0.006	0.008

Table 6:Biases and MSEs for R^*_{SELF} and R^*_{LINEX} estimates based on ERSS data at $(n,m) = (4,4)$.

ρ	Biases			MSEs		
	R^*_{SELF}	R^*_{LINEX}		R^*_{SELF}	R^*_{LINEX}	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.023	-0.031	-0.017	0.007	0.008	0.006
0.33	-0.019	-0.027	-0.014	0.008	0.009	0.008
0.50	-0.018	-0.029	-0.007	0.011	0.012	0.010
1	-0.002	-0.015	0.010	0.013	0.013	0.013
2	0.013	0.002	0.024	0.011	0.010	0.012
3	0.018	0.009	0.027	0.008	0.007	0.009
4	0.020	0.013	0.027	0.006	0.006	0.007
5	0.020	0.014	0.026	0.005	0.005	0.006
6	0.019	0.015	0.024	0.004	0.004	0.005

Table 7: Efficiencies for $\widehat{R}_{LINEX}$, $\grave{R}_{SELF}$ and $\grave{R}_{LINEX}$ estimates based on SRS and ERSS at $(n,m)=(2,2)$

ρ	SRS		ERSS		
	$\widehat{R}_{LINEX}$		$\grave{R}_{SELF}$	$\grave{R}_{LINEX}$	$\grave{R}_{LINEX}$
	$\vartheta=2$	$\vartheta=-2$		$\vartheta=2$	$\vartheta=-2$
0.25	0.787	1.290	1.509	1.206	1.877
0.33	0.793	1.259	1.421	1.163	1.724
0.50	0.833	1.185	1.312	1.129	1.503
1	0.967	1.001	1.178	1.159	1.168
2	1.159	0.844	1.138	1.291	0.988
3	1.252	0.791	1.156	1.386	0.956
4	1.294	0.769	1.182	1.452	0.957
5	1.316	0.756	1.208	1.501	0.967
6	1.329	0.748	1.233	1.539	0.980

Table 8: Efficiencies for $\widehat{R}_{LINEX}$, $\grave{R}_{SELF}$ and $\grave{R}_{LINEX}$ estimates based on SRS and ERSS at $(n,m)=(3,3)$

ρ	SRS		ERSS		
	$\widehat{R}_{LINEX}$		R^*_{SELF}	R^*_{LINEX}	R^*_{LINEX}
	$\vartheta=2$	$\vartheta=-2$		$\vartheta=2$	$\vartheta=-2$
0.25	0.806	1.229	2.035	1.737	2.357
0.33	0.824	1.199	1.912	1.653	2.181
0.50	0.866	1.134	1.788	1.600	1.962
1	0.990	0.982	1.732	1.710	1.710
2	1.142	0.861	1.890	2.077	1.687
3	1.207	0.821	2.077	2.377	1.787
4	1.237	0.803	2.247	2.614	1.906
5	1.250	0.794	2.399	2.807	2.026
6	1.257	0.789	2.538	2.971	2.142

Table 9: Efficiencies for $\widehat{R}_{LINEX}$, $\grave{R}_{SELF}$ and $\grave{R}_{LINEX}$ estimates based on SRS and ERSS at $(n,m)=(4,4)$

ρ	SRS		ERSS		
	$\widehat{R}_{LINEX}$		$\grave{R}_{SELF}$	$\grave{R}_{LINEX}$	$\grave{R}_{LINEX}$
	$\vartheta=2$	$\vartheta=-2$		$\vartheta=2$	$\vartheta=-2$
0.25	0.833	1.190	2.349	2.089	2.618
0.33	0.847	1.166	2.199	1.970	2.427
0.50	0.884	1.112	2.043	1.875	2.195
1	0.991	0.985	1.959	1.932	1.950
2	1.117	0.881	2.126	2.266	1.966
3	1.168	0.847	2.334	2.561	2.102
4	1.190	0.832	2.525	2.801	2.254
5	1.200	0.826	2.697	3.002	2.402
6	1.204	0.823	2.852	3.173	2.543

5. Conclusion

In this paper, Bayesian estimator based on ERSS technique for $R = P(X < Y)$, when X and Y follow Burr XII distribution using non informative Jeffery prior under SELF and LINEX loss functions, is obtained and compared with corresponding estimators based on SRS technique. From the numerical results, it can be concluded that, The MSEs of all estimates are decreasing and the efficiencies are increasing as the set size increasing in almost all cases. In almost all cases, MSEs of all estimates based on ERSS are smaller than the corresponding estimates based on SRS and the efficiencies of all estimators based on ERSS techniques are larger than the corresponding estimates based on SRS technique.

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