

## Half-Zumkeller Labeling for Some Cartesian product Graphs

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### ABSTRACT

A positive integer  $x$  is said to be a half-Zumkeller number if the proper positive divisors of  $x$  can be partitioned into two disjoint non-empty subsets of equal sum. Half-Zumkeller labeling of a graph  $\Gamma = (V(\Gamma), E(\Gamma))$  with  $\alpha = |V(\Gamma)|$  vertices and  $\beta = |E(\Gamma)|$  edges, is an injective mapping  $\psi$  of the vertex set  $V(\Gamma)$  into the set of natural number such that the induced mapping  $\psi^*: E(\Gamma) \rightarrow Z^+ \cup \{0\}$ , given by  $\psi^*(\lambda\mu) = \psi(\lambda)\psi(\mu)$ , is a half-Zumkeller number for all  $\lambda\mu \in E(\Gamma)$ ,  $\lambda, \mu \in V(\Gamma)$ . The graph that admits a half-Zumkeller labeling is called a half-Zumkeller graph. In this paper, we present half-Zumkeller labeling of the graphs: the stacked book graph  $SB_{m,n}$ , the cylinder grid graph  $C_{m,n}$  and the prisms of the following graphs: ladder graph  $L_n$ , the grid graph  $G_{m,n}$ , the gear graph  $G_n$ , flower graph  $FL_n$ . Furthermore, if  $H$  a non-totally disconnected subgraph of  $\Gamma$  then  $H$  is also, a half-Zumkeller graph.

### 1. Introduction

The graphs considered here will be finite, undirected, and simple connected graphs, for a graph  $\Gamma = (V, E)$ , let  $\alpha = |V(\Gamma)|$  be the cardinality of vertices  $V(\Gamma)$  and  $\beta = |E(\Gamma)|$  be that of  $E(\Gamma)$ .

In graph theory, numerous graphs can generate from given ones by using graph operation. The Cartesian product of two graphs  $\Gamma$  and  $Y$ , denoted  $\Gamma \square Y$ , is the graph with vertex set  $V(\Gamma \square Y) = V(\Gamma) \times V(Y)$  and the edge  $\{(\lambda_1, \mu_1), (\lambda_2, \mu_2)\}$  is presented in the product whenever  $\lambda_1 \lambda_2 \in E(\Gamma)$  and  $\mu_1 = \mu_2$  or symmetrically  $\mu_1 \mu_2 \in E(Y)$  and  $\lambda_1 = \lambda_2$  [1]. The Cartesian product of a graph  $\Gamma$  with  $P_2$  is called a prism over  $\Gamma$ .

A labeling of a graph is a mapping that carries a graph element (edges or vertices, or both) to positive integers, subject to certain constraints. A labeling of a graph is called edge labeling if the domain of the mapping is the edge set.

Graph labeling methods are used for application problems in communication network addressing system, for fastening Communication in sensor networks, for designing fault-tolerant systems with facility Graphs, in coding theory for the design of good radar type codes and can also use for issues in mobile Ad hoc networks [2,3].

In the early 1960s, Rosa [4] introduced the idea of graph labeling. Following this paper different techniques have been studied in graph labelings. In 1985, Edge graceful labeling was introduced by Lo [5]. An edge even graceful labeling of a graph introduced by Elsonbaty and Daoud in 2017 [6]. In 2020 Zeen El Deen [7] introduced the definition of an edge  $\delta$ -graceful labeling of many graph for any positive integer  $\delta$ . In [8] Zeen El Deen et al. introduced new classes of graphs with edge  $\delta$  graceful labeling. For a detailed survey on graph labeling refer to a dynamic survey of graph labeling [9].

### 2. Zumkeller and half-Zumkeller numbers

A positive integer  $x$  is said to be perfect if the sum of its positive factors  $\theta(x)$  is equal to  $2x$ . Generalization the concept of perfect number termed as Zumkeller number. In 2003 Zumkeller observed a sequence of integers that their positive factors can be partitioned into two disjoint subsets with equal sum, i.e., each subset with

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sum  $\theta(x)/2$  [10,11]. All the following numbers 6; 12; 20; 24; 28; 30; 40,... are Zumkeller number.

**2.1 Properties of Zumkeller Numbers:** [12,13]

1. Let  $x$  be a Zumkeller number and  $\omega$  be a prime where  $(x, \omega) = 1$  then  $x \omega^k$  is a Zumkeller number for any positive integer  $k$ .
2. For any prime  $\omega \neq 2$  and positive integer  $j$  with  $\omega \leq 2^{j+1} - 1$  the number  $2^j \omega$  is a Zumkeller Number.
3. Let  $x = 2^j \omega^k$  be a positive integer. Then  $x$  is a Zumkeller number if and only if  $\omega \leq 2^{j+1} - 1$  and  $k$  is an odd number.
4. If the prime factorization of an even Zumkeller number  $x$  is  $2^j \omega_1^{j_1} \omega_2^{j_2} \dots \dots \omega_m^{j_m}$ . Hence, at least one of  $j_i$  must be an odd number.
5. If  $\omega_1^{j_1} \omega_2^{j_2} \dots \dots \omega_m^{j_m}$  is the prime factorization of the Zumkeller number  $x$ . Hence, for any positive integers  $k_1, k_2, \dots, k_m$ . The number  $\omega_1^{k_1+l_1(k_1+1)} \omega_2^{k_2+l_2(k_2+1)} \dots \dots \omega_m^{k_m+l_m(k_m+1)}$  is also Zumkeller number.

Newly, a recent type of labeling is introduced by Balamurugan et. al. [14,15,16] called Zumkeller labeling. They studied many graphs which have Zumkeller labeling.

**Definition 2.1.1.** Zumkeller labeling of a graph  $\Gamma = (V(\Gamma), E(\Gamma))$  with  $\alpha = |V(\Gamma)|$  vertices and  $\beta = |E(\Gamma)|$  edges, is an one to one mapping  $\psi$  of the vertex set  $v(\Gamma)$  into the set of natural number  $Z^+ \cup \{0\}$  such that the induced mapping  $\psi^*: E(\Gamma) \rightarrow Z^+ \cup \{0\}$ , given by  $\psi^*(\lambda \mu) = \psi(\lambda) \psi(\mu)$ , is a Zumkeller number for all  $\lambda \mu \in E(\Gamma)$ ,  $\lambda, \mu \in v(\Gamma)$ . The graph that admits a Zumkeller labeling is called a Zumkeller graph.

Extra outcomes have been published on Zumkeller labeling of simple graphs see, [ 17,18,19,20].

**2.2 Properties of half- Zumkeller Numbers:** [10, 13]

**Definition 2.2.1.** [10] A positive integer  $x$  is said to be a **half-Zumkeller** number if the proper positive divisors of  $x$  can be partitioned into two disjoint non-empty subsets of equal sum. A half-Zumkeller partition for a half-Zumkeller number  $x$  is a partition  $\{A, B\}$  of the set of proper positive divisors of  $x$  so that each of  $A$  and  $B$  sums to the same value.

1. A positive integer  $x$  is half-Zumkeller if and only if  $\frac{\theta(x) - x}{2}$  is the sum of some distinct positive proper positive divisors of  $x$ , an example of half-Zumkeller number is 70. The proper divisors of 70 are  $a_i = \{1, 2, 5, 7, 10, 14, 35\}$  which can be partitioned into

two disjoint subset  $A = \{1, 5, 7, 10, 14\}$  and  $B = \{2, 35\}$  of equal sum which is equal 37. So

$$\theta(70) = \sum a_i = 144, \text{ then } \frac{\theta(x) - x}{2} = \frac{144 - 70}{2} = 37.$$

2. A positive even integer  $x$  is half-Zumkeller if and only if  $\frac{\theta(x) - 2x}{2}$  is the sum (possibly empty sum) of some distinct positive divisors of  $x$  excluding  $x$  and  $\frac{x}{2}$ . For example, let  $x = 70$ ,  $\frac{\theta(70) - 2(70)}{2} = \frac{144 - 140}{2} = 2$ .

3. Let  $x$  be **even number**, then  $x$  is half-Zumkeller if and only if  $x$  admits a Zumkeller partition such that  $x$  and  $\frac{x}{2}$  are in distinct subsets. Therefore, if  $x$  is an even half-Zumkeller number then  $x$  is Zumkeller. For example of an even half-Zumkeller number is 836, since the divisors of 836 are  $\{1, 2, 4, 11, 19, 22, 38, 44, 76, 209, 418, 836\}$  which can be partitioned into two disjoint subset of equal sum  $A = \{836, 4\}$  and  $B = \{1, 2, 11, 19, 22, 38, 44, 76, 209, 418\}$  where 836 is in one subset and 418 in the other subset, so 836 is Zumkeller number.

4. Let  $x$  be an even Zumkeller number. If  $\theta(x) < 3x$ , then  $x$  is half-Zumkeller, an example of an even Zumkeller number which is half-Zumkeller is 54, since  $\theta(54) = 120 < 3(54) = 162$

5. If 6 divides  $x$ ,  $x$  is Zumkeller, and  $\theta(x) < \frac{10x}{3}$ , then  $x$  is half-Zumkeller, an example of Zumkeller number which is divisible by 6 is 24. The proper divisors of 24 are  $\{1, 2, 3, 4, 6, 8, 12\}$  which can be partitioned into two disjoint subsets of equal sum,  $A = \{12, 6\}$  and  $B = \{1, 2, 3, 4, 8\}$  so  $x = 24$  is half-Zumkeller and  $\theta(24) = 60 < \frac{10x}{3} = 80$ .

6. If  $x$  is Zumkeller, then  $2x$  is half-Zumkeller, An example of Zumkeller number is  $x=30$ , then  $2x = 60$  is half-Zumkeller because the proper divisors of 60 are  $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30\}$  which can be partitioned into two disjoint subset of equal sum,  $A = \{30, 20, 4\}$  and  $B = \{1, 2, 3, 5, 6, 10, 12, 15\}$ .

7. An even Zumkeller number  $x = 2^j \omega_1^{j_1} \omega_2^{j_2} \dots \dots \omega_m^{j_m}$ , is a half- Zumkeller if  $y = 2^{j-1} \omega_1^{j_1} \omega_2^{j_2} \dots \dots \omega_m^{j_m}$ , is Zumkeller number and  $\omega_1, \omega_2, \dots, \omega_m \leq 2^j - 1$ .

An example of an even Zumkeller number is,  $x = 80$  the prime factorization of 80 is  $2^4 \cdot 5$ . To prove that it is enough to prove that Zumkeller -is a half  $x = 80$

are  $y = 2^3 \cdot 5$  divisors of The, umkellerZa is  $y = 2^3 \cdot 5$  which can be partitioned into two  $\{1, 2, 4, 5, 8, 10, 20, 40\}$  disjoint subset of equal sum,  $A = \{40, 5\}$  and  $B = \{1, 2, 4, 8, 10, 20\}$

### 3. Half-Zumkeller labeling of the stacked book graph $SB_{m,n}$

**Definition 3.1.** Half-Zumkeller labeling of a graph  $\Gamma = (V(\Gamma), E(\Gamma))$  with  $\alpha = |V(\Gamma)|$  vertices and

$\beta = |E(\Gamma)|$  edges, is an injective mapping  $\psi$  of the vertex set  $v(\Gamma)$  into the set of natural number such that the induced mapping  $\psi^*: E(\Gamma) \rightarrow Z^+ \cup \{0\}$ , given by  $\psi^*(\lambda \mu) = \psi(\lambda) \psi(\mu)$ , is a half- Zumkeller number for

all  $\lambda \mu \in E(\Gamma)$ ,  $\lambda, \mu \in v(\Gamma)$ . The graph that admits a half- Zumkeller labeling is called a half- Zumkeller graph.

**Definition 3.2.** [ 1 ] A stacked book graph denoted by  $SB_{m,n}$  is defined as the cartesian product of a star graph  $S_m$  on  $m + 1$  vertices with a path  $p_n$  on  $n$  vertices, i.e.,  $SB_{m,n} \cong S_{m+1} \square p_n$  where the symbol  $\square$  used to denote the cartesian product of two graphs. The stacked book graph  $SB_{m,n}$  contains  $m(n + 1)$  vertices and  $n(2m + 1) - (m + 1)$  edges. The vertex set  $V(SB_{m,n}) = \{\lambda_l, \mu_l^\omega : 1 \leq l \leq n, 1 \leq \omega \leq m\}$  and the edge set  $E(SB_{m,n}) = \{\lambda_l \lambda_{l+1} : 1 \leq l \leq n - 1\} \cup \{\mu_l^\omega \mu_{l+1}^\omega : 1 \leq l \leq n - 1, 1 \leq \omega \leq m\} \cup \{\lambda_l \mu_l^\omega : 1 \leq l \leq n, 1 \leq \omega \leq m\}$ . A typical picture of staked book graph  $SB_{m,n}$  is given in Fig. 1 .

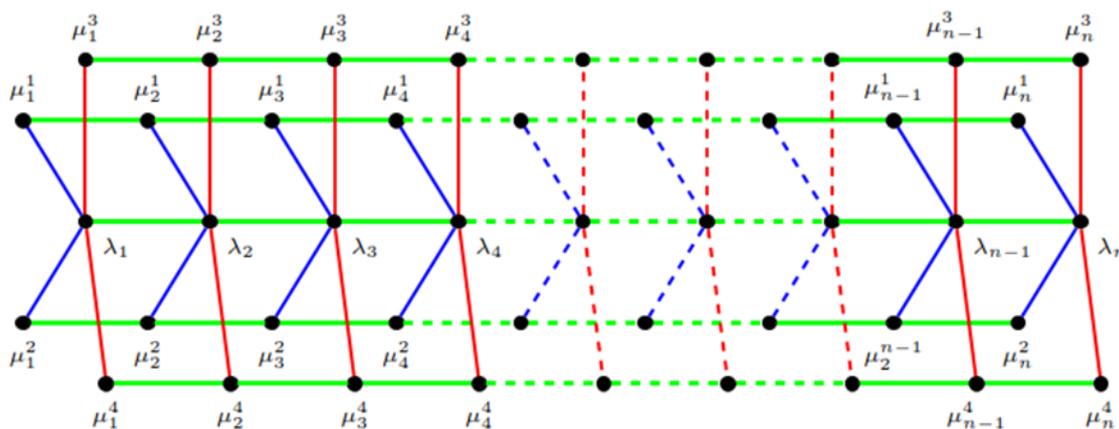


Figure 1: The stacked book graph denoted  $SB_{4,n}$

**Theorem 3.4.** The stacked book graph  $SB_{m,n}$  admits half-Zumkeller labeling.

**Proof.** An injective map  $\psi : V[SB_{m,n}] \rightarrow Z^+ \cup \{0\}$  defined on vertex set as follows

$$\psi(\lambda_l) = \frac{k}{2^2} p^l \quad 1 \leq l \leq n,$$

$$\psi(\mu_l^j) = \frac{k}{2^{2+\omega}} p^{2n+1-l} \quad 1 \leq l \leq n, \quad 1 \leq \omega \leq m.$$

Where  $p$  is an odd prime number,  $p \leq 2^k - 1$  and  $k$  is an even number  $\geq 2$ .

The induced edge labels can be calculated as follows

$$\psi^*(\lambda_l \lambda_{l+1}) = 2^k p^{2l+1} \quad 1 \leq l \leq n - 1,$$

$$\psi^*(\mu_l^\omega \mu_{l+1}^\omega) = 2^{k+2\omega} p^{4n-2l+1} \quad 1 \leq l \leq n - 1, \quad 1 \leq \omega \leq m,$$

$$\psi^*(\lambda_l \mu_l^\omega) = 2^{k+\omega} p^{2n+1} \quad 1 \leq l \leq n, \quad 1 \leq \omega \leq m.$$

Hence, from the properties of half-Zumkeller number, we can see that all the edge labeling of the stacked book graph  $SB_{m,n}$  are half-Zumkeller numbers, for example  $2^k p^{2l+1}$  is half-Zumkeller number since  $2^{k-1} p^{2l+1}$  is Zumkeller number such that  $p \leq 2^k - 1$ . Thus, the graph  $SB_{m,n}$  admits half-Zumkeller labeling.

**Illustration:** In Fig. 2 we present a half-Zumkeller labeling of stacked book graph  $SB_{4,9}$  with  $p = 7, k = 4$ .

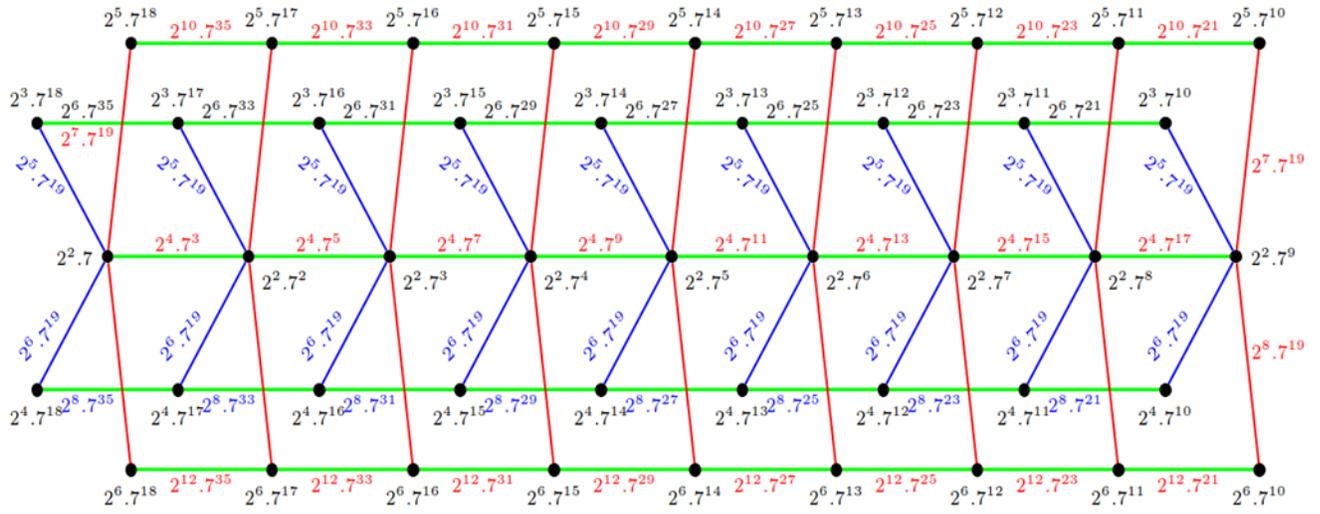


Figure 2: A half-Zumkeller labeling of stacked book graph  $SB_{4,9}$  with  $p = 7, k = 4$ .

### 3. Half-Zumkeller labeling of the Cylinder grid graph $C_{m,n}$

**Definition 4.1.** [1] The Cylinder grid graph  $C_{m,n}$  is the graph formed from the Cartesian product  $P_m \square C_n$  of the path graph  $P_m$  and the cycle graph  $C_n$ . That is the cylinder grid graph consists of  $m$  copies of  $C_n$  which represented by circles and will be numbered from the inner most circle to

the outer circle as  $C_n^1, C_n^2, C_n^3, \dots, C_n^{m-1}, C_n^m$  and we call them circles, and  $n$  copies of  $P_m$  which represented by paths transverse the  $m$  circles and will be numbered clockwise as  $P_m^1, P_m^2, P_m^3, \dots, P_m^{n-1}, P_m^n$  and we call them paths, see Fig. 3.

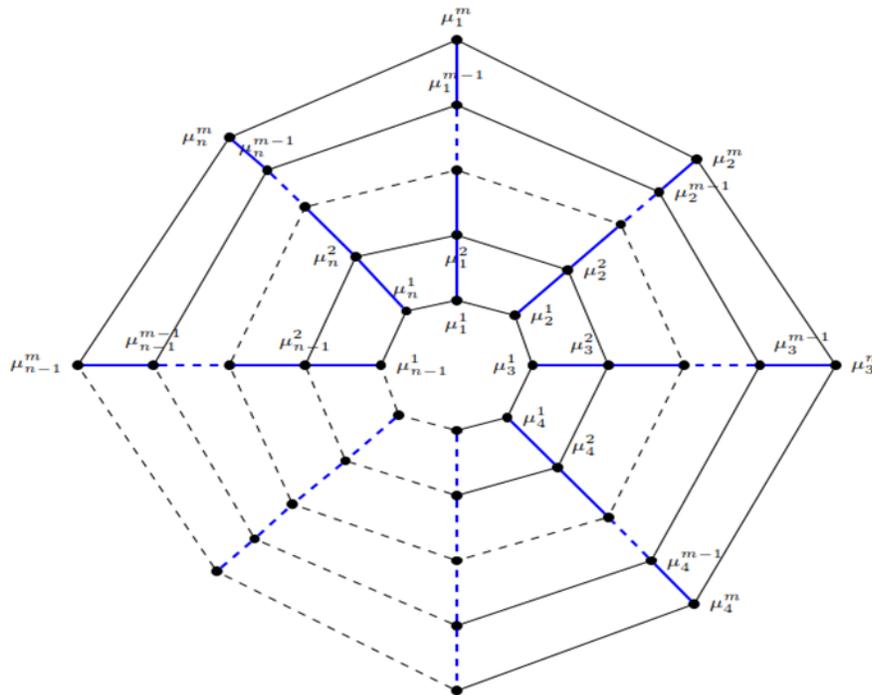


Figure 3: The cylinder grid graph  $C_{m,n}$

**Theorem 4.2.** The cylinder grid graph  $C_{m,n}$  is half-Zumkeller graph.

**Proof.** The cylinder grid graph  $C_{m,n}$ , contains  $nm$  vertices and  $n(2m - 1)$  edges, the vertex set  $V(C_{m,n})$  have the elements  $\{\mu_l^\omega : 1 \leq l \leq n, 1 \leq \omega \leq m\}$  and the edge set  $E[C_{m,n}] = \{\mu_l^\omega \mu_{l+1}^\omega : 1 \leq l \leq n - 1, 1 \leq \omega \leq m\} \cup \{\mu_n^\omega \mu_1^\omega : 1 \leq \omega \leq m\} \cup \{\mu_l^\omega \mu_l^{\omega+1} : 1 \leq l \leq n, 1 \leq \omega \leq m - 1\}$ .

We consider following two cases

**Case (1):** when  $n$  is even.

An injective map  $\psi : V [C_{m,n}] \rightarrow Z^+ \cup \{0\}$  defined on vertex set as follows

$$\forall 1 \leq l \leq n, \quad \psi(\mu_l^\omega) = \begin{cases} 2^{\frac{\omega+k-1}{2}} p^l & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\frac{\omega+k-2}{2}} q^{l+1} & \text{if } \omega \equiv 0 \pmod 2; \end{cases}$$

Where  $p$  is an odd prime number,  $p, q \leq 2^{k+\omega} - 1$  and  $k$  is an even number  $\geq 2$ .

The induced edge labels can be calculated as follows

$$\begin{aligned} \psi^*(\mu_l^\omega \mu_{l+1}^\omega) &= 2^{\omega+k-1} p^l q^{l+1}, \quad 1 \leq l \leq n \text{ and } 1 \leq \omega \leq m - 1, \\ \forall 1 \leq l \leq n - 1, \quad \psi^*(\mu_l^\omega \mu_{l+1}^\omega) &= \begin{cases} 2^{\omega+k-1} p^{2l+1} & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\omega+k-2} q^{2l+3} & \text{if } \omega \equiv 0 \pmod 2; \end{cases} \\ \psi^*(\mu_n^\omega \mu_1^\omega) &= \begin{cases} 2^{\omega+k-1} p^{n+1} & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\omega+k-2} q^{n+3} & \text{if } \omega \equiv 0 \pmod 2; \end{cases} \end{aligned}$$

**Case (2):** When  $n$  is odd. An injective map  $\psi : V [C_{m,n}] \rightarrow Z^+ \cup \{0\}$  defined on vertex set as follows

$$\begin{aligned} \psi(\mu_1^\omega) &= \begin{cases} 2^{\frac{\omega+k-1}{2}} q & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\frac{\omega+k-2}{2}} p & \text{if } \omega \equiv 0 \pmod 2; \end{cases} \\ \forall 2 \leq l \leq n, \quad \psi(\mu_l^\omega) &= \begin{cases} 2^{\frac{\omega+k-1}{2}} p^l & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\frac{\omega+k-2}{2}} q^{l+1} & \text{if } \omega \equiv 0 \pmod 2; \end{cases} \end{aligned}$$

The induced edge labels can be calculated as follows

$$\begin{aligned} \psi^*(\mu_1^\omega \mu_1^{\omega+1}) &= 2^{\omega+k-1} p q, \quad 1 \leq \omega \leq m - 1, \\ \psi^*(\mu_l^\omega \mu_l^{\omega+1}) &= 2^{\omega+k-1} p^l q^{l+1}, \quad 1 \leq l \leq n \text{ and } 1 \leq \omega \leq m - 1, \\ \psi^*(\mu_1^\omega \mu_2^\omega) &= \begin{cases} 2^{\omega+k-1} p^2 q & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\omega+k-2} p q^3 & \text{if } \omega \equiv 0 \pmod 2; \end{cases} \\ \forall 2 \leq l \leq n - 1, \quad \psi^*(\mu_l^\omega \mu_{l+1}^\omega) &= \begin{cases} 2^{\omega+k-1} p^{2l+1} & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\omega+k-2} q^{2l+3} & \text{if } \omega \equiv 0 \pmod 2; \end{cases} \\ \psi^*(\mu_n^\omega \mu_1^\omega) &= \begin{cases} 2^{\omega+k-1} p^n q & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\omega+k-2} p q^{n+1} & \text{if } \omega \equiv 0 \pmod 2. \end{cases} \end{aligned}$$

Hence, from the properties of half-Zumkeller number, we can see that all the edge labeling of the Cylinder grid graph  $C_{m,n}$  are half-Zumkeller numbers, for example  $2^{\omega+k-1} p q$  is half-Zumkeller number since  $2^{\omega+k-2} p q$  is Zumkeller number such that  $p \leq 2^{\omega+k-1} - 1$ . Thus the graph  $C_{m,n}$  admits half-Zumkeller labeling.

**Illustration:** In Fig. 4 we present the cylinder grid graph  $C_{5,8}$  with half-zumkeller labeling.

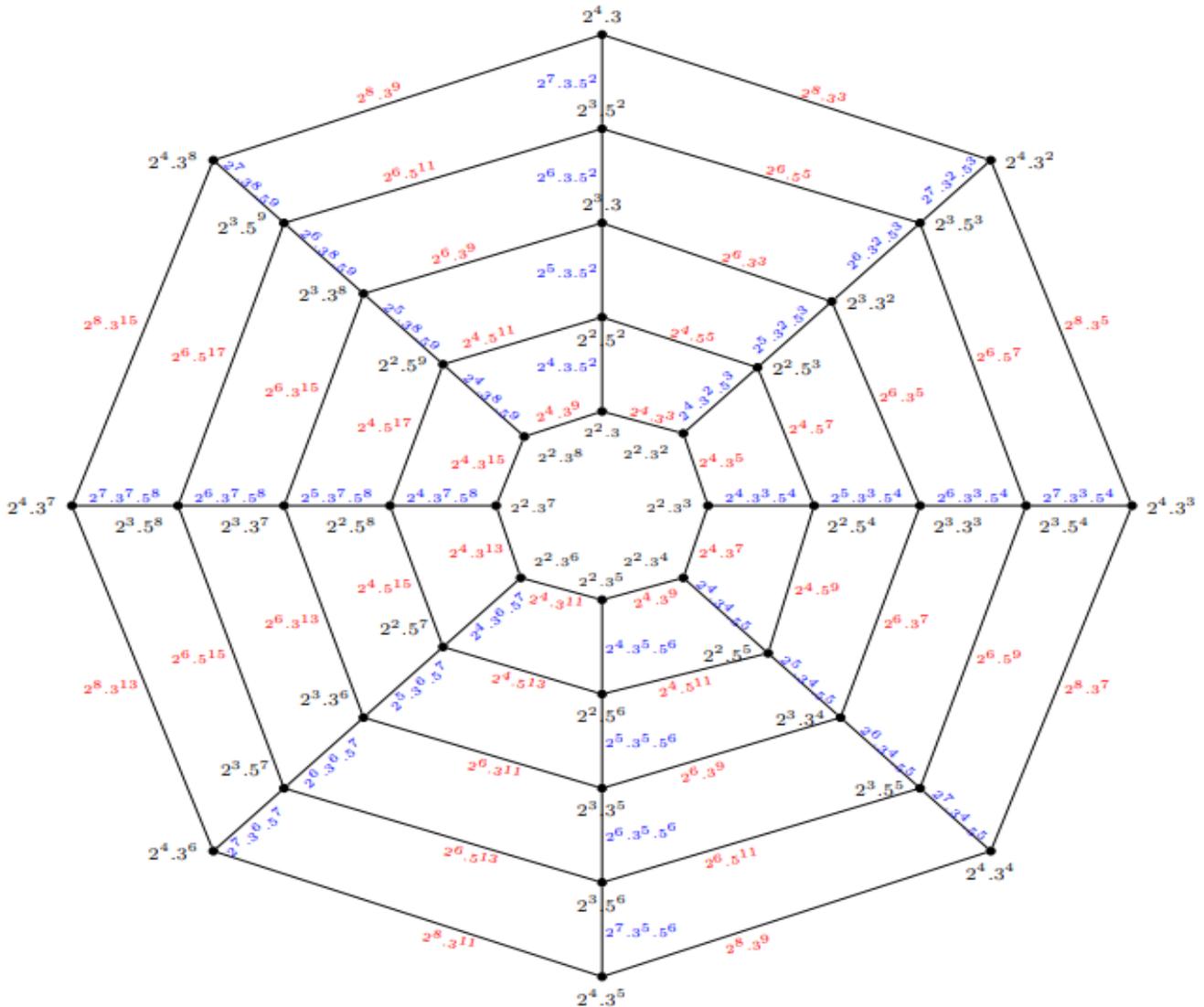


Figure 4: Half-Zumkeller labeling of the cylinder grid graph  $C_{5,8}$  with  $p = 3, q = 5$  and  $k = 4$ .

5. Half-Zumkeller labeling of the prism of the ladder  $P(L_n)$

**Definition 5.1.** The prism of the ladder graph  $P(L_n) = L_n \square P_2$  contains  $4n$  vertices and  $8n - 4$  edges,  $n \geq 2$ , see Fig. 5. The vertex set  $V(P(L_n))$  have the elements  $\{\lambda_l, \lambda'_l, \mu_l, \mu'_l : 1 \leq l \leq n\}$  and the edge set  $E[P(L_n)] = \{\lambda_l \lambda_{l+1}, \lambda'_l \lambda'_{l+1}, \mu_l \mu_{l+1}, \mu'_l \mu'_{l+1} : 1 \leq l \leq n - 1\} \cup \{\lambda_l \lambda'_l, \mu_l \mu'_l, \lambda_l \mu_l, \lambda'_l \mu'_l : 1 \leq l \leq n\}$ .

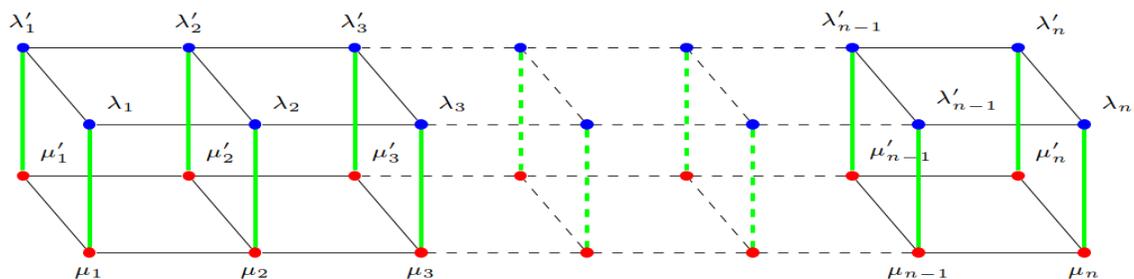


Figure 5: The prism  $P(L_n) = L_n \square P_2$  of the ladder graph  $L_n$

**Theorem 5.2.** The prism of the ladder graph  $P(L_n)$  admits half-Zumkeller labeling.

**Proof.** An injective map  $\psi : V [P(L_n)] \rightarrow Z^+ \cup \{0\}$  defined on vertex set as follows

$$\begin{aligned} \psi (\lambda'_l) &= 2^{\frac{k}{2}+1} p^l & 1 \leq l \leq n, \\ \psi (\lambda_l) &= 2^{\frac{k}{2}+1} p^{2n+1-l} & 1 \leq l \leq n, \\ \psi (\mu_l) &= 2^{\frac{k}{2}} p^l & 1 \leq l \leq n, \\ \psi (\mu'_l) &= 2^{\frac{k}{2}} p^{2n+1-l} & 1 \leq l \leq n. \end{aligned}$$

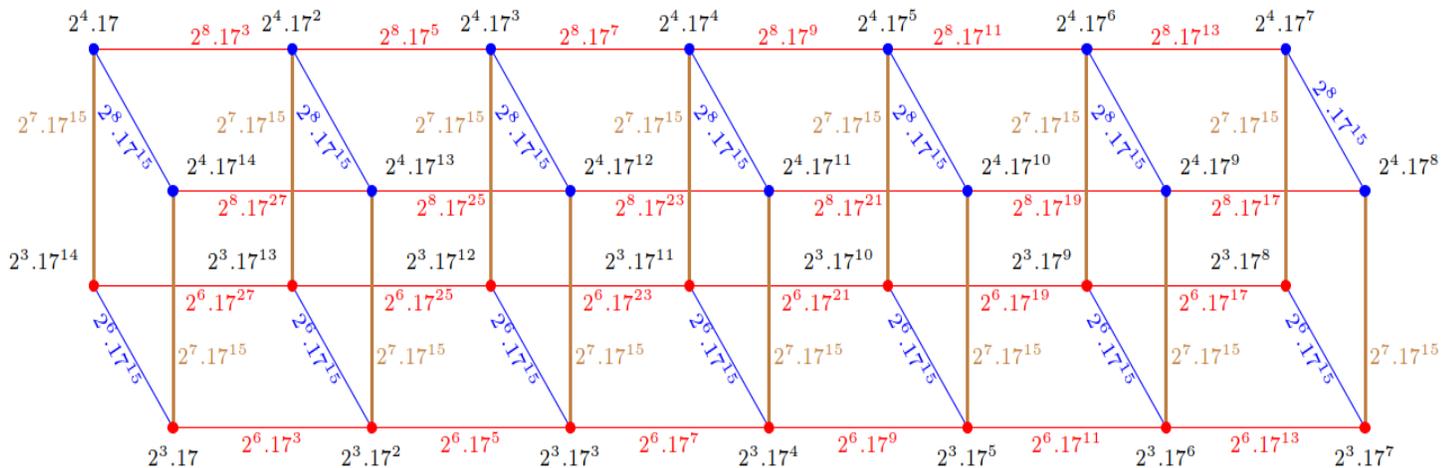
Where  $p$  is an odd prime number,  $p \leq 2^k - 1$  and  $k$  is an even number .

In view of the above defined labeling pattern, the induced edge labels  $\psi^*$  can be obtained as follows:

$$\begin{aligned} \psi^* (\lambda'_l \lambda'_{l+1}) &= 2^{k+2} p^{2l+1} & 1 \leq l \leq n-1, \\ \psi^* (\lambda_l \lambda_{l+1}) &= 2^{k+2} p^{4n-2l+1} & 1 \leq l \leq n-1, \\ \psi^* (\mu_l \mu_{l+1}) &= 2^k p^{2l+1} & 1 \leq l \leq n-1, \\ \psi^* (\mu'_l \mu'_{l+1}) &= 2^k p^{4n-2l+1} & 1 \leq l \leq n-1, \\ \psi^* (\lambda_l \lambda'_l) &= 2^{k+2} p^{2n+1} & 1 \leq l \leq n, \\ \psi^* (\mu_l \mu'_l) &= 2^k p^{2n+1} & 1 \leq l \leq n, \\ \psi^* (\lambda_l \mu_l) &= \psi^* (\lambda'_l \mu'_l) = 2^{k+1} p^{2n+1} & 1 \leq l \leq n. \end{aligned}$$

Hence, from the properties of half-Zumkeller number, we can see that all the edge labeling of the prism of ladder graph  $P(L_n)$  are all half-Zumkeller numbers, for example  $2^k p^{2n+1}$  is half-Zumkeller number since  $2^{k-1} p^{2n+1}$  is Zumkeller number and  $p \leq 2^k - 1$ . Thus, the prism of ladder graph  $P(L_n)$  admits Half-Zumkeller labeling.

**Illustration:** In Fig. 6 we present a half-Zumkeller labeling of  $P(L_7)$  with  $p = 17, k = 6$ .



**Figure 6:** Half-Zumkeller labeling of prism  $P(L_7)$  of the ladder graph with  $p = 17, k = 6$ .

### 6. Half-Zumkeller labeling of the prism of the grid graph $G_{m,n}$

**Definition 6.1.** [1] The grid graph  $G_{m,n}$  is the Cartesian product  $P_m \square P_n$ , and the prism of the grid graph  $(G_{m,n})$   $G_{m,n} \square P_2 = P_m \square P_n \square P_2 = L_m \square P_2$ , so  $L_m \square P_n$  is the prism of the grid graph  $(G_{m,n})$ , see Fig. 7. The prism of the grid graph  $P(G_{m,n})$ , contains  $2nm$  vertices and  $5mn - 2(m + n)$  edges. The vertex set

$V [P(G_{m,n})]$  have the elements  $\{\lambda_l^\omega, \mu_l^\omega : 1 \leq l \leq n, 1 \leq \omega \leq m\}$  and the edge set  $E [P(G_{m,n})] = \{\lambda_l^\omega \lambda_{l+1}^\omega, \mu_l^\omega \mu_{l+1}^\omega : 1 \leq l \leq n-1, 1 \leq \omega \leq m\} \cup \{\lambda_l^\omega \mu_l^\omega : 1 \leq l \leq n, 1 \leq \omega \leq m\} \cup \{\lambda_l^\omega \lambda_l^{\omega+1}, \mu_l^\omega \mu_l^{\omega+1} : 1 \leq l \leq n, 1 \leq \omega \leq m-1\}$ .

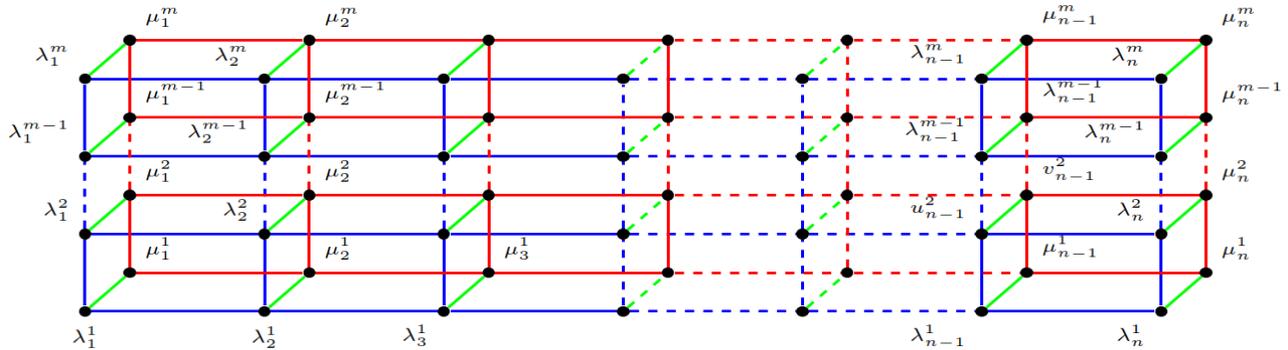


Figure 7: The prism  $P(G_{m,n})$  of the grid graph  $G_{m,n}$

**Theorem 6.2.** The prism of the grid graph  $P(G_{m,n})$ , admits half-Zumkeller labeling.

**Proof.** An injective map  $\psi : V [P(G_{m,n})] \rightarrow Z^+ \cup \{0\}$  defined on vertex set as follows

$$\psi (\lambda_l^\omega) = \begin{cases} 2^{\frac{k}{2}-1+\omega} p^l & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\frac{k}{2}-1+\omega} p^{2n+1-l} & \text{if } \omega \equiv 0 \pmod 2; \end{cases}$$

$$\psi (\mu_l^\omega) = \begin{cases} 2^{\frac{k}{2}-1+\omega} p^{2n+1-l} & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{\frac{k}{2}-1+\omega} p^l & \text{if } \omega \equiv 0 \pmod 2; \end{cases}$$

Where  $p$  is an odd prime number,  $p \leq 2^{k+2\omega-2} - 1$  and  $k$  is an even number  $\geq 2$ .

In view of the above defined labeling pattern, the induced edge labels  $\psi^*$  can be obtained as follows:

$$\psi^* (\lambda_l^\omega \mu_l^\omega) = 2^{k-2+2\omega} p^{2n+1},$$

$$\psi^* (\lambda_l^\omega \lambda_{l+1}^\omega) = \psi^* (\mu_l^\omega \mu_{l+1}^\omega) = 2^{k-1+2\omega} p^{2n+1},$$

$$\psi^* (\lambda_l^\omega \lambda_{l+1}^\omega) = \begin{cases} 2^{k-2+2\omega} p^{2l+1} & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{k-2+2\omega} p^{4n+1-2l} & \text{if } \omega \equiv 0 \pmod 2; \end{cases}$$

$$\psi^* (\mu_l^\omega \mu_{l+1}^\omega) = \begin{cases} 2^{k-2+2\omega} p^{4n+1-2l} & \text{if } \omega \equiv 1 \pmod 2; \\ 2^{k-2+2\omega} p^{2l+1} & \text{if } \omega \equiv 0 \pmod 2. \end{cases}$$

Hence, from the properties of half-Zumkeller number, we can see that all the edge labeling of the prism  $P(G_{m,n})$  of the grid graph  $G_{m,n}$  are all half-Zumkeller numbers, for example  $2^{k+2\omega-2} p^{2n+1}$  is half-Zumkeller number since  $2^{k+2\omega-3} p^{2n+1}$  is Zumkeller number such that  $p \leq 2^{k+2\omega-2} - 1$ . Thus, the prism of the grid graph  $P(G_{m,n})$  admits Half-Zumkeller labeling which complete the proof.

**Illustration:** In Fig. 8, we present a half-Zumkeller labeling of prism  $P(G_{7,4})$  of the grid graph  $G_{7,4}$  with  $p = 31, k = 8$ .

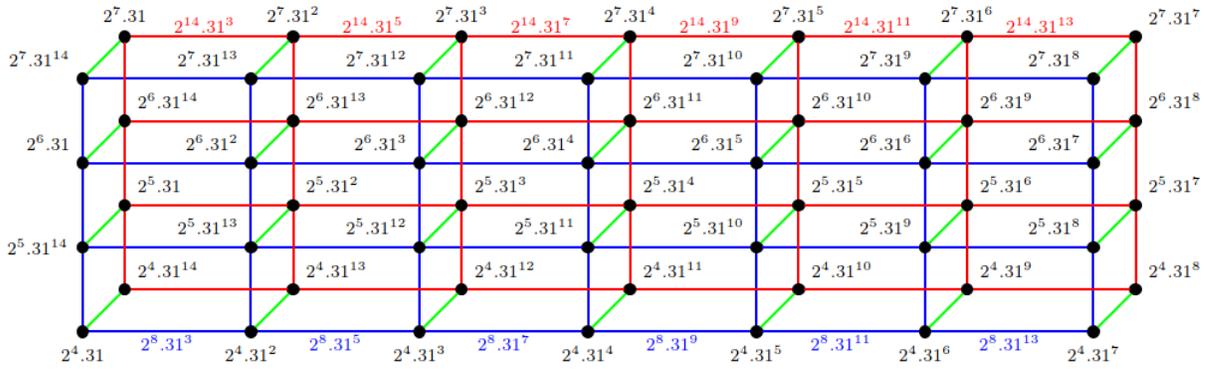


Figure 8: Half-Zumkeller labeling of prism  $P(G_{7,4})$  of the grid graph  $G_{7,4}$  with  $p = 31, k = 8$ .

7. Half-Zumkeller labeling of the prism of the gear graph  $G_n$

Definition 7.1. The prism  $P(G_n)$  of the gear graph  $G_n$  has number of vertices  $|V(P(G_n))| = 4n + 2$  and edges  $|E(P(G_n))| = 8n + 1$ , see Fig.9. The vertex set  $V(P(G_n)) = \{\mu_0, \lambda_0, \mu_l, \lambda_l, w_l, t_l : 1 \leq l \leq n\}$  and the edge set

$$E[P(G_n)] = \{\mu_0 \lambda_0, w_n \mu_1, t_n \lambda_1\} \cup \{\mu_0 \mu_l, \lambda_0 \lambda_l, w_l \mu_{l+1}, t_l \lambda_{l+1}, \mu_l w_l, \lambda_l t_l, \mu_l \lambda_l, w_l t_l : 1 \leq l \leq n\}$$

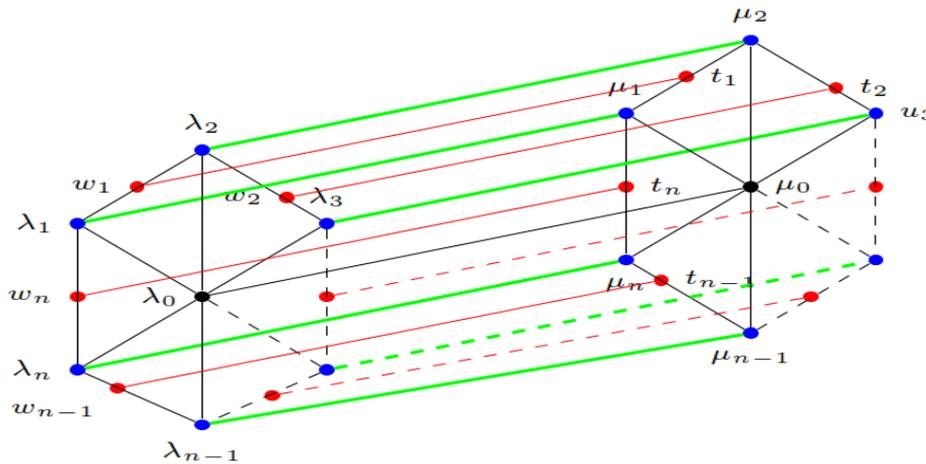


Figure 9: The prism  $P(G_n)$  of the gear graph  $G_n$

Theorem 7.2. The prism of the gear graph  $P(G_n)$ , admits half-Zumkeller labeling.

Proof. An injective map  $\psi : V[P(G_n)] \rightarrow \mathbb{Z}^+ \cup \{0\}$  defined on vertex set as follows

$$\begin{aligned} \psi(\mu_0) &= 2^{\frac{k}{2}} q, \\ \psi(\lambda_0) &= 2^{\frac{k}{2}} q^2, \\ \psi(\mu_l) &= 2^{\frac{k}{2}} p^{2l}, \\ \psi(\lambda_l) &= 2^{\frac{k}{2}} p^{2l-1}, \\ \psi(w_l) &= 2^{\frac{k}{2}} q^{2l+1}, \\ \psi(t_l) &= 2^{\frac{k}{2}} q^{2l+2}. \end{aligned}$$

Where  $p$  and  $q$  are distinct odd prime numbers and  $p, q \leq 2^k - 1$  and  $k$  is an even number.

In view of the above defined labeling pattern, the induced edge labels  $\psi^*$  can be obtained as follows:

$$\begin{aligned} \psi^*(\mu_0 \lambda_0) &= 2^k q^3, \\ \psi^*(\mu_0 \mu_l) &= 2^k p^{2l} q, \\ \psi^*(w_l \mu_{l+1}) &= 2^k p^{2l+2} q^{2l+1}, \\ \psi^*(\mu_l w_l) &= 2^k p^{2l} q^{2l+1}, \\ \psi^*(w_n \mu_1) &= 2^k p^2 q^{2n+1}, \\ \psi^*(\lambda_0 \lambda_l) &= 2^k p^{2l-1} q^2, \\ \psi^*(t_l \lambda_{l+1}) &= 2^k p^{2l+1} q^{2l+2}, \\ \psi^*(\lambda_l t_l) &= 2^k p^{2l-1} q^{2l+2}, \\ \psi^*(t_n \lambda_1) &= 2^k p q^{2n+2}, \end{aligned}$$

$$\psi^*(\mu_l \lambda_l) = 2^k p^{4l-1},$$

$$\psi^*(w_l t_l) = 2^k q^{4l+3}.$$

Hence, from the properties of half-Zumkeller number, we can see that all the edge labeling of the prism  $P(G_n)$  of the gear graph  $G_n$  are all half-Zumkeller numbers, for example  $2^k p^{2l} q$  is half-Zumkeller number since

$2^{k-1} p^{2l} q$  is Zumkeller number and  $p, q \leq 2^k - 1$ . Thus, the prism of the gear graph  $P(G_n)$  admits Half-Zumkeller labeling which complete the proof.

**Illustration:** In Fig. 10, we present a half-Zumkeller labeling of prism  $P(G_6)$  of the gear graph  $G_6$  with  $p = 13, q = 29, k = 6$ .

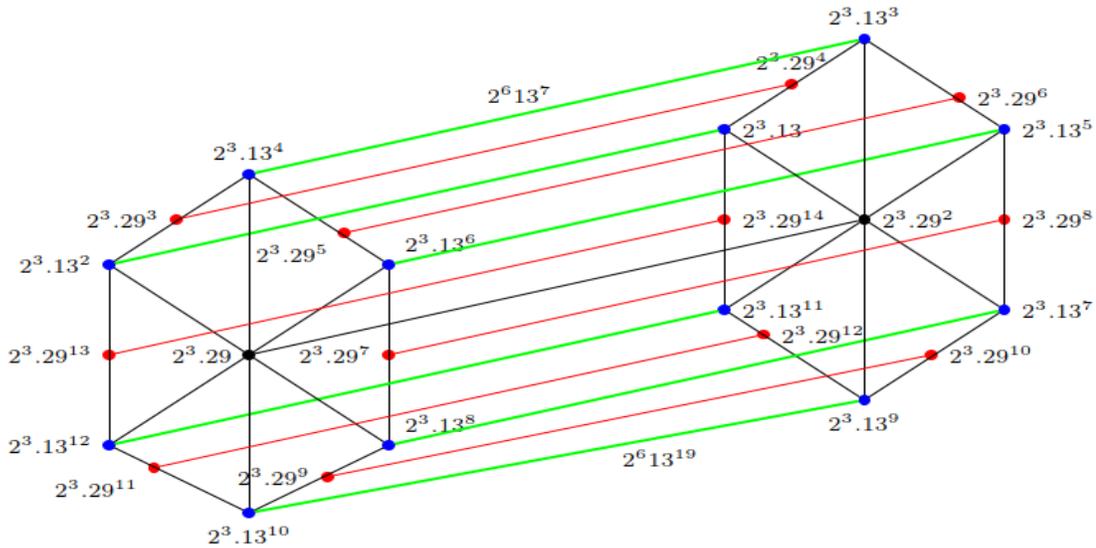


Figure 10: Half-Zumkeller labeling of  $P(G_6)$  of the gear graph  $G_6$  with  $p = 13, q = 29$  and  $k = 6$ .

### 8. Half-Zumkeller labeling of the prism of the flower graph $FL_n$

**Definition 8.1.** The flower graph  $FL_n$  ( $n \geq 3$ ) is the graph obtained from a helm  $H_n$  by joining each pendant vertex to the center of the helm [3]. The prism  $P(FL_n)$  of the flower graph  $FL_n$  has number of vertices  $|V(P(FL_n))| = 4n + 2$  and edges  $|E(FL_n)| = 10n + 1$ , see Fig. 11. The vertex set  $V(P(FL_n)) = \{\mu_0, \lambda_0, \mu_l, \lambda_l, w_l, t_l : 1 \leq l \leq n\}$  and the edge set

$$E[P(FL_n)] = \{\mu_0 \lambda_0, \mu_n \mu_1, \lambda_n \lambda_1\} \cup \{\mu_0 \mu_l, \lambda_0 \lambda_l, \mu_0 w_l, \lambda_0 t_l, \mu_l \mu_{l+1}, \lambda_l \lambda_{l+1}, \mu_l w_l, \lambda_l t_l, \mu_l \lambda_l, w_l t_l : 1 \leq l \leq n\}$$

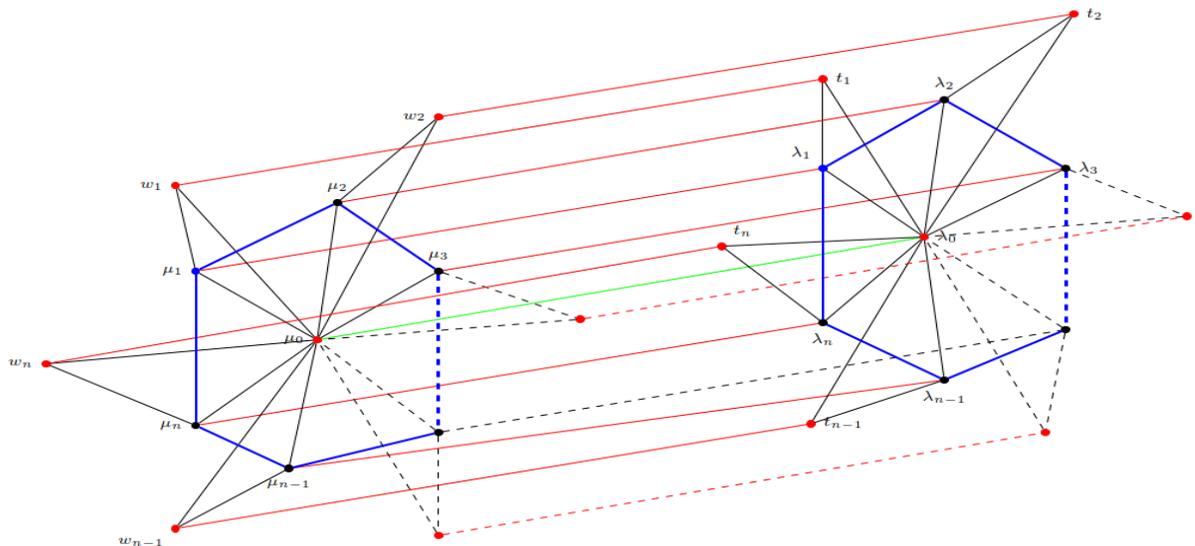


Figure 11: The prism  $P(FL_n)$  of the flower graph  $FL_n$

**Theorem 8.2.** The prism of the flower graph  $P(FL_n)$ , admits half-Zumkeller labeling.

**Proof.** An injective map  $\psi : V [P(FL_n)] \rightarrow \mathbb{Z}^+ \cup \{0\}$  defined on vertex set as follows

$$\begin{aligned} \psi(\mu_0) &= 2^{\frac{k}{2}} r, \\ \psi(\lambda_0) &= 2^{\frac{k}{2}} q, \\ \psi(\mu_l) &= 2^{\frac{k}{2}} p^l, \\ \psi(\lambda_l) &= 2^{\frac{k}{2}} p^{2n+1-l}, \\ \psi(w_l) &= 2^{\frac{k}{2}} q^{2l+1}, \\ \psi(t_l) &= 2^{\frac{k}{2}} r^{2l+1}. \end{aligned}$$

Where  $p, q$  and  $r$  are distinct odd prime numbers and  $p, q, r \leq 2^k - 1$  and  $k$  is an even number.

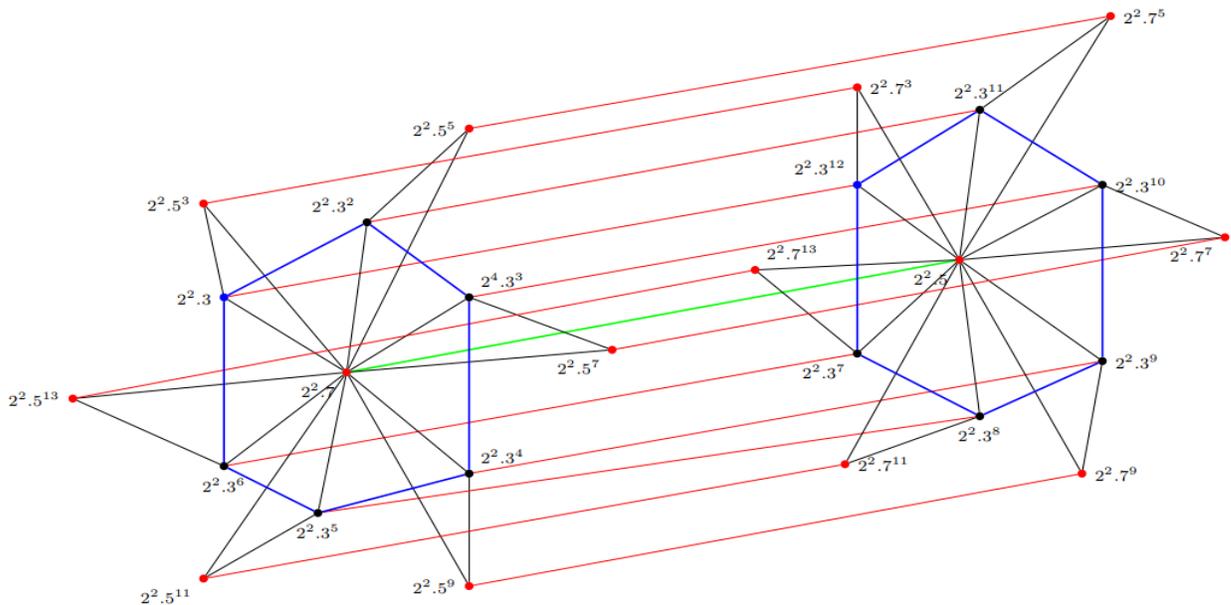
The edge labels that are induced can be obtained as follows considering the labelling pattern that was previously defined:

$$\begin{aligned} \psi^*(\mu_0 \lambda_0) &= 2^k q r, \\ \psi^*(\mu_0 \mu_l) &= 2^k p^l r, \\ \psi^*(\mu_0 w_l) &= 2^k q^{2l+1} r, \\ \psi^*(\mu_l \mu_{l+1}) &= 2^k p^{2l+1}, \\ \psi^*(\mu_n \mu_1) &= 2^k p^{n+1}, \end{aligned}$$

$$\begin{aligned} \psi^*(\mu_l w_l) &= 2^k p^l q^{2l+1}, \\ \psi^*(\mu_l \lambda_l) &= 2^k p^{2n+1}, \\ \psi^*(w_l t_l) &= 2^k q^{2l+1} r^{2l+1}, \\ \psi^*(\lambda_0 \lambda_l) &= 2^k p^{2n+1-l} q, \\ \psi^*(\lambda_0 t_l) &= 2^k q r^{2l+1}, \\ \psi^*(\lambda_l \mu_{l+1}) &= 2^k p^{4n-2l+1}, \\ \psi^*(\lambda_n \lambda_1) &= 2^k p^{3n+1}, \\ \psi^*(\lambda_l t_l) &= 2^k p^{2n+1-l} r^{2l+1}. \end{aligned}$$

Hence, from the properties of half-Zumkeller number, we can see that all the edge labeling of the prism  $P(FL_n)$  of the flower graph  $FL_n$  are all half-Zumkeller numbers, for example  $2^k q r$  is half-Zumkeller number since  $2^{k-1} q r$  is Zumkeller number and  $q, r \leq 2^k - 1$ . Thus, the prism of the flower graph  $P(FL_n)$  admits Half-Zumkeller labeling which complete the proof.

**Illustration:** In Fig. 12, we present a half-Zumkeller labeling of prism  $P(FL_n)$  of the flower graph  $FL_6$  with  $p = 3, q = 5, r = 7, k = 4$ .



**Figure 12:** Half-Zumkeller labeling of  $P(FL_6)$ , with  $p = 3, q = 5, r = 7$  and  $k = 4$

**Lemma 8.3.** Let  $\Gamma$  be a half- Zumkeller graph and  $H$  a non-totally disconnected subgraph of  $\Gamma$ . Then  $H$  is also, a half- Zumkeller graph.

Using **Lemma (8.3)**, we can see that the graphs: star graph, the path graph, ladder graph, grid graph, the gear graph, and the flower graph are half-Zumkeller graphs.

### Conclusion

In the past few years, Zumkeller labeling of graphs have been studied intensely and these topics continue to be an attractive in the field of graph theory and discrete mathematics. A great number of published papers and results exist. So far, many graphs are unknown if it is half-Zumkeller or not. In this work, a new type of labeling called half-Zumkeller labeling is defined, we studied the notations of half-Zumkeller numbers and some of their properties, then we defined a new type of labeling called half-Zumkeller labeling. Half-Zumkeller labeling of some special classes of graphs like the stacked book graph, the Cylinder grid graph, the prism of the ladder graph, the prism of the grid graph, the prism of the gear graph and the prism of the flower graph are investigated.

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