Alternative Quadratic Exponential Form to Optimal Control Problem

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Abstract

This paper provides an extension of the optimal control problem of multi-item inventory model with deteriorating items using the negative value of natural logarithm of deterioration and spoilage function, depending on the alternative quadratic exponential form. The solution of optimal control problem of this model can be carried out using the Pontryagin principle. The controlled system of non-linear differential equations will be solved numerically. Different functions of the demand rates are used to study the effect of demand rates on the optimal inventory levels, optimal production rates and then measure the optimal value of objective function.

Keywords: State Variables, Co-state Variables, Control Variables, Multi-item Inventory Model, Pontryagin Principle, Boundary Conditions, Quadratic Exponential Form, Runge-Kutta Method.

1 Introduction

Most of the classical inventory models deal with single item. But in real world situation, a single item inventory seldom occurs. It is a common experience that: the presence of a second item in an inventory favors the demand of the first and vice-versa. This is why the companies and retailers deal with several items and stock them in their warehouses. This leads to the idea of multi-item inventory problems. The purpose remains the same for single-item as well as for multi-item inventory. It is either maximization of total average profit or minimization of total average cost during a given period. Consequently, the analysis for a single-item inventory is almost parallel to that of multi-item inventory. Also, the results obtained are almost identical in single-item and in multi-item inventory. As the inventory is containing of deteriorating items, so they are sold out, better than returned. The literature on multi-item dynamic inventory models is really sparse, since most of the classical studies are concerned with a single item inventory model. We cite some of the most recent ones, in order to give an idea on the wide range of optimal control applications in the multi-item inventory-production system. Multi-item classical inventory models under resource constraints are available in well known books [9, 10, 13, 16]. Ben-Daya and Raouf [3] have developed approach for a more realistic, they consider a multi-item with budgetary and floor- or shelf- space constraints. They assumed that, the demand of the items follows uniform probability distribution. Also they have discussed multi-item inventory model with stochastic demand subject to the restrictions on available space and budget. Bhattacharya [4] has studied a two-item inventory for deteriorating items with a linear stock dependent demand rates. Lenard and Roy [12] defined another approach for the determination of inventory policies based on the notion of efficient policy surface and extended to multi-item inventory control by defining the concepts of family and aggregate item. As in the case of a single item inventory, Kar et al. [11] have also considered density dependent demand rate for multi-item inventory. Sulem [17] has been determined the optimal ordering policy for impulse control of a deterministic two product inventory system subject to constant demand rates, linear storage and shortage costs and economies of joint ordering. Padmanabhan and Vrat [14] have presented an EQQ model for items with stock dependent consumption rate and exponential decay. Rosenblatt [15] has discussed multi-item inventory system with budgetary constraint comparison between the Lagrangian and the fixed cycle approach. Different mathematicians like Worell and Hall [18] have applied different programming methods to solve multi-items inventory problems. El-Gohary and El-sayed [5] have presented an optimal control of two-item inventory model with deteriorating items using the objective function in quadratic form which represented the total cost. Foul et al. [7], and Al-Babtain [2] have presented an optimal control of HMMS reverse logistic model with deteriorating items with different types of demand and return rates using total cost function. Grain and El-sayed [8] presented an optimal control problem with natural deterioration function also with quadratic form for the total cost. Al-Babtain [1] presented a necessary condition for multi-item inventory model using the total profits function. As we see, most of the previous studies used the quadratic form or integral total cost or profit functions in the optimal control problems that represent the objective function.

In this paper, we use the quadratic exponential form [19], and its development [6], to formulate the objective function (minimization problem) which represents the negative value of natural logarithm of deterioration and spoilage function. Also, we explain the effect of different types of demand rates on the optimal trajectory for the inventory levels, the production rates and the objective function. This paper is organized as follows, In section 2, we consider the formulation of the quadratic exponential form to show how it is used to describe the deterioration function.

Section 3 introduces the optimal control problem under some constraints using the minimum Pontryagin principle. Section 4 presents the numerical solution for the model using different types of demand rates and displayed that using many curves and one table depending on Maple program. Finally, section 5 presents conclusion.

2 Alternative Quadratic Exponential Form

Normally, we use the quadratic form or integral total cost in the optimal control problem of inventory models for describing the objective function, but in this study we will use the quadratic exponential form, because it provides the purpose and give us the simplicity to deal with the differential equations as we see later.

The Quadratic exponential form (QEF) for the two correlated variables X_1, X_2 is:

$$f(x_1,x_2)=\frac{1}{\sum_{x_1,x_2}\exp\left\{\theta_1 x_1+\theta_2 x_2+\theta_{12} x_1 x_2\right\}}\exp\left\{\theta_1 x_1+\theta_2 x_2+\theta_{12} x_1 x_2\right\}. \quad (1)$$

Where, θ_1, θ_2 are natural parameters, θ_{12} is associated parameter and all θ's > 0.

El-Sayed et al. [6] presented a simple form for the previous function, (1), using the normalizing term, it is named alternative quadratic exponential form (AQEF) for two binary variables X_1, X_2:

$$f(x_1,x_2)=\exp\left\{\theta_1 x_1+\theta_2 x_2+\theta_{12} x_1 x_2-\log(1+\theta_1+\theta_2+\theta_{12})\right\}. \quad (2)$$

In this study we will use the AQEF to formulate the objective function of this problem, which we aim to reach a minimum value of the negative value of natural logarithm of the deterioration and spoilage function:

$$f(x_1,x_2)=\exp\left\{\theta_1 x_1+\theta_2 x_2+\theta_{12} x_1 x_2-\log(1+\psi_1 u_1+\psi_2 u_2+\psi_{12} u_1 u_2)\right\}. \quad (3)$$

Since, in this study we supposed that, for the normalizing term, θ_1 represents $\psi_1 u_1$, θ_2 represents $\psi_2 u_2$ and θ_{12} represents $\psi_{12} u_1 u_2$, where the parameters ψ_1, ψ_2 and ψ_{12} depend on the control variables U_1 and U_2.

3 Optimal Control Problem

In this section we be concerned with describe the problem of two inventory model with different types of deterioration. The results of this study extend the studies of Bhattacharya [4], and El-Gohary and El-Sayed [5], but with different function for the objective function and different functions for the demand rates. The objective function can be represented as a negative value of natural logarithm of deterioration and spoilage function. The demand rates also can be represented in four types: constant, linear, logistic and periodic functions. The problem is presented as optimal control problem with two state variables (inventory levels) and two control variables (production rates). The constraints represent the change rates of inventory levels for two items affected by many types of deterioration and spoilage. An important part of any optimal control problem is the process of putting the mathematical form of the system under some considerations. We consider in this study the firm produces two items and puts the finished products at one warehouse. We suppose that the all productions are presented in the warehouse. There are two types of deterioration, natural deterioration and damage happens for one item as a result of presence another item in one warehouse. Also, we suppose that there is a spoilage rate for each item and spoilage rate for both. Let us define the following parameters which are used in the mathematical formulation of the model:

$X_i(t)$: The inventory levels at time t.

$U_i(t)$: The production rates at time t.

T : The length of planning period.

x_{i0} : The initial inventory levels.

a_{ii} : The deteriration coefficient due to self-contact of item x_i.

a_{ij} : The deteriration coefficient of x_i due to presence a unit of x_j, $i \neq j = 1,2$.

$D_i(x_1, x_2, t)$: The demand rates of (x_1, x_2).

ψ_1 : The spoilage rate of of x_1, $\psi_1 > 0$.

ψ_2 : The spoilage rate of of x_2, $\psi_2 > 0$.

ψ_{12} : The spoilage rate of (x_1, x_2), jointly, $\psi_{12} > 0$.

θ_1 : The natural deterioration rate of x_1, $\theta_1 > 0$.

θ_2 : The natural deterioration rate of x_2, $\theta_2 > 0$.

θ_{12} : The natural deterioration rate of (x_1, x_2), jointly, $\theta_{12} > 0$.

Next, we will use the obove assumptions and Pontrygin principle to describe the optimal control problem of two-inventory model with deterioration.

The main purpose of this subsection is describe the mathematical form of the optimal control problem using the objective function with two constraints as shown below:

The negative value of natural logarithm of the function (3), that represents the deterioration and spoilage function, can be obtained as:

$$J = -\ln f(x_1, x_2) = -\theta_1 x_1 - \theta_2 x_2 - \theta_{12} x_1 x_2 + \log(1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12} u_1 u_2). \tag{4}$$

So, the problem can be formulated as

$$\text{Minimize } \left\{ J = -\theta_1 x_1 - \theta_2 x_2 - \theta_{12} x_1 x_2 + \log(1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12} u_1 u_2) \right\}, \tag{5}$$

subject to:

$$\begin{aligned} \dot{x}_1 &= -x_1 \left(\theta_1 + a_{12} x_2 + a_{11} x_1 \right) - D_1 + u_1, \\ \dot{x}_2 &= -x_2 \left(\theta_2 + a_{21} x_2 + a_{22} x_2 \right) - D_2 + u_2, \end{aligned} \tag{6}$$

and

$$x_1 = x_1(t) \ge 0, \quad x_2 = x_2(t) \ge 0, \quad u_1 = u_1(t) \ge 0, \quad u_2 = u_2(t) \ge 0. \tag{7}$$

Where,

$$t \in T, \quad D = D(x_1, x_2, t) \ge 0, \quad \theta_1, \theta_2, \theta_{12} > 0, \quad \psi_1, \psi_2, \psi_{12} > 0.$$

Using the Pontryagin principle, let us define $J = \dot{x}_0$, and introduce the co-state variables λ_0, λ_1 and λ_2 corresponding to the state variables X_0, X_1 and X_2 respectively. From (5) and (6), we can write the Hamiltonian function as follows:

$$H = \lambda_0 \dot{x}_0 + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2, \tag{8}$$

In addition, to obtain the co-state equations and the Lagrange multipliers associated with the constraints (6), we form the Lagrange function as follows:

$$L = H + \mu_1(t)x_1 + \mu_2(t)x_2 + \mu_3(t)u_1 + \mu_4(t)u_2 \quad , \tag{9}$$

where, $\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t)$ are known as Lagrange multipliers.

These Lagrange multipliers satisfy the conditions:

$$\mu_1(t) \geq 0, \mu_2(t) \geq 0, \mu_3(t) \geq 0, \mu_4(t) \geq 0, \quad \mu_1 x_1(t) = 0, \quad \mu_2 x_2(t) = 0, \quad \mu_3 u_1(t) = 0, \quad \mu_4 u_2(t) = 0. \tag{10}$$

From (9), we can easily obtain the co-state equations

$$\dot{\lambda}_i(t) = -\frac{\partial L}{\partial x_i}, \quad i = 0,1,2, \tag{11}$$

then,

$$\dot{\lambda}_0(t) = -\frac{\partial L}{\partial x_0} = 0, \quad \dot{\lambda}_1(t) = -\frac{\partial L}{\partial x_1}, \quad \dot{\lambda}_2(t) = -\frac{\partial L}{\partial x_2}, \tag{12}$$

The first equation of the system (12) shows that the co-state variable $\lambda_0(t)$ remains constant along the optimal trajectory, and the Pontryagin principle requires that this constant should be a negative value as

$$\lambda_0(t) = -1. \tag{13}$$

Here, we fixed the value of this co-state variable $\lambda_0(t)$ to concentrate on the effect of different types of demand rates along the optimal trajectory. Substituting from (5), (6),(8) and (13) in (9) we can write L in the form

$$\begin{aligned} L = {} & \theta_1 x_1 + \theta_2 x_2 + \theta_{12} x_1 x_2 - \log(1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12} u_1 u_2) \\ & + \lambda_1\left[-x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - D_1 + u_1\right] + \lambda_2\left[-x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - D_2 + u_2\right]. \\ & + \mu_1 x_1 + \mu_2 x_2 + \mu_3 u_1 + \mu_4 u_2. \end{aligned} \tag{14}$$

From conditions (7) and (10), we get

$$\mu_1(t) = \mu_2(t) = \mu_3(t) = \mu_4(t) = 0. \tag{15}$$

Substituting from (13) and (14) into (12) we get

$$\dot{\lambda}_1 = \lambda_1(\frac{\partial D_1}{\partial x_1} + \theta_1 + a_{12}x_2 + 2a_{11}x_1) + \lambda_2(\frac{\partial D_2}{\partial x_1} + a_{21}x_2) - \theta_1 - \theta_{12}x_2, \quad (16)$$

$$\dot{\lambda}_2 = \lambda_2(\frac{\partial D_2}{\partial x_2} + \theta_2 + a_{21}x_1 + 2a_{22}x_2) + \lambda_1(\frac{\partial D_1}{\partial x_2} + a_{12}x_1) - \theta_2 - \theta_{12}x_1, \quad (17)$$

with boundary conditions

$$\lambda_i(T) \neq 0, \quad i = 1,2. \quad (18)$$

Where T is the length of planning period can be suggested.

To obtain the optimal production rates (control variables) U_i, $i = 1,2$, we differentiate the Lagrange function (14) with respect to u_1, u_2 respectively and putting equal to zero, we get

$$\frac{\partial L}{\partial u_1} = -\frac{\psi_1 + \psi_{12}u_2}{1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12}u_1 u_2} + \lambda_1 = 0.$$

$$\frac{\partial L}{\partial u_2} = -\frac{\psi_2 + \psi_{12}u_1}{1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12}u_1 u_2} + \lambda_2 = 0.$$

Then,

$$u_1^*(t) = \frac{1}{\lambda_1} - \frac{1 + \psi_2 \hat{u}_2}{\psi_1 + \psi_{12}\hat{u}_2}, \quad \lambda_1 \neq 0, \quad (19)$$

$$u_2^*(t) = \frac{1}{\lambda_2} - \frac{1 + \psi_1 \hat{u}_1}{\psi_2 + \psi_{12}\hat{u}_1}, \quad \lambda_2 \neq 0 \quad (20)$$

Since, $\hat{u}_1$ and $\hat{u}_2$ are goal levels of production rates at the end of planning period, T. Using the equations (6), (16) and (17) have the controlled system of non-linear ordinary differential equations:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - D_1 + u_1 \\
\dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - D_2 + u_2 \\
\dot{\lambda}_1 &= \lambda_1\left(\frac{\partial D_1}{\partial x_1} + \theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + \lambda_2\left(\frac{\partial D_2}{\partial x_1} + a_{21}x_2\right) - \theta_1 - \theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(\frac{\partial D_2}{\partial x_2} + \theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + \lambda_1\left(\frac{\partial D_1}{\partial x_2} + a_{12}x_1\right) - \theta_2 - \theta_{12}x_1
\end{aligned}\right\}, \qquad (21)$$

This system can be used to describe the time evolution of inventory levels and production rates. The analytical solution of this system is very difficult and we can solve it numerically.

4 Numerical Solution

The solution of optimal control problem of this model will be carried out using Pontryagin Principle, see [16]. The numerical solution is to be necessary when the analytical solution absence for the non-linear system (21). In this solution we solve the non-linear ordinary differential equations using Runge-Kutta method and Maple program, using the initial and boundary values for $x_1(t), x_2(t), \lambda_1(t)$ and $\lambda_2(t)$.

The numerical solution can be explained by different types of demand rates as:

1. The demand rates are constant:

   $D(x_1, x_2, t) = \gamma_i$.

2. The demand rates are linear functions of inventory levels:

   $D(x_1, x_2, t) = \gamma_i + w_i x_i$.

3. The demand rates are logistic functions of inventory levels:

   $D(x_1, x_2, t) = 2x_i(\kappa_i - x_i)$.

4. The demand rates are periodic functions of time:

   $D(x_1, x_2, t) = 1 - b_i \cos(t)$.

where γ_i, w_i, κ_i and $b_i (i = 1,2)$ are positive constants.

Table 1 presents the values of system parameters and the initial states which are used in the numerical examples in the four cases of demand rate functions as follow:

Table 1. Values and initial states of system parameters

$\hat{u}_1$	$\hat{u}_2$	θ_1	θ_2	θ_{12}	a_{12}	a_{21}	a_{11}	a_{22}	γ_1
20	20	0.05	0.07	0.06	0.8	0.9	0.05	0.04	0.6
x_{10}	x_{20}	w_1	w_2	κ_1	κ_2	b_1	b_2	T	γ_2
5	5	0.9	0.8	0.4	0.5	0.7	0.8	5	0.7
ψ_1	ψ_2	ψ_{12}	$\lambda_1(T)$	$\lambda_2(T)$					
0.05	0.07	0.06	1	1					

Hint: In the logistic function demand rates, we will use the next values

$\hat{u}_1$	$\hat{u}_2$	T	$\lambda_1(0)$	$\lambda_2(0)$
200	200	1	1	1

The reset values are fixed same as other cases (constant, linear and periodic).

The next subsections explain the controlled system for each case of the demand rates functions (constant, linear, logistic and periodic) as shown below.

4.1 Constant Rates

In this subsection, we will present the model with demand function as constant demand rates, $D(x_1, x_2, t) = \gamma_i$. Substituting in the controlled system (21) by the constant demand rates, we have

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - \gamma_1 + \hat{u}_1 \\
\dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - \gamma_2 + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\left(\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + a_{21}\lambda_2 x_2 - \theta_1 - \theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - \theta_{12}x_1
\end{aligned}\right\}, \qquad (22)$$

Solving the controlled system (22) numerically, using Maple program, we get some results can be displayed by figures (1.1:1.5).

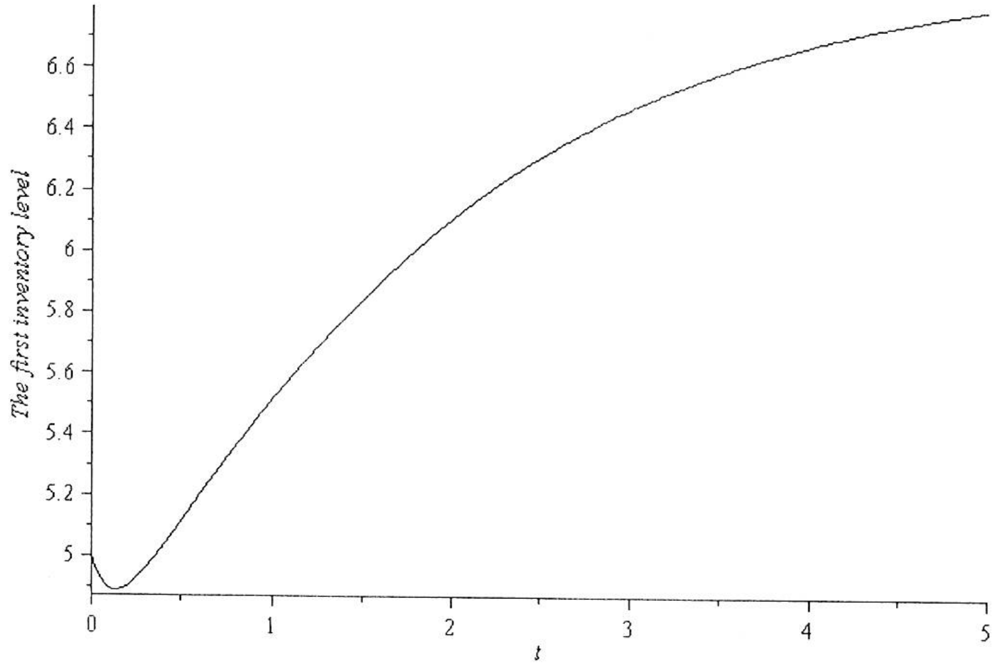

Fig.1.1 The first inventory level against time *t*

Figure 1.1 indicates that the first inventory level starts from initial value and tends to be increased as time increases.

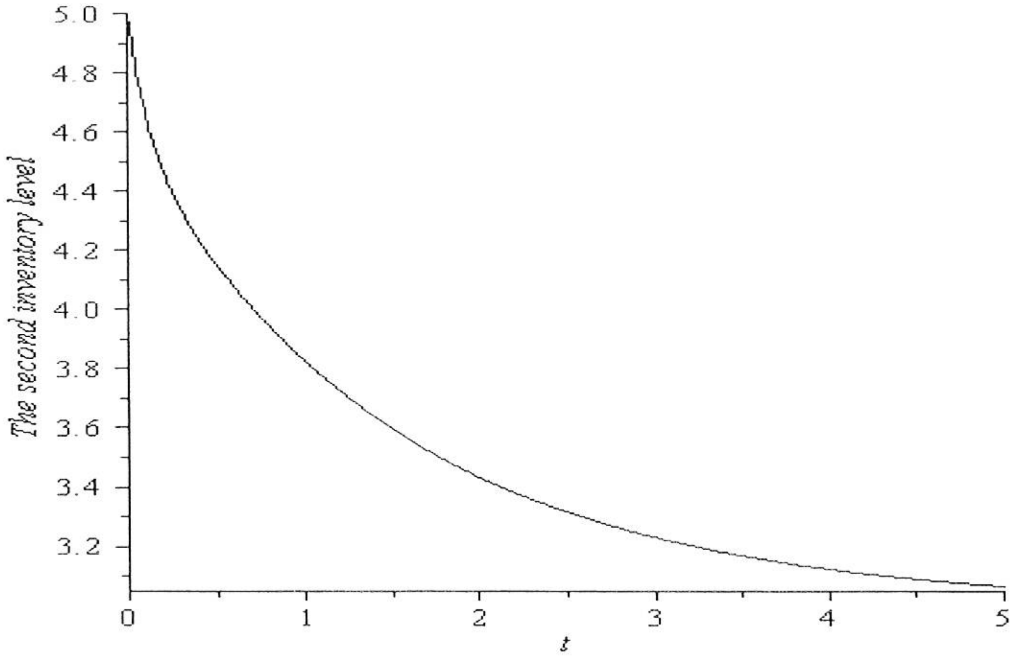

Fig.1.2 The second inventory level against time *t*

Figure 1.2 indicates that the second inventory level starts from initial value and tends to be decreased as time increases.

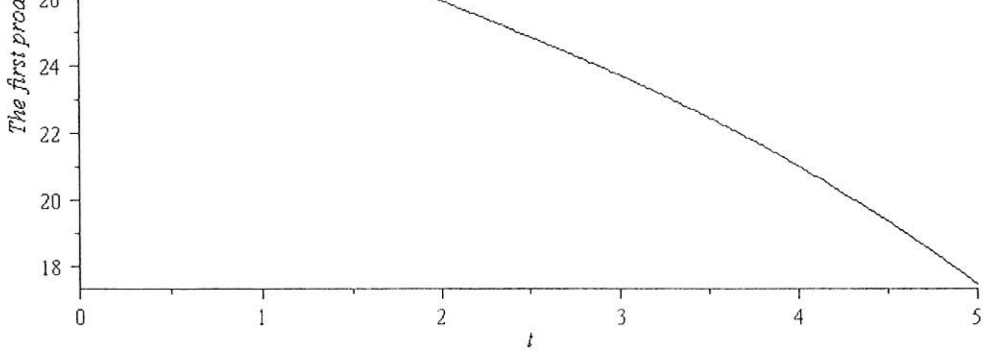

Fig.1.3 The first production rate against time *t*

Figure 1.3 indicates that first production rate is time decreasing and also tends to its goal rate (20) before the end of planning period.

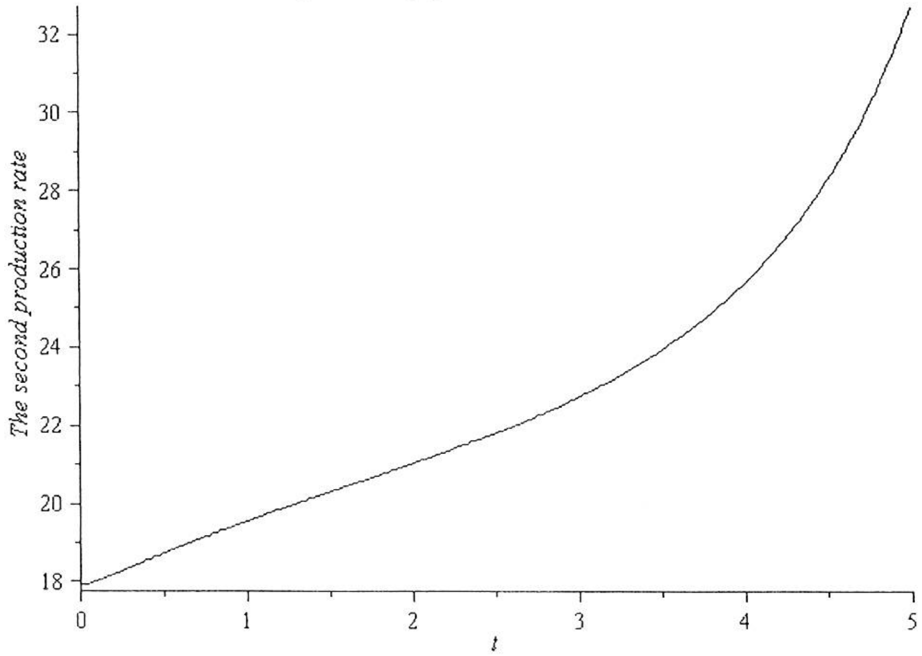

Fig.1.4 The second production rate against time *t*

Figure 1.4 indicates that the second production rate is time increasing and also tends to its goal rate (20) before the end of planning period.

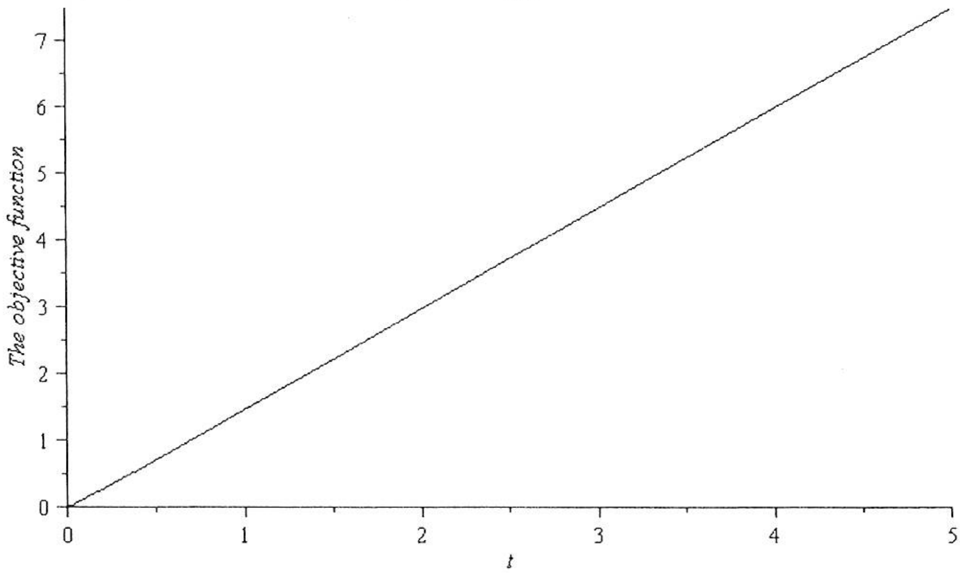

Fig.1.5 The objective function against time *t*

Figure 1.5 indicates that the increasing of the objective function, as time increased, returns to the increasing of the first inventory level and decreasing of the first production rate.

4.2 Linear Rates

In this subsection, we will present the model with demand function as linear demand rates, $D(x_1, x_2, t) = \gamma_i + w_i x_i$. Substituting in the controlled system (21) by the linear demand rates, we have

$$\left.\begin{aligned} \dot{x}_1 &= -x_1\left(\omega_1 + \theta_1 + a_{12}x_2 + a_{11}x_1\right) - \gamma_1 + \hat{u}_1 \\ \dot{x}_2 &= -x_2\left(\omega_2 + \theta_2 + a_{21}x_2 + a_{22}x_2\right) - \gamma_2 + \hat{u}_2 \\ \dot{\lambda}_1 &= \lambda_1\left(\omega_1 + \theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + a_{21}\lambda_2 x_2 - \theta_1 - \theta_{12}x_2 \\ \dot{\lambda}_2 &= \lambda_2\left(\omega_2 + \theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - \theta_{12}x_1 \end{aligned}\right\}, \tag{23}$$

Solving the controlled system (23) numerically, using Maple program, we get some results can be displayed by figures (2.1:2.5).

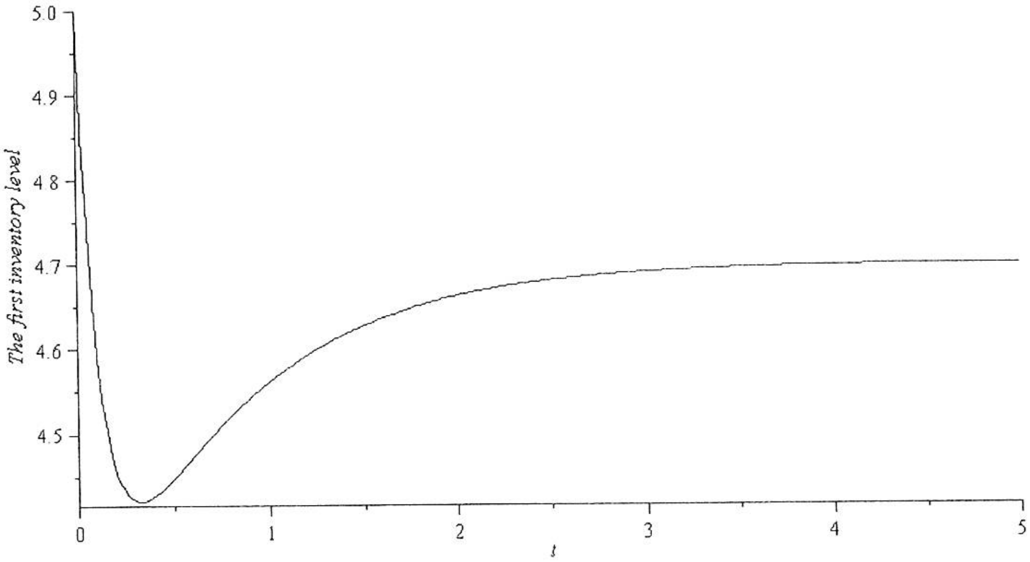

Fig.2.1 The first inventory level against time *t*

Figure 2.1 indicates that the first inventory level starts from initial value, decreases and tends to be increased as time increases.

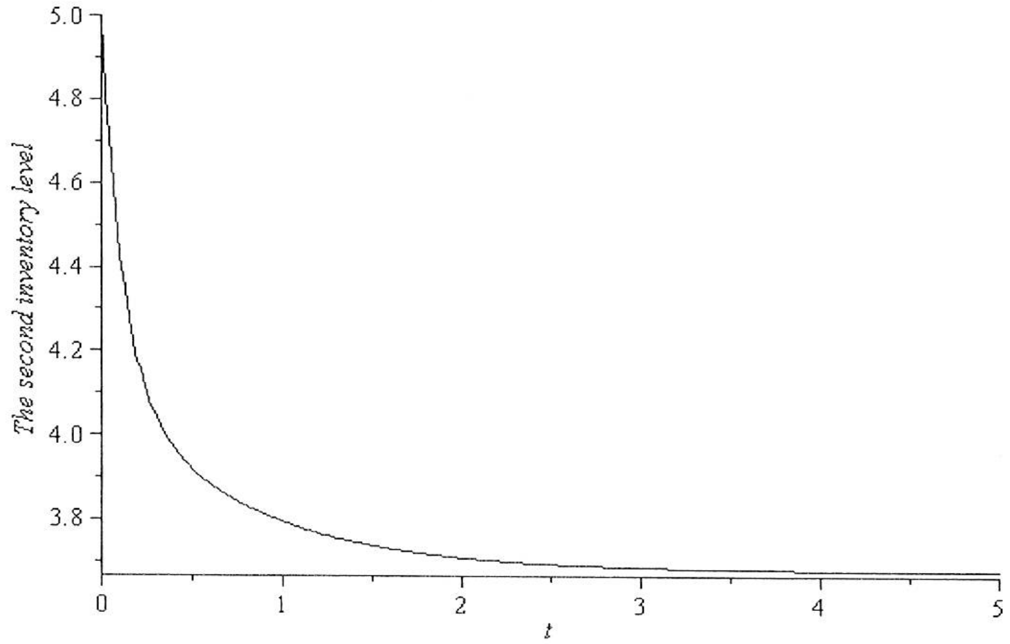

Fig.2.2 The second inventory level against time *t*

Figure 2.2 indicates that the second inventory level starts from initial value and tends to be decreased as time increases.

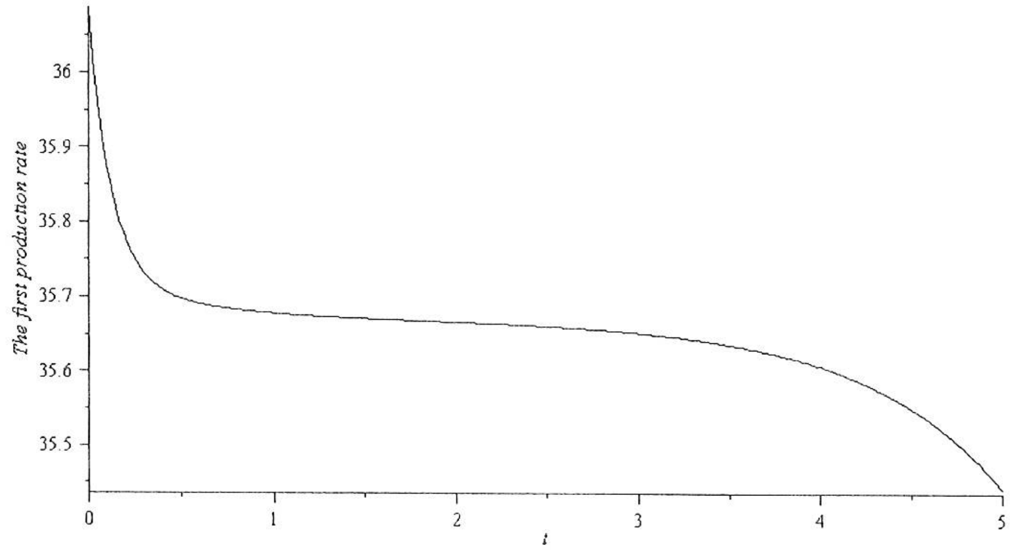

Fig.2.3 The first production rate against time *t*

Figure 2.3 indicates that first production rate is time decreasing and also tends to its goal rate (20) at the end of planning period.

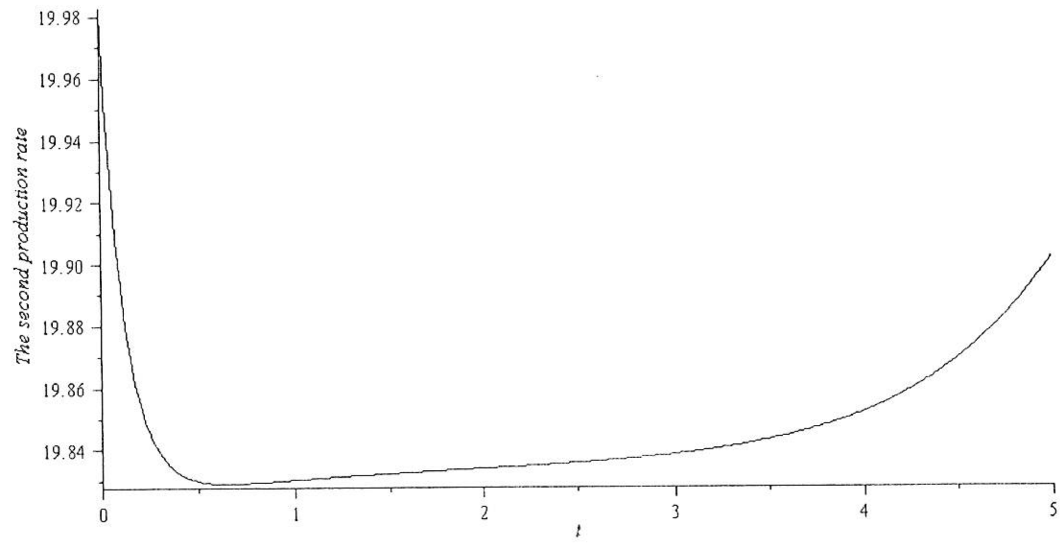

Fig.2.4 The second production rate against time t

Figure 2.4 indicates that second production rate decreases, changes to be time increasing and also tends to its goal rate (20) at the end of planning period.

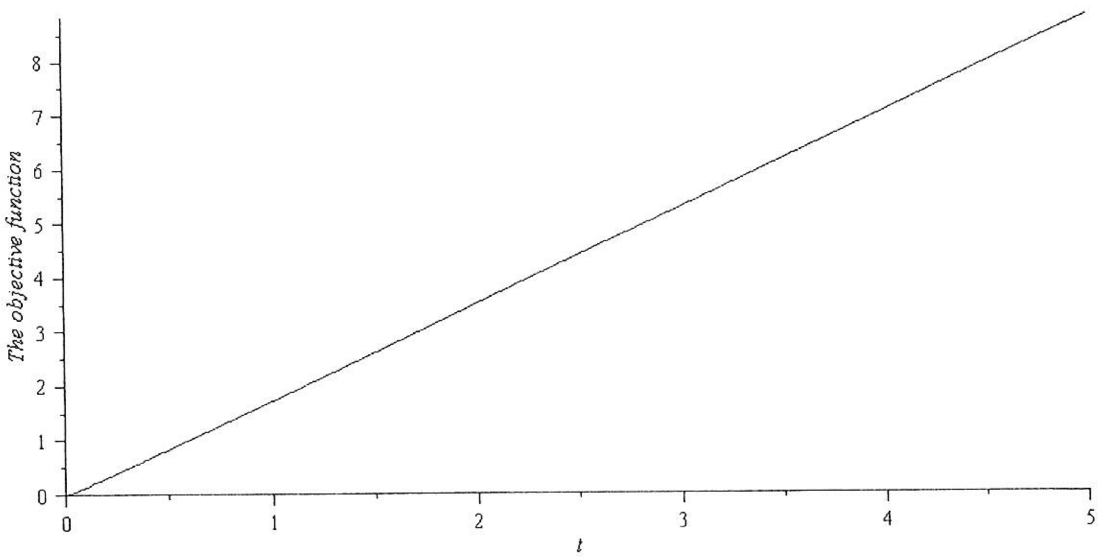

Fig.2.5 The objective function against time t

Figure 2.5 indicates that the increasing of the objective function, as time increased, returns to the increasing of the first inventory level and decreasing of the first production rate.

4.3 Logistic Rates

In this subsection, we will present the model with demand function as logistic demand rates, $D(x_1, x_2, t) = 2x_i(\kappa_i - x_i)$. Substituting in the controlled system (21) by the logistic demand rates, we have

$$\left.\begin{aligned}\dot{x}_1 &= -x_1\left(2(\kappa_1 - x_1) + \theta_1 + a_{12}x_2 + a_{11}x_1\right) + \hat{u}_1 \\ \dot{x}_2 &= -x_2\left(2(\kappa_2 - x_2) + \theta_2 + a_{21}x_2 + a_{22}x_2\right) + \hat{u}_2 \\ \dot{\lambda}_1 &= \lambda_1\left(2(\kappa_1 - 2x_1 + a_{11}x_1) + \theta_1 + a_{12}x_2\right) + a_{21}\lambda_2 x_2 - \theta_1 - \theta_{12}x_2 \\ \dot{\lambda}_2 &= \lambda_2\left(2(\kappa_2 - 2x_2 + a_{22}x_2) + \theta_2 + a_{21}x_1\right) + a_{12}\lambda_1 x_1 - \theta_2 - \theta_{12}x_1\end{aligned}\right\}, \qquad (24)$$

Solving the controlled system (24) numerically, using Maple program, we get some results can be displayed by figures (3.1:3.5).

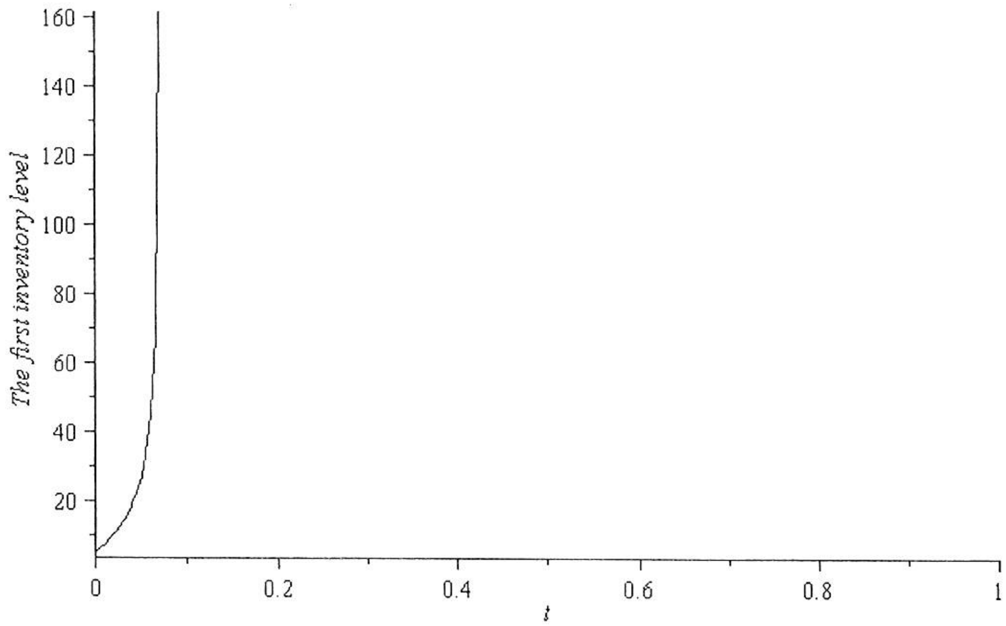

Fig.3.1 The first inventory level against time *t*

Figure 3.1 indicates that the first inventory level starts from initial value and tends to be increased as time increases.

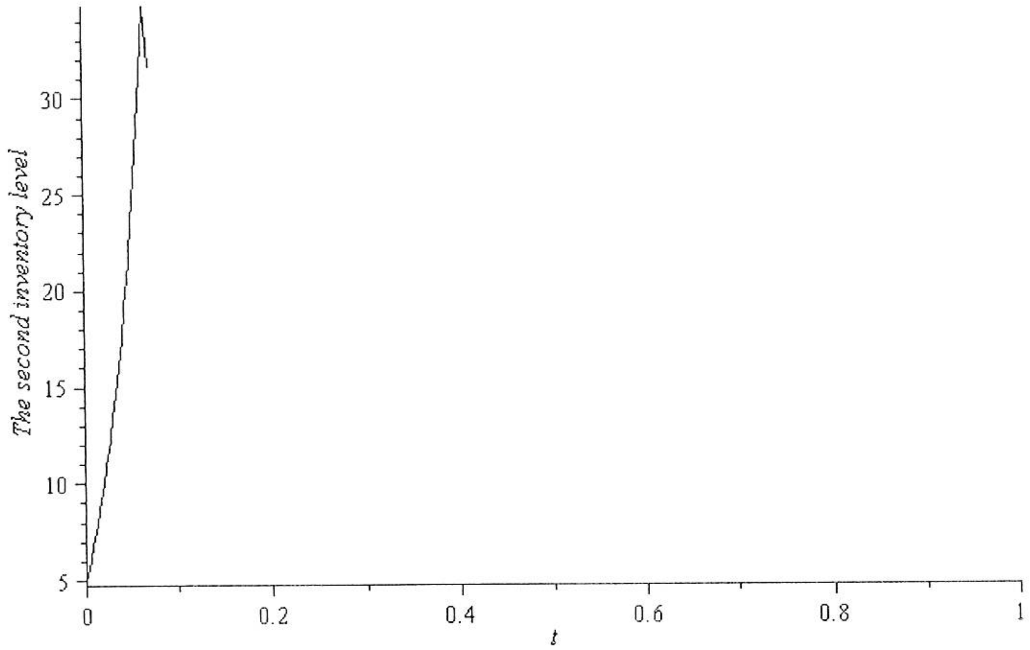

Fig.3.2 The second inventory level against time *t*

Figure 3.2 indicates that the second inventory level starts from initial value and tends to be increased as time increases quickly, them tends to be decreased.

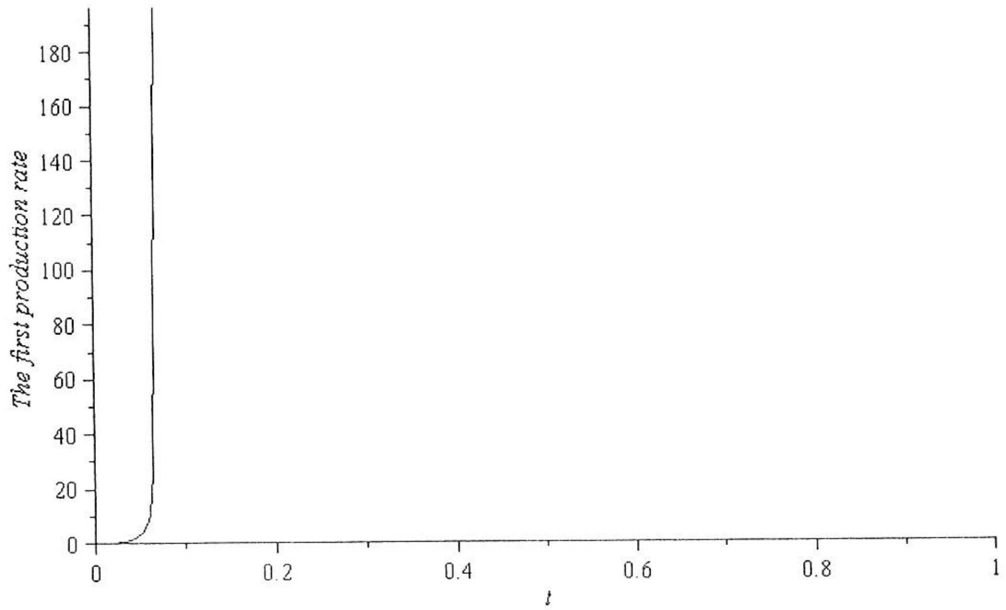

Fig.3.3 The first production rate against time *t*

Figure 3.3 indicates that first production rate is time increasing and also tends to its goal rate (200) quickly.

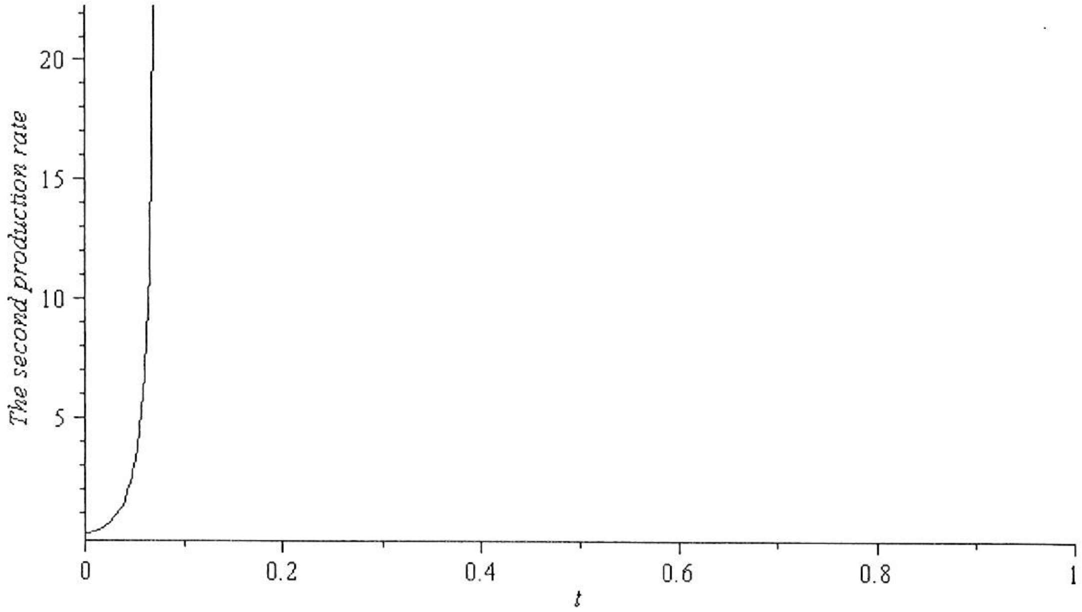

Fig.3.4 The second production rate against time *t*

Figure 3.4 indicates that the second production rate is quickly time increasing, and cannot be reached its goal rate (200).

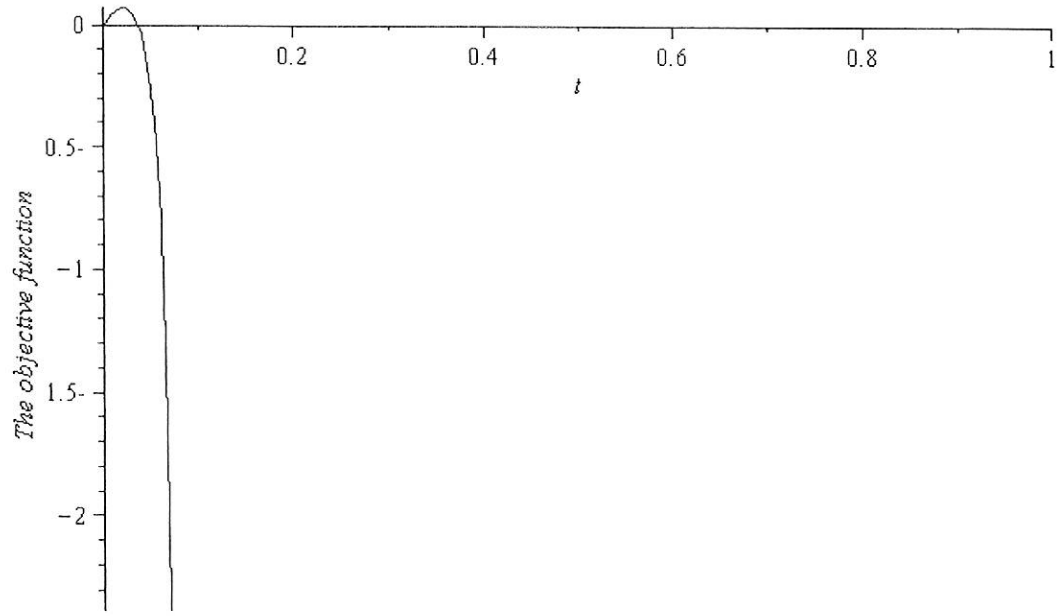

Fig.3.5 The objective function against time *t*

Figure 3.5 indicates that the decreasing of the objective function, as time increased, returns to the increasing of both first and second inventory levels, and increasing of both first and second production rates.

4.4 Periodic Rates

In this subsection, we will present the model with demand function as periodic demand rates, $D(x_1, x_2, t) = 1 - b_i \cos(t)$. Substituting in the controlled system (21) by the periodic demand rates, we have

$$\left.\begin{aligned} \dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - 1 + b_1\cos(t) + \hat{u}_1 \\ \dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - 1 + b_2\cos(t) + \hat{u}_2 \\ \dot{\lambda}_1 &= \lambda_1\left(\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + a_{21}\lambda_2 x_2 - \theta_1 - \theta_{12}x_2 \\ \dot{\lambda}_2 &= \lambda_2\left(\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - \theta_{12}x_1 \end{aligned}\right\}, \tag{25}$$

Solving the controlled system (25) numerically, using Maple program, we get some results can be displayed by (figures 4.1:4.5).

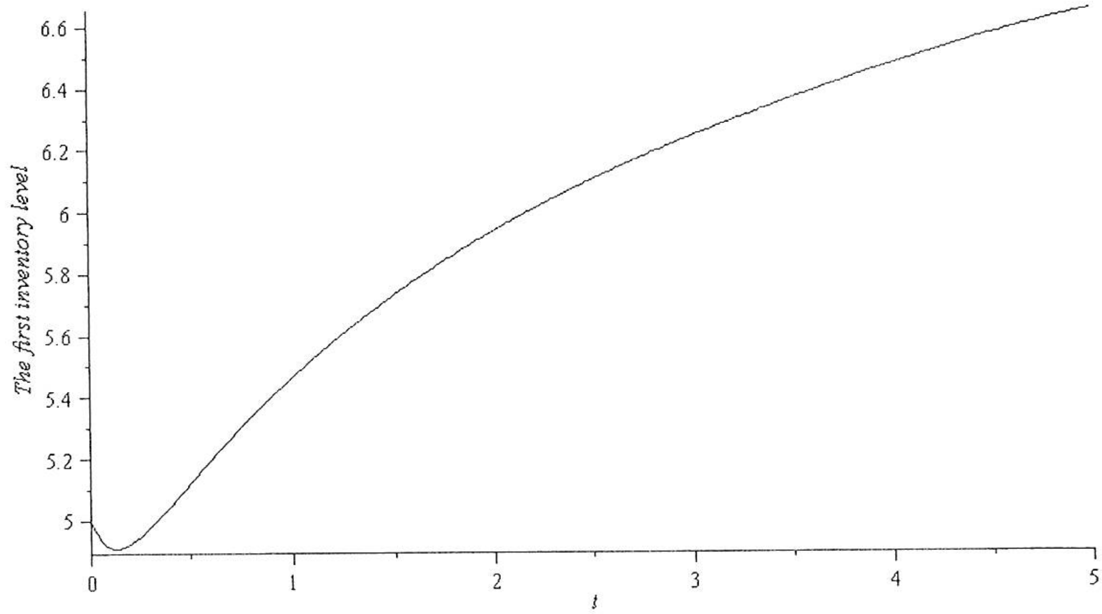

Fig.4.1 The first inventory level against time t

Figure 4.1 indicates that the first inventory level starts from initial value and tends to be increased as time increases.

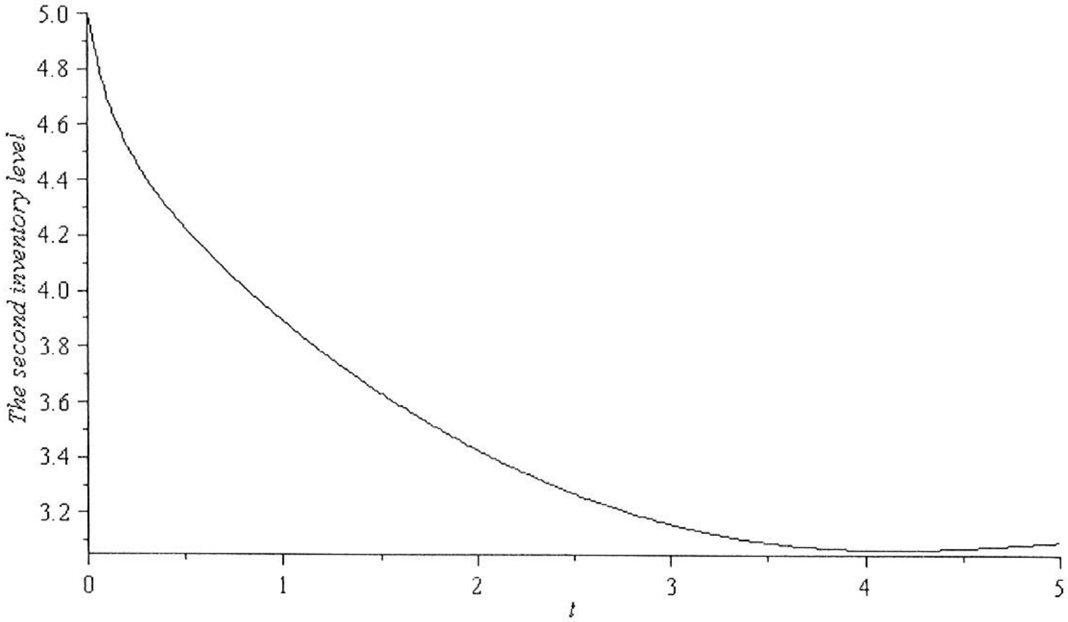

Fig.4.2 The second inventory level against time *t*

Figure 4.2 indicates that the second inventory level starts from initial value and tends to be decreased as time increases.

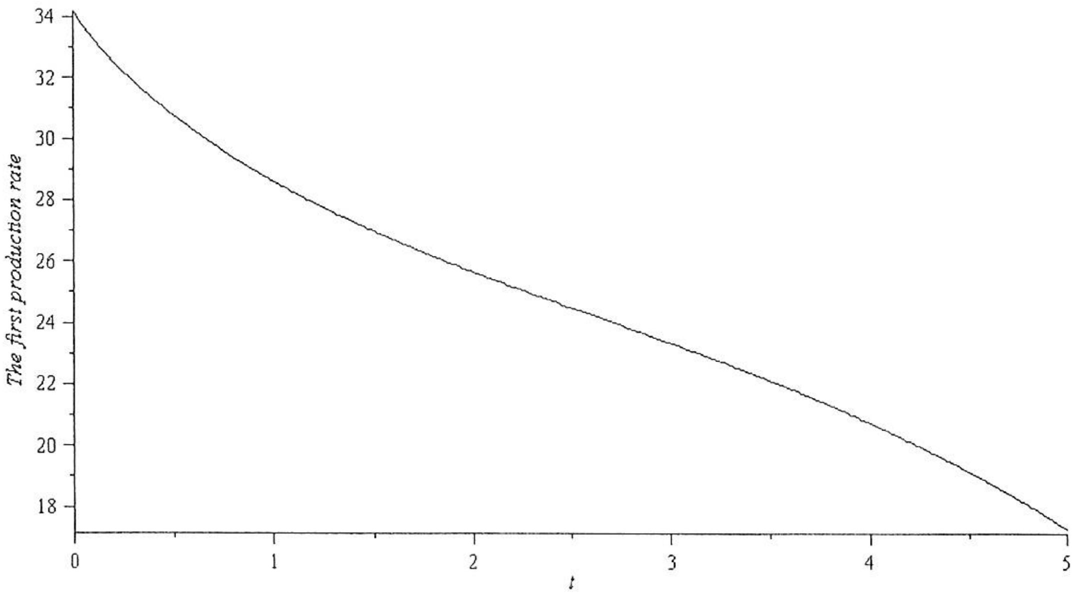

Fig.4.3 The first production rate against time *t*

Figure 4.3 indicates that first production rate is time decreasing and also tends to its goal rate (20) before the end of the planning period.

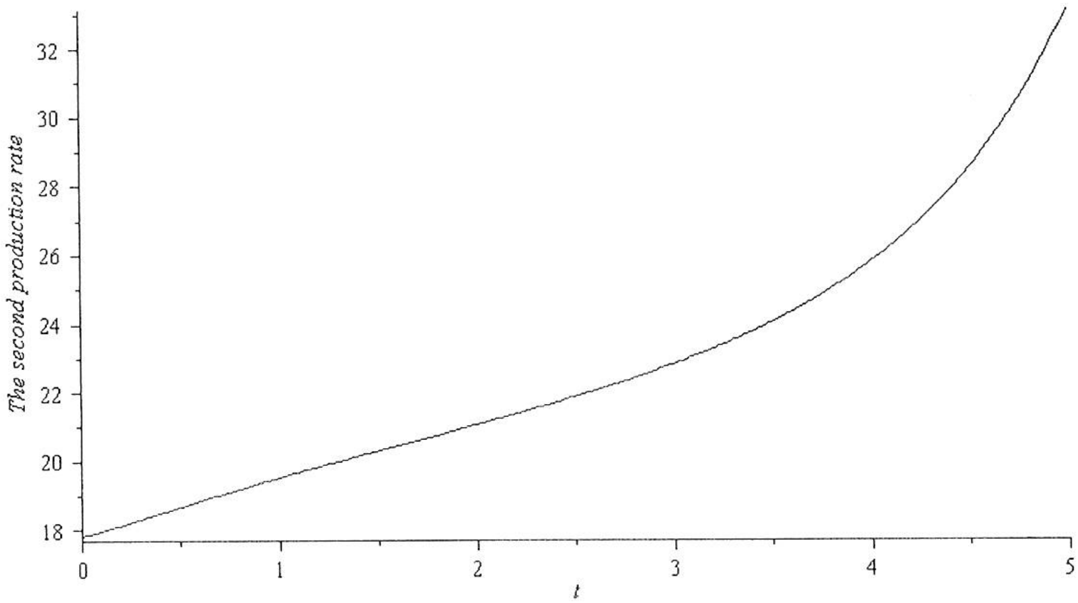

Fig.4.4 The second production rate against time *t*

Figure 4.4 indicates that the second production rate is time increasing and also tends to its goal rate (20) quickly.

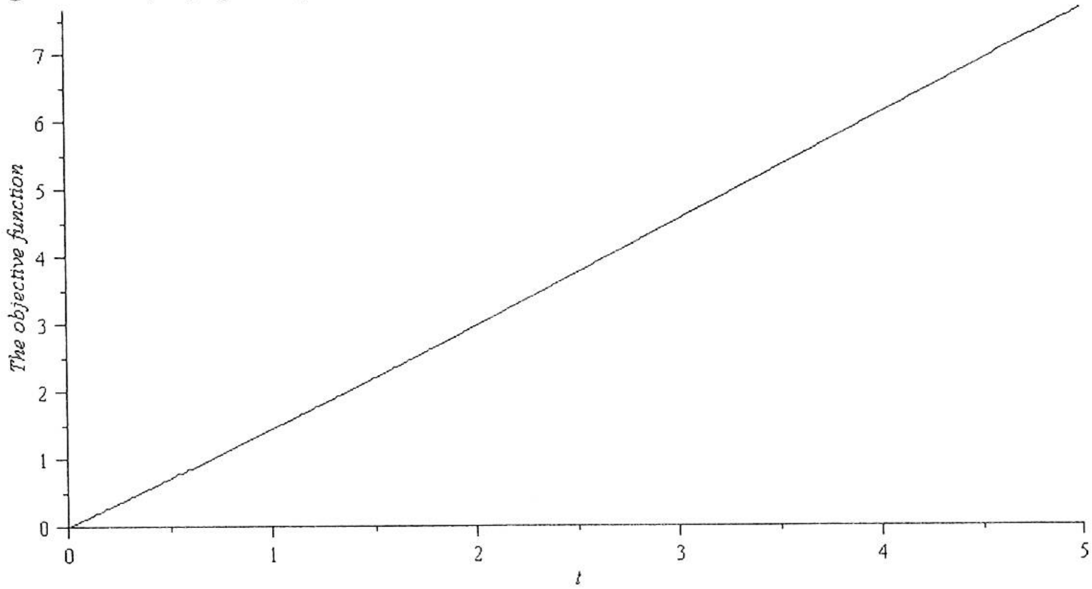

Fig.4.5 The objective function against time *t*

Figure 4.5 indicates that the increasing of the objective function, as time increased, returns to the increasing of the first inventory level, and decreasing of the first production rate.

We can summarize the values of the optimal inventory levels, the optimal production rates and the optimal objective function at end of the planning period, for four cases of the demand rates in Table 2 as follows:

Table 2. Optimal Solution

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.79	3.07	17.48	32.71	7.49
Linear	4.70	3.69	35.44	19.90	8.83
Logistic	161.50	34.80	196.14	22.27	-2.38
Periodic	6.66	3.10	17.28	33.13	7.68

From previous examples, as explanid from Figures 1.1 to 4.5, and from the results of which obtained from Table 2, we have found that:

For the objective function, that represents the negative value of natural logarithm of deterioration and spoilage function, we have found that the optimum (minimum) value (7.49) happens in the constant demand rates. Then, according to this optimal value, the optimal values of inventory levels are $[x_1^*(5), x_2^*(5)]$ = [6.79, 3.07], and the optimal production rates are $[u_1^*(5), u_2^*(5)]$ = [17.48, 32.71].

For the logistic demand rates are not suitable for the optimal control model, in this case, becaues some reasons:

1. It needs the large values of goal levels of production rates $[\hat{u}_1, \hat{u}_2]$ = [200, 200].

2.The boundary condition of co-state variables, λ_1 and λ_2, did not achieved at the end of planning period, T=5, just achieved when T=0, because the boundary conditions in this case were $\lambda_1(0)=1,\ \lambda_2(0)=1$.

3. All optimal paths are achieved only at T=0.07 (un-explaned).

4. The objective function is negative value (-2.38) and this is un-acceptable value.

5.The second optimal inventory level $x_2^*(5)=34.8$ and the second optimal production rate $u_2^*(5)$=22.27 are acceptable.

6.The first optimal inventory level $x_1^*(5)=161.5$ and the first optimal production rate $u_1^*(5)$=196.14 are un-acceptable (too much).

It is cleared that all results can be cganged if we choose other constant values for the co-sate variable $\lambda_0(t)$ in equation (13), but as mentioned before we fixed this value and concentrate here on the effect of different types of demand rates on the optimal trajectory. Also, we show that the increasing direction of the curves of first inventory level and decreasind direction of the curves of first and second production rates are stable, just on the objective function for all demand rates except in the logistic case.

5 Conclusion

In this paper, we discussed the optimal control problem using the deterioration and spoilage function and taking the negative value of natural logarithm as objective function using the AQEF. We used the Pontryagin principle to solve the optimal control problem under some conditions. The controlled system is solved numerically via Maple program using four cases of demand rates (constant, linear, logistic and periodic) to study the effect of demand rates on the inventory levels and the production rates, and then on the optimal value of objective function. The logistic demand rate function is not suitable for this problem under these constraints, but the other demand rates functions are suitable. The optimum value of objective function is achieved on the constant demand rate.

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