Chance Constrained Programming with Exponential Input Coefficients

Nada Hafez*

Prof. Afaf El-Dash†

Dr. Nagwa Albehery‡

Abstract

In this paper, we consider Chance constrained programming (CCP) technique when at least two of the LHS input coefficients are random with two-parameter exponential distribution.

Two approaches are introduced to transform CCP into deterministic: (i) approach 1 assumes independence between exponential input coefficients and (ii) approach 2 assumes that random input coefficients are dependent with correlation coefficient ρ.

Approach 1 for independence is an extension of Biswal's approach deals with m independent two-parameter exponential input coefficients instead of single-parameter ones. Approach 2 of dependence uses Downton bivariate exponential (DBE) distribution under two cases; the first introduced case assumes that dependent input coefficients have single-parameter exponential marginals and the second introduced case is an extension of DBE distribution for two-parameter exponentials.

It was shown that the equivalent deterministic transformation of the extension of approach 2 is a generalization of both approach 1 for $m = 2$ when $\rho = 0$ and first case of approach 2 for single-parameter exponential marginal when the second parameter is zero.

Keywords: Bessel function, Bishwal approach, chance constrained programming, Downton bivariate exponential distribution, input coefficients, Non-linear programming, probabilistic programming, Stochastic Programming, two-parameter exponential distribution.

1. Introduction

In the nineteen forties, Mathematical Programming (MP) approaches underwent a rapid development, one of these approaches is Stochastic Programming approach where some or all of the model parameters are random variables, taking into account the probability distribution of the random parameters in the underlying problem (Prékopa, 1995). Randomness may exists in the RHS parameter or in some (or all) LHS input coefficients, or in some (or all) objective function coefficients. In this paper we are concerned with the case when two of LHS input coefficients are random.

Consider the following linear programming model:

* **Teaching Assistant**, Department of Mathematics , Insurance and Applied Statistics, Faculty of commerce and Business Administration, Helwan University

† **Professor of operations Research**; Department of Mathematics , Insurance and Applied Statistics, Faculty of commerce and Business Administration, Helwan University

‡ **Assistant Professor**; Department of Mathematics , Insurance and Applied Statistics, Faculty of commerce and Business Administration, Helwan University

$$Max. z = \sum_{j=1}^{m} c_j x_j \tag{1.1}$$

$$\text{S.t.; } \sum_{j=1}^{\dot{m}} \tilde{a}_{ij} x_j + \sum_{j=\dot{m}+1}^{m} a_{ij} x_j \leq b_i, \; i = 1,2,\dots,n; \;\; ; x_j \geq 0 \quad j = 1,2,\dots,m \tag{1.2}$$

we refer to the stochastic parameters as $\tilde{a}_{ij}$. Chance constrained programming (CCP) technique is one of the stochastic programming techniques developed by Charnes and Cooper (1959, 1962, 1963) which offers powerful means of modeling stochastic decision problems with assumption that the stochastic constraints will hold with probability at least α, where α refers to the confidence level provided as an appropriate safety margin by the decision-maker and is called tolerance measure (Liu, 2002). CCP aims to transform the stochastic model into a deterministic one, then solving the deterministic model using a suitable mathematical programming technique. For theoretical background see (Prékopa, 2003).

Several contributions have been suggested by many researchers assuming different probability distributions for random parameters, for example; the Normal distribution (Charnes and Cooper, 1962; Symonds, 1967; Prekopa, 1974; Jagannathan, 1974; Ackooij et al., 2011; Houda and Lisser, 2015), the Chi-square distribution (Sengupta, 1972; El-Dash, 1984), the Gamma distribution (Lingaraj and Wolfe, 1974; Atalay and Apaydin, 2011), the Exponential distribution (Sengupta, 1972; Biswal et al., 1998), and the Dirichlet distribution (Gouda and Szántai 2010).

The exponential distribution is our concern is this paper, and it is applicable for a wide class of economic, demographic and reliability models where the input coefficients and the resources have to be nonnegative, which require distributions with nonnegative range. As for exponentially distributed random input coefficients $\tilde{a}_{ij}$, constraint (1.2) is first manipulated in literature by Sengupta (1972) assuming $\tilde{a}_{ij}'s$ are independent two-parameter exponential random variables, he suggested the use of central χ^2 approximation. Then Biswal et al. (1998) used different approach assuming $\tilde{a}_{ij}'s$ are independent single-parameter exponential random variables; giving the exact distribution of the variable $\sum_{j=1}^{m} \tilde{a}_{ij} x_j$, from which the deterministic equivalent of the CC is derived. In this paper; we propose an approach to transform CCP into deterministic equivalent through introducing an extension of Biswal's approach by hypothesizing two-parameter exponential marginal distribution instead of the single parameter exponential which doesn't suit in most real life applications due to having its mode at the origin.

Moreover; It is remarkable that the independence assumption was used by both approaches; Sengupta and Biswal approaches. However; in reality the assumption of independence in most situations is unrealistic, and parameters usually reflect dependent relationships among each other, where the assumption of independence is considered a special case of the dependence case. As for the bivariate exponential distribution, many different forms were presented in literature. Gumbel (1960) presented the first type of bivariate exponential distribution, followed by Freund (1961), then many other forms were presented after that, see (Kotz et al., 2000). However; no attempts were introduced to manipulate CCP with dependent exponential LHS parameters. In this paper; based on Downton's Bivariate exponential distribution introduced by Downton (1970), we introduce an approach for transforming CCP into deterministic with bivariate dependent exponential LHS input coefficients. Besides; we introduce an extension of Downton bivariate exponential distribution assuming two-parameter exponential marginal, and of course the deterministic equivalent of CC is obtained.

Throughout this paper; we are concerned with individual chance constraints taking one of the following forms:

$$P(\sum_{j=1}^{m} \tilde{a}_{ij}x_j \leq b_i) \geq \alpha_i; \quad i = 1,2,\dots,n'; j = 1,\dots,m \tag{1.3}$$

$$P\left(\sum_{j=1}^{\dot{m}} \tilde{a}_{ij}x_j + \sum_{j=\dot{m}+1}^{m} a_{ij}x_j \leq b_i\right) \geq \alpha_i, \quad i = 1,2,\dots,n; \; j = 1,2,\dots,m \tag{1.4}$$

where for $i = 1,2,\dots,n'$; it is assumed that all LHS parameters $\tilde{a}_{ij}$ of the i^{th} constaint are random variables , and for $i = (n'+1),\dots,n$ only some of the LHS parameters $\tilde{a}_{ij}$; $j = 1,\dots,m'$are random variables and the remaining LHS parameters a_{ij} ; $j = m'+1,\dots,m$ are deterministic.

The following sections are organized as follows; section 2 illustrates the proposed approach for dealing with CCP when some or all of LHS input coefficients are independent random variables distributed with two-parameter exponential distribution. Section 3 explains the proposed approach for dealing with CCP under the assumption that only two of the LHS input coefficients are random variables and distributed as bivariate Downton exponential distribution with correlation coefficient ρ. In section 4 we introduce an extension of the Downton bivariate exponential distribution to deal with two-parameter exponential random variables. Section 5 illustrates some numerical examples, and finally the summary and conclusions are provided.

2. Approach 1: CCP with Independent Input Coefficients

As mentioned before Biswal et al (1998) considered constraint (1.3) assuming that $\tilde{a}_{ij}'s$ are independent single-parameter exponential random variables; However; the single-parameter exponential distribution doesn't suit in most real life applications due to having its mode at the origin; therefore in this section we introduce an extension of Biswal's approach for two-parameter exponential random parameters, which overcome the previous drawback and is considered a generalization of Biswal's approach.

Recall constraint (1.3) and assume that $\tilde{a}_{ij}'s$ are independently two-parameter exponentially distributed random variables, with Pdf defined as;

$$f(\tilde{a}_{ij}) = \lambda_{ij} \exp\{-\lambda_{ij}(\tilde{a}_{ij} - \gamma_{ij})\} \qquad ; \lambda_{ij} > 0;\ 0 \le \gamma_{ij} \le \tilde{a}_{ij} \tag{2.1}$$

with known mean and variance:

$$E(\tilde{a}_{ij}) = \frac{1}{\lambda_{ij}} + \gamma_{ij} \ , \quad V(\tilde{a}_{ij}) = \left(\frac{1}{\lambda_{ij}}\right)^2 \tag{2.2}$$

Where γ_{ij} is the location parameter, λ_{ij} is the scale parameter. By using the same approach of Biswal et al., now we introduce an extension of Biswal et al approach to deal with two-parameter exponential random variates in the LHS of constraints (1.3) or (1.4) through the following lemmas.

Lemma (2.1): *let* $\tilde{y}_i = \sum_{j=1}^{2} \tilde{a}_{ij} x_j$; *where* $\tilde{a}_{ij}$; $j = 1,2$ *are independent exponential random variables with two parameters* $(\lambda_{ij}; \gamma_{ij})$ *and with known means* $\left(\frac{1}{\lambda_{ij}} + \gamma_{ij}\right)$ *and variances* $\left(\frac{1}{\lambda_{ij}}\right)^2$; $\lambda_{ij} > 0, 0 \le \gamma_{ij} \le \tilde{a}_{ij}$, *then for some scalars* $x_j; j = 1,2$ *the probability density function of the random variable* $\tilde{y}_i$ *is given by:*

$$g(\tilde{y}_i) = \exp\{\textstyle\sum_{j=1}^{2} \lambda_{ij}\gamma_{ij}\} \Big(\frac{\lambda_{i1}\lambda_{i2}}{(x_1\lambda_{i2} - x_2\lambda_{i1})} \exp\left\{-\frac{\lambda_{i1}}{x_1}\tilde{y}_i - \left(\frac{x_1\lambda_{i2} - x_2\lambda_{i1}}{x_1}\right)\gamma_{i2}\right\} + \frac{\lambda_{i1}\lambda_{i2}}{(x_2\lambda_{i1} - x_1\lambda_{i2})} \exp\left\{-\frac{\lambda_{i2}}{x_2}\tilde{y}_i - \left(\frac{x_2\lambda_{i1} - x_1\lambda_{i2}}{x_2}\right)\gamma_{i1}\right\}\Big); \ \tilde{y}_i \ge \gamma_{i1}x_{i1} + \gamma_{i2}x_{i2} \tag{2.3}$$

Proof: Let the i^{th} probabilistic constraint be

$$P(\tilde{a}_{i1}x_1 + \tilde{a}_{i2}x_2 \le b_i) > \alpha_i \quad (2.4)$$

then the cumulative density function (c.d.f.) of the random variable $\tilde{y}_i = \sum_{j=1}^{2} \tilde{a}_{ij}x_j$ can be computed as:

$$G(\tilde{y}_i) = \int_{\tilde{a}_{i2}=\gamma_{i2}}^{\frac{\tilde{y}_i-\gamma_{i1}x_1}{x_2}} \int_{\tilde{a}_{i1}=\gamma_{i1}}^{\frac{\tilde{y}_i-\tilde{a}_{i2}x_2}{x_1}} \lambda_{i1}\lambda_{i2} \exp\{-\lambda_{i1}\tilde{a}_{i1} - \lambda_{i2}\tilde{a}_{i2} + \textstyle\sum_{k=1}^{2} \lambda_{ik}\gamma_{ik}\} \, d\tilde{a}_{i1} \, d\tilde{a}_{i2}$$

$$= \exp\{\textstyle\sum_{j=1}^{2} \lambda_{ij}\gamma_{ij}\} \Big(\exp\{-\textstyle\sum_{j=1}^{2} \lambda_{ij}\gamma_{ij}\} - \frac{x_1\lambda_{i2}}{(x_1\lambda_{i2}-x_2\lambda_{i1})} \exp\Big\{-\frac{\lambda_{i1}}{x_1}\tilde{y}_i - \Big(\frac{x_1\lambda_{i2}-x_2\lambda_{i1}}{x_1}\Big)\gamma_{i2}\Big\} + \frac{x_2\lambda_{i1}}{(x_1\lambda_{i2}-x_2\lambda_{i1})} \exp\Big\{-\frac{\lambda_{i2}}{x_2}\tilde{y}_i - \Big(\frac{x_2\lambda_{i1}-x_1\lambda_{i2}}{x_2}\Big)\gamma_{i1}\Big\}\Big)$$

By differentiation with respect to $\tilde{y}_i$; we obtain the pdf of $\tilde{y}_i$ in (2.3) ■

Lemma (2.2): *let $\tilde{y}_i = \sum_{j=1}^{3} \tilde{a}_{ij}x_j$; where $\tilde{a}_{ij}$; $j = 1,2,3$ are independent exponential random variables with two parameters$(\lambda_{ij}; \gamma_{ij})$ and with known means $\left(\frac{1}{\lambda_{ij}} + \gamma_{ij}\right)$ and variances $\left(\frac{1}{\lambda_{ij}}\right)^2$; $\lambda_{ij} > 0, 0 \le \gamma_{ij} \le \tilde{a}_{ij}$, then for some scalars $x_j; j = 1,2,3$ the probability density function of the random variable $\tilde{y}_i$ is given by:*

$$g(\tilde{y}_i) = \exp\{\textstyle\sum_{j=1}^{3} \lambda_{ij}\gamma_{ij}\}\Bigg(\frac{\lambda_{i1}\lambda_{i2}\lambda_{i3}}{(x_1\lambda_{i2}-x_2\lambda_{i1})(x_1\lambda_{i3}-x_3\lambda_{i1})} \exp\Big\{-\frac{\lambda_{i1}}{x_1}\tilde{y}_i - \sum_{j=2}^{3}\Big(\frac{x_1\lambda_{ij}-x_j\lambda_{i1}}{x_1}\Big)\gamma_{ij}\Big\} + \frac{\lambda_{i1}\lambda_{i2}\lambda_{i3}}{(x_2\lambda_{i1}-x_1\lambda_{i2})(x_2\lambda_{i3}-x_3\lambda_{i2})} \exp\Big\{-\frac{\lambda_{i2}}{x_2}\tilde{y}_i - \sum_{\substack{j=1\\ j\ne 2}}^{3}\Big(\frac{x_2\lambda_{ij}-x_j\lambda_{i2}}{x_2}\Big)\gamma_{ij}\Big\} + \frac{\lambda_{i1}\lambda_{i2}\lambda_{i3}}{(x_3\lambda_{i1}-x_1\lambda_{i3})(x_3\lambda_{i2}-x_2\lambda_{i3})} \exp\Big\{-\frac{\lambda_{i3}}{x_3}\tilde{y}_i - \sum_{j=1}^{2}\Big(\frac{x_3\lambda_{ij}-x_j\lambda_{i3}}{x_3}\Big)\gamma_{ij}\Big\}\Bigg); \tilde{y}_i \ge \textstyle\sum_{j=1}^{3} \gamma_{ij}x_j$$

Proof: Similar to proof of lemma (2.1). ■

The following theorem provides the general form of the density function of the random variable $\tilde{y}_i = \sum_{j=1}^{m} \tilde{a}_{ij}x_j$; where $\tilde{a}_{ij}'s$ are independent exponential random variables with two parameters. And the deterministic transformation of constraint (1.3)

Theorem (2.1): *let the i^{th} constraint be of the form of (1.3) .and let $\tilde{y}_i = \sum_{j=1}^{m} \tilde{a}_{ij}x_j$ where $\tilde{a}_{ij}$; $j = 1, \dots, m$, are independent two-parameter exponential random variables with $(\lambda_{ij}; \gamma_{ij})$ and with known means $\left(\frac{1}{\lambda_{ij}} + \gamma_{ij}\right)$ and variances $\left(\frac{1}{\lambda_{ij}}\right)^2$; $\lambda_{ij} > 0$, $0 \leq \gamma_{ij} \leq \tilde{a}_{ij}$, Then:*

i) The probability density function of the random variable $\tilde{y}_i$ for some scalars x_j; $j = 1; \dots; m$ is given by $g(\tilde{y}_i)$

$$g(\tilde{y}_i) = \left(\prod_{j=1}^{m} \lambda_{ij}\right) \exp\left\{\sum_{j=1}^{m} \lambda_{ij}\gamma_{ij}\right\} \left[\sum_{j=1}^{m} \frac{x_j^{m-2}\, exp\left\{-\frac{\lambda_{ij}\tilde{y}_i}{x_j} - \sum_{\substack{l=1\\ l\neq j}}^{m} \left(\frac{x_j\lambda_{il} - x_l\lambda_{ij}}{x_j}\right)\gamma_{il}\right\}}{\prod_{\substack{l=1\\ l\neq j}}^{m} (x_j\lambda_{il} - x_l\lambda_{ij})}\right];$$

$$\tilde{y}_i \geq \sum_{j=1}^{m} \gamma_{ij}x_{ij} \qquad (2.5)$$

ii) The deterministic equivalent of constraint (1.3) is:

$$-\sum_{j=1}^{m} \frac{x_j^{m-1} \exp\left\{-\frac{\lambda_{ij}b_i}{x_j} - \sum_{\substack{l=1\\ l\neq j}}^{m} \left(\frac{x_j\lambda_{il} - x_l\lambda_{ij}}{x_j}\right)\gamma_{il}\right\}}{\lambda_{ij} \prod_{\substack{l=1\\ l\neq j}}^{m} (x_j\lambda_{il} - x_l\lambda_{ij})} \geq \frac{\alpha_i}{\left(\prod_{j=1}^{m} \lambda_{ij}\right) \exp\left\{\sum_{j=1}^{m} \lambda_{ij}\gamma_{ij}\right\}};$$

$$i = 1, \dots, n' \qquad (2.6)$$

Proof: The proof of the first part is based on mathematical induction, therefore the proof consists of three steps.

1) At first we want to prove the relation (2.5) is true for $m = 1$ and $m = 2$:

- For $m = 1$:

$$\text{LHS} = g(\tilde{y}_i = \tilde{a}_{i1}x_1) = \frac{\lambda_{i1}}{x_1} \exp\left\{-\frac{\lambda_{i1}y_i}{x_1} + \lambda_{i1}\gamma_{i1}\right\} = \text{RHS}$$

Therefore the relation is true for $m = 1$

- For $m = 2$:

$$\text{RHS} = \lambda_{i1}\lambda_{i2}e^{\sum_{j=1}^{2} \lambda_{ij}\gamma_{ij}} \left(\frac{1}{(x_1\lambda_{i2} - x_2\lambda_{i1})} \exp\left\{-\frac{\lambda_{i1}}{x_1}\tilde{y}_i - \left(\frac{x_1\lambda_{i2} - x_2\lambda_{i1}}{x_1}\right)\gamma_{i2}\right\} + \frac{1}{(x_2\lambda_{i1} - x_1\lambda_{i2})} \exp\left\{-\frac{\lambda_{i2}}{x_2}\tilde{y}_i - \left(\frac{x_2\lambda_{i1} - x_1\lambda_{i2}}{x_2}\right)\gamma_{i1}\right\}\right)$$

which equals to the LHS for $m = 2$ by lemma (2.1)

2) Assume that the relation (2.5) is true for $m = r$; Then the Pdf of $\tilde{y}_i = \sum_{j=1}^{r} \tilde{a}_{ij} x_j$ is:

$$g(\tilde{y}_i) = C_r \left[\sum_{j=1}^{r} \frac{x_j^{r-2} \exp\{-\frac{\lambda_{ij}\tilde{y}_i}{x_j} - \sum_{\substack{l=1 \\ l \neq j}}^{r} (\frac{x_j\lambda_{il} - x_l\lambda_{ij}}{x_j}) \gamma_{il}\}}{\prod_{\substack{l=1 \\ l \neq j}}^{r} (x_j\lambda_{il} - x_l\lambda_{ij})} \right] ; \tilde{y}_i \geq \sum_{j=1}^{r} \gamma_{ij} x_j \tag{2.7}$$

where $C_r = \left(\prod_{j=1}^{r} \lambda_{ij}\right) \exp\left\{\sum_{j=1}^{r} \lambda_{ij}\gamma_{ij}\right\}$

3) Now; we need to proof that the relation holds for $m = r + 1$

Let $\tilde{y}_i = \sum_{j=1}^{r+1} \tilde{a}_{ij} x_j = \tilde{a} + \tilde{a}_{i\,(r+1)} x_{(r+1)}$ where the pdf of $\tilde{a} = \sum_{j=1}^{r} \tilde{a}_{ij} x_j$ and $\tilde{a}_{i\,(r+1)}$ are given by $g(\tilde{a})$ and $f(\tilde{a}_{i\,(r+1)})$ respectively:

$$g(\tilde{a}) = C_r . \left[\sum_{j=1}^{r} \frac{x_j^{r-2} \exp\left\{-\frac{\lambda_{ij}\tilde{a}}{x_j} - \sum_{\substack{l=1 \\ l \neq j}}^{r} (\frac{x_j\lambda_{il} - x_l\lambda_{ij}}{x_j}) \gamma_{il}\right\}}{\prod_{\substack{l=1 \\ l \neq j}}^{r} (x_j\lambda_{il} - x_l\lambda_{ij})} \right] ; \tilde{a} \geq \sum_{j=1}^{r} \gamma_{ij} x_j \tag{2.8}$$

$$f(\tilde{a}_{i\,(r+1)}) = \lambda_{i\,(r+1)}\, e^{-\lambda_{i\,(r+1)}\left(\tilde{a}_{i\,(r+1)} - \gamma_{i\,(r+1)}\right)} ; \tilde{a}_{i\,(r+1)} \geq \gamma_{i\,(r+1)} \tag{2.9}$$

The CDF of $\tilde{y}_i = \tilde{a} + \tilde{a}_{i\,(r+1)} x_{(r+1)}$ is given by

$$G(\tilde{y}_i) = \int_{\tilde{a}_{i\,(r+1)} = \gamma_{i\,(r+1)}}^{\frac{(\tilde{y}_i - \sum_{j=1}^{r} \gamma_{ij} x_j)}{x_{(r+1)}}} \int_{\tilde{a} = \sum_{j=1}^{r} \gamma_{ij} x_j}^{\tilde{y}_i - \tilde{a}_{i\,(r+1)} x_{(r+1)}} g(\tilde{a}) f(\tilde{a}_{i\,(r+1)})\, d\tilde{a}\, d\tilde{a}_{i\,(r+1)} \tag{2.10}$$

$$= C_{r+1} . \int_{\tilde{a}_{i\,(r+1)} = \gamma_{i\,(r+1)}}^{\frac{(\tilde{y}_i - \sum_{j=1}^{r} \gamma_{ij} x_j)}{x_{(r+1)}}} \int_{\tilde{a} = \sum_{j=1}^{r} \gamma_{ij} x_j}^{\tilde{y}_i - \tilde{a}_{i\,(r+1)} x_{(r+1)}} \left[\sum_{j=1}^{r} \frac{x_j^{r-2}}{\prod_{\substack{l=1 \\ l \neq j}}^{r} (x_j\lambda_{il} - x_l\lambda_{ij})} exp\{-\lambda_{i\,(r+1)} \tilde{a}_{i\,(r+1)}\} . \; exp\left\{-\frac{\lambda_{ij}\tilde{a}}{x_j} - \sum_{\substack{l=1 \\ l \neq j}}^{r} (\frac{x_j\lambda_{il} - x_l\lambda_{ij}}{x_j}) \gamma_{il}\right\} \right] d\tilde{a}\, d\tilde{a}_{i\,(r+1)} \tag{2.11}$$

Where $C_{r+1} = \left(\prod_{j=1}^{r+1} \lambda_{ij}\right) exp\left\{\sum_{j=1}^{r+1} \lambda_{ij}\gamma_{ij}\right\}$

$$= C_{r+1} \int_{\tilde{a}_{i\,(r+1)}=\gamma_{i\,(r+1)}}^{\frac{(\tilde{y}_i - \sum_{j=1}^{r} \gamma_{ij} x_j)}{x_{(r+1)}}} \sum_{j=1}^{r} \frac{x_j^{r-1}}{\lambda_{ij} \prod_{\substack{l=1 \\ l\neq j}}^{r} (x_j \lambda_{il} - x_l \lambda_{ij})} \cdot \Bigg[exp\left\{ -\lambda_{i\,(r+1)} \tilde{a}_{i\,(r+1)} - \frac{\lambda_{ij}}{x_j} \sum_{j=1}^{r} \gamma_{ij} x_j - \sum_{\substack{l=1 \\ l\neq j}}^{r} \left(\frac{x_j \lambda_{il} - x_l \lambda_{ij}}{x_j}\right) \gamma_{il} \right\} - exp\left\{ -\left(\frac{x_j \lambda_{i\,(r+1)} - x_{(r+1)} \lambda_{ij}}{x_j}\right) \tilde{a}_{i\,(r+1)} - \frac{\lambda_{ij}}{x_j} \tilde{y}_i - \sum_{\substack{l=1 \\ l\neq j}}^{r} \left(\frac{x_j \lambda_{il} - x_l \lambda_{ij}}{x_j}\right) \gamma_{il} \right\} \Bigg] d\tilde{a}_{i\,(r+1)} \tag{2.12}$$

$$= C_{r+1} \sum_{j=1}^{r} \Bigg[\frac{x_j^{r-1}}{\lambda_{ij} \lambda_{i\,(r+1)} \prod_{\substack{l=1 \\ l\neq j}}^{r} (x_j \lambda_{il} - x_l \lambda_{ij})} \Bigg(exp\left\{ -\lambda_{i\,(r+1)} \gamma_{i\,(r+1)} - \frac{\lambda_{ij}}{x_j} \sum_{j=1}^{r} \gamma_{ij} x_j - \sum_{\substack{l=1 \\ l\neq j}}^{r} \left(\frac{x_j \lambda_{il} - x_l \lambda_{ij}}{x_j}\right) \gamma_{il} \right\} - exp\left\{ -\frac{\lambda_{i\,(r+1)}}{x_{(r+1)}} \tilde{y}_i - \sum_{\substack{l=1 \\ l\neq j}}^{r+1} \left(\frac{x_j \lambda_{il} - x_l \lambda_{ij}}{x_j}\right) \gamma_{il} \right\} \Bigg) - \frac{x_j^{r}}{\lambda_{ij} \prod_{\substack{l=1 \\ l\neq j}}^{r+1} (x_j \lambda_{il} - x_l \lambda_{ij})} \Bigg(exp\left\{ -\frac{\lambda_{ij}}{x_j} \tilde{y}_i - \sum_{\substack{l=1 \\ l\neq j}}^{r+1} \left(\frac{x_j \lambda_{il} - x_l \lambda_{ij}}{x_j}\right) \gamma_{il} \right\} - exp\left\{ -\frac{\lambda_{i\,(r+1)}}{x_{(r+1)}} \tilde{y}_i - \sum_{\substack{l=1 \\ l\neq j}}^{r+1} \left(\frac{x_j \lambda_{il} - x_l \lambda_{ij}}{x_j}\right) \gamma_{il} \right\} \Bigg) \Bigg] \tag{2.13}$$

The Pdf of $\tilde{y}_i$ can be obtained by differntialting $G(\tilde{y}_i)$ with respect to $\tilde{y}_i$:

$$g(\tilde{y}_i) = C_{r+1} \left[\sum_{j=1}^{r+1} \frac{x_j^{(r+1)-2} \exp\left\{ -\frac{\lambda_{ij} y_i}{x_j} - \sum_{\substack{l=1 \\ l\neq j}}^{r+1} \left(\frac{x_j \lambda_{il} - x_l \lambda_{ij}}{x_j}\right) \gamma_{il} \right\}}{\prod_{\substack{l=1 \\ l\neq j}}^{r+1} (x_j \lambda_{il} - x_l \lambda_{ij})} \right] ; \; y_i \geq \sum_{j=1}^{r+1} \gamma_{ij} x_j \tag{2.14}$$

which is the same as the RHS of (2.5) for $m = r + 1$. This completes the proof of the first part.

The second part of theorem (2.1) is the transformation of constraint (1.3) into deterministic through integrating the Pdf of $\tilde{y}_i$ as stated below

$$\int_{\sum_{j=1}^{m} \gamma_{ij} x_j}^{b_i} g(y_i) dy_i \geq \alpha_i \quad ; \quad i = 1, \ldots, n' \tag{2.15}$$

$$-\sum_{j=1}^{m} \frac{x_j^{m-1} \exp\left\{ -\frac{\lambda_{ij} b_i}{x_j} - \sum_{\substack{l=1 \\ l\neq j}}^{m} \left(\frac{x_j \lambda_{il} - x_l \lambda_{ij}}{x_j}\right) \gamma_{il} \right\}}{\lambda_{ij} \prod_{\substack{l=1 \\ l\neq j}}^{m} (x_j \lambda_{il} - x_l \lambda_{ij})} \geq \frac{\alpha_i}{\left(\prod_{j=1}^{m} \lambda_{ij}\right) \exp\left\{\sum_{j=1}^{m} \lambda_{ij} \gamma_{ij}\right\}} ; i = 1, \ldots, n' \tag{2.16}$$

■

Now; Consider constraint (1.4), the deterministic equivalent can be obtained by integrating the pdf. of $\tilde{y}_i$ as stated by the following corollary.

Corollary (2.1): Let the i^{th} CC be of the form of (1.4), *and let* $\tilde{y}_i = \sum_{j=1}^{m} \tilde{a}_{ij}x_j$ *where* $\tilde{a}_{ij}$; $j = 1, \ldots, m$, *are independent two-parameter exponential random variables with* $(\lambda_{ij}; \gamma_{ij})$ *and with known means* $\left(\frac{1}{\lambda_{ij}} + \gamma_{ij}\right)$; $\lambda_{ij} > 0, 0 \le \gamma_{ij} \le \tilde{a}_{ij}$, *then for some scalars* x_j; $j = 1; \ldots; m$, the deterministic equivalent of the CC (1.4) is:

$$-\sum_{j=1}^{m} \frac{x_j^{m-1} \exp\left\{-\frac{\lambda_{ij}\left(b_i - \sum_{j=\dot{m}+1}^{m} a_{ij}x_j\right)}{x_j} - \sum_{\substack{l=1\\ l\neq j}}^{m}\left(\frac{x_j\lambda_{il} - x_l\lambda_{ij}}{x_j}\right)\gamma_{il}\right\}}{\lambda_{ij}\prod_{\substack{l=1\\ l\neq j}}^{m}(x_j\lambda_{il} - x_l\lambda_{ij})} \ge$$

$$\frac{\alpha_i}{\left(\prod_{j=1}^{m}\lambda_{ij}\right)\exp\left\{\sum_{j=1}^{m}\lambda_{ij}\gamma_{ij}\right\}}; \quad i = n' + 1, \ldots, n \tag{2.17}$$

Proof: based on the The Pdf of $\tilde{y}_i$ which is obtained in theorem (2.1), then the deterministic equivalent of (1.4) could be obtained by integrating the p.d.f. of $\tilde{y}_i$ as below

$$\int_{\sum_{j=1}^{m}\gamma_{ij}x_j}^{b_i - \sum_{j=\dot{m}+1}^{m} a_{ij}x_j} g(y_i)dy_i \ge \alpha_i \quad ; \quad i = n' + 1, \ldots, n \tag{2.18}$$

$$-\sum_{j=1}^{m} \frac{x_j^{m-1} \exp\left\{-\frac{\lambda_{ij}\left(b_i - \sum_{j=\dot{m}+1}^{m} a_{ij}x_j\right)}{x_j} - \sum_{\substack{l=1\\ l\neq j}}^{m}\left(\frac{x_j\lambda_{il} - x_l\lambda_{ij}}{x_j}\right)\gamma_{il}\right\}}{\lambda_{ij}\prod_{\substack{l=1\\ l\neq j}}^{m}(x_j\lambda_{il} - x_l\lambda_{ij})} \ge \frac{\alpha_i}{\left(\prod_{j=1}^{m}\lambda_{ij}\right)\exp\left\{\sum_{j=1}^{m}\lambda_{ij}\gamma_{ij}\right\}};$$

$$i = n' + 1, \ldots, n \tag{2.19}$$ ■

Special cases

1) Consider the special case where $m = 2$ that is :

$$\mathrm{P}(\tilde{a}_{i1}x_1 + \tilde{a}_{i2}x_2 \le b_i) \ge \alpha_i \tag{2.20}$$

Then; the deterministic transformation of the chance constraint (2.20) could be obtained by (2.16) as:

$$-\left[\frac{x_1 \exp\left\{\frac{-(x_1\lambda_{i2} - x_2\lambda_{i1})\gamma_{i2} - \lambda_{i1}b_i}{x_1}\right\}}{\lambda_{i1}(x_1\lambda_{i2} - x_2\lambda_{i1})} + \frac{x_2 \exp\left\{\frac{-(x_2\lambda_{i1} - x_1\lambda_{i2})\gamma_{i1} - \lambda_{i2}b_i}{x_2}\right\}}{\lambda_{i2}(x_2\lambda_{i1} - x_1\lambda_{i2})}\right] \ge \frac{\alpha_i}{\lambda_{i1}\lambda_{i2}\exp\left\{\sum_{j=1}^{2}\lambda_{ij}\gamma_{ij}\right\}};$$

$$i = 1, \ldots, n \tag{2.21}$$

2) Consider the special case of only two input coefficients are random and the remaining coefficients are deterministic; that is :

$$P(\tilde{a}_{i1}x_1 + \tilde{a}_{i2}x_2 + \textstyle\sum_{j=3}^{m} a_{ij}x_j \leq b_i) \geq \alpha_i \qquad (2.22)$$

Then the deterministic equivalent of constraint (2.22) is given by (2.17) as:

$$-\left[\frac{x_1 \exp\left\{\frac{-(x_1\lambda_{i2}-x_2\lambda_{i1})\gamma_{i2}-\lambda_1\left(b_i-\sum_{j=3}^{m} a_{ij}x_j\right)}{x_1}\right\}}{\lambda_{i1}(x_1\lambda_{i2}-x_2\lambda_{i1})} + \frac{x_2 \exp\left\{\frac{-(x_2\lambda_{i1}-x_1\lambda_{i2})\gamma_{i1}-\lambda_{i2}(b_i-\sum_{j=3}^{m} a_{ij}x_j)}{x_2}\right\}}{\lambda_{i2}(x_2\lambda_{i1}-x_1\lambda_{i2})}\right] \geq$$

$$\frac{\alpha_i}{\lambda_{i1}\lambda_{i2}\exp\left\{\sum_{j=1}^{2}\lambda_{ij}\gamma_{ij}\right\}} \;; \quad i = 1, \ldots, n \qquad (2.23)$$

3) Consider the case when $\gamma_{ij} = 0 , j = 1, \ldots, m$; then the pdf of $\tilde{y}_i = \sum_{j=1}^{m} \tilde{a}_{ij}x_j$ obtained in theorm (2.1) is reduced to the corresponding pdf of single-parameter exponential variates obtained by Biswal. This indicates that the proposed extension introduced in this section for two-parameter exponential variates is a gneralization of Biswal's approach.

3. Approach 2 for Dependent Single-Parameter Exponential Input Coefficients

Consider again constraints (1.3) and (1.4), Our objective in this section is to find the deterministic equivalent of the probabilistic constraints (1.3) and (1.4) when $\tilde{a}_{ij}$; $j = 1,2$ are dependent random parameters and distributed according to Downton's Bivariate exponential distribution – introduced by Downton (1970). The approximated Downton bivariate exponential joint density function $f(\tilde{a}_1, \tilde{a}_2)$ is:

$$f(\widetilde{a_1}, \widetilde{a_2}) = \frac{\lambda_1\lambda_2}{(1-\rho)^2}\exp\left\{-\frac{\lambda_1\widetilde{a_1}+\lambda_2\widetilde{a_2}}{1-\rho}\right\}; \; \widetilde{a_1}, \widetilde{a_2} > 0 \;; \; \lambda_1, \lambda_2 > 0 \;; 0 \leq \rho \leq 1 \qquad (3.1)$$

where ρ is the correlation between $\widetilde{a_1}, \widetilde{a_2}$ (Hafez, 2017).

Theorem (3.1): *Let the i^{th} CC be of the form of (1.3), and let* $\widetilde{y_i} = \widetilde{a_{i1}}x_1 + \widetilde{a_{i2}}x_2$ $(a_{i1}, a_{i2})^T$ *are approximate Downton bivariate exponential random variables with joint density function* $f(\widetilde{a_{i1}}, \widetilde{a_{i2}})$ *defined in (3.1) , then;*

i) $\tilde{y}_i$ *is distributed with Probability density function* $g(\widetilde{y_i})$*:*

$$g(\tilde{y}_i) = \frac{\lambda_{i1}\lambda_{i2}}{(1-\rho_i)\,(\lambda_{i2}x_1-\lambda_{i1}x_2)} \exp\left\{-\frac{\lambda_{i1}}{(1-\rho_i)\,x_1}\left(\tilde{y}_i\right)\right\} + \frac{\lambda_{i1}\,\lambda_{i2}}{(1-\rho_i)(\lambda_{i1}x_2-\lambda_{i2}\,x_1)} \exp\left\{-\frac{\lambda_{i2}}{(1-\rho_i)\,x_2}\left(\tilde{y}_i\right)\right\} \tag{3.2}$$

ii) the deterministic equivalent of the CC (1.3) is:

$$\frac{\lambda_{i1}x_2}{(\lambda_{i1}x_2-\lambda_{i2}\,x_1)} \exp\left\{-\frac{\lambda_{i2}}{(1-\rho_i)}\left(\frac{b_i}{x_2}\right)\right\} + \frac{\lambda_{i2}x_1}{(\lambda_{i2}x_1-\lambda_{i1}x_2)} \exp\left\{-\frac{\lambda_{i1}}{(1-\rho_i)}\left(\frac{b_i}{x_1}\right)\right\} \leq 1-\alpha_i \tag{3.3}$$

Proof: The proof is straightforward, in the first part we use the method of distribution functions through firstly deriving the distribution function of $\tilde{y}$ by double integrating $f(\widetilde{a_{i1}}, \widetilde{a_{i2}})$ as follows:

$$G(\tilde{y}_i) = \int_{\widetilde{a_2}=0}^{\frac{\tilde{y}_i}{x_2}} \int_{\widetilde{a_1}=0}^{\frac{\tilde{y}_i-\widetilde{a_2}x_2}{x_1}} \frac{\lambda_1\lambda_2}{(1-\rho_i)^2} \exp\left\{-\frac{\lambda_1\widetilde{a_{i1}}+\lambda_2\widetilde{a_{i2}}}{(1-\rho_i)}\right\} d\widetilde{a_{i1}}\, d\widetilde{a_{i2}} \tag{3.4}$$

then Differentiating $G(\tilde{y})$ with respect to $\tilde{y}$ to obtain the P.d.f. of $\tilde{y}$ as $g(\tilde{y}_i)$.

For the second part; the deterministic equivalent of constraint (1.3) could be found by integrating $g(\tilde{y}_i)$ as follows:

$$\int_0^{b_i} g(\tilde{y}_i)d\tilde{y}_i \geq \alpha_i \quad ; \quad i = 1, \dots, n \tag{3.5}$$ ■

Corollary (3.1): Let the i^{th} CC be of the form of (1.4), and let $(a_{i1}, a_{i2})^T$ are approximate Downton bivariate exponential random variables with joint density function $f(\widetilde{a_{i1}}, \widetilde{a_{i2}})$ defined in (3.1), then the deterministic equivalent of the CC (1.4) is:

$$\frac{\lambda_{i1}x_2}{(\lambda_{i1}x_2-\lambda_{i2}\,x_1)} \exp\left\{-\frac{\lambda_{i2}}{(1-\rho_i)}\left(\frac{b_i-\sum_{j=3}^{m} a_{ij}x_j}{x_2}\right)\right\} + \frac{\lambda_{i2}x_1}{(\lambda_{i2}x_1-\lambda_{i1}x_2)} \exp\left\{-\frac{\lambda_{i1}}{(1-\rho_i)}\left(\frac{b_i-\sum_{j=3}^{m} a_{ij}x_j}{x_1}\right)\right\} \leq 1-\alpha_i \tag{3.6}$$

4. Approach 2 for Dependent Two-Parameter Exponential Input Coefficients

. In this section we will propose an extension of the approximated Downton bivariate exponential distribution which is based on two-parameter exponential marginal, where we assume the marginal pdf of $\tilde{a}_{ij}$ is

$$f(\tilde{a}_{ij}) = \lambda_{ij}\exp\{-\lambda_{ij}(\tilde{a}_{ij}-\gamma_{ij})\} \qquad ;\lambda_{ij}>0;\ 0\le\gamma_{ij}\le\tilde{a}_{ij} \qquad (4.1)$$

The suggested extension of the approximated pdf of Downton bivariate exponential distribution in (3.1) for the case of two-parameter exponential marginals could be expressed as (4.2):

$$f(\tilde{a}_{i1},\tilde{a}_{i2}) = \frac{\lambda_{i1}\lambda_{i2}}{(1-\rho_i)^2}\exp\left\{-\frac{\lambda_{i1}(\tilde{a}_{i1}-\gamma_{i1})+\lambda_{i2}(\tilde{a}_{i2}-\gamma_{i2})}{1-\rho_i}\right\};\lambda_{ij}>0,0\le\gamma_{ij}\le\tilde{a}_j, \quad 0\le\rho_i\le 1; j=1,2 \qquad (4.2)$$

Similar to previous derivations in case of Downton bivariate single-parameter exponential distribution in sections 3, we will derive the distribution of $\sum_{j=1}^{2}\tilde{a}_{ij}x_j$, Assuming that $\widetilde{a_{i1}},\widetilde{a_{i2}}$ are distributed as Downton bivariate two-parameter exponential random variates with Pdf defined in (4.2). And transform the CC (1.3) and (1.4) into deterministic.

<u>Theorem (4.1):</u> Let the i^{th} CC be of the form of (1.3) with $j=1,2$, and let $\tilde{y}_i=\sum_{j=1}^{2}\tilde{a}_{ij}x_j$.; where$(\widetilde{a_{i1}},\widetilde{a_{i2}})^T$ are the approximated Downton bivariate two-parameter exponential random variables with approximated joint density function $f(\widetilde{a_{i1}},\widetilde{a_{i2}})$ defined in (4.2), Then:

i) $\tilde{y}_i$ is distributed with Probability density function $g(\tilde{y}_i)$:

$$g(\tilde{y}_i) = \frac{\lambda_{i1}\lambda_{i2}}{(1-\rho_i)}\exp\left\{\frac{\sum_{j=1}^{2}\lambda_{ij}\gamma_{ij}}{(1-\rho_i)}\right\}\left(-\frac{1}{x_1\lambda_{i2}-x_2\lambda_{i1}}\exp\left\{-\frac{\lambda_{i2}}{(1-\rho_i)x_2}\tilde{y}_i-\left(\frac{x_2\lambda_{i1}-x_1\lambda_{i2}}{(1-\rho_i)\,x_2}\right)\gamma_{i1}\right\}+\frac{1}{x_1\lambda_{i2}-x_2\lambda_{i1}}\exp\left\{-\frac{\lambda_{i1}}{(1-\rho_i)x_1}\tilde{y}_i-\left(\frac{x_1\lambda_{i2}-x_2\lambda_{i1}}{(1-\rho_i)\,x_1}\right)\gamma_{i2}\right\}\right) \quad ;\tilde{y}_i\ge\gamma_{i1}x_1+\gamma_{i2}x_2 \qquad (4.3)$$

ii) the deterministic equivalent of the CC (1.3) is:

$$-\left[\frac{x_1\exp\left\{\frac{-(x_1\lambda_{i2}-x_2\lambda_{i1})\gamma_{i2}-\lambda_{i1}b_i}{(1-\rho_i)\,x_1}\right\}}{\lambda_{i1}(x_1\lambda_{i2}-x_2\lambda_{i1})}+\frac{x_2\exp\left\{\frac{-(x_2\lambda_{i1}-x_1\lambda_{i2})\gamma_{i1}-\lambda_{i2}b_i}{(1-\rho_i)\,x_2}\right\}}{\lambda_{i2}(x_2\lambda_{i1}-x_1\lambda_{i2})}\right]\ge\frac{\alpha_i}{\lambda_{i1}\lambda_{i2}\exp\left\{\frac{\sum_{j=1}^{2}\lambda_{ij}\gamma_{ij}}{(1-\rho_i)}\right\}} \quad ;i=1,\dots,n \qquad (4.4)$$

<u>Proof:</u> the same idea of the proof of theorem (3.1); Firstly, we use the method of distribution functions to derive the distribution function of $\tilde{y}_i$ by double Integrating $f(\widetilde{a_{i1}},\widetilde{a_{i2}})$as follows:

$$G(\tilde{y}_i) = \int_{\widetilde{a_{i2}}=\gamma_{i2}}^{\frac{\tilde{y}_i-\gamma_1 x_1}{x_2}} \int_{\widetilde{a_{i1}}=\gamma_{i1}}^{\frac{\tilde{y}_i-\widetilde{a_{i2}}x_2}{x_1}} \frac{\lambda_{i1}\lambda_{i2}}{(1-\rho_i)^2} \exp\left\{\frac{\sum_{j=1}^{2}\lambda_{ij}\gamma_{ij}}{(1-\rho_i)}\right\} \exp\left\{-\frac{\lambda_{i1}\widetilde{a_{i1}}+\lambda_{i2}\widetilde{a_{i2}}}{(1-\rho_i)}\right\} d\widetilde{a_{i1}}\, d\widetilde{a_{i2}} \quad (4.5)$$

Then, obtain the pdf $g(\tilde{y}_i)$ by differentiation with respect to $\tilde{y}_i$.

Secondly; a deterministic equivalent of constraint (1.3) could be found by integrating $g(\tilde{y}_i)$ as follows:

$$\int_{\gamma_{i1}x_1+\gamma_{i2}x_2}^{b_i} g(\tilde{y}_i)d\tilde{y}_i \geq \alpha_i \quad ; \quad i = 1, \dots, n \qquad (4.6)\blacksquare$$

Corollary (4.1): Let the i^{th} CC be of the form of (1.4); $j = 1,2$, and let $(\tilde{a}_{i1}, \tilde{a}_{i2})^T$ are approximate Downton bivariate two-parameter exponential random variables with joint density function $f(\widetilde{a_{i1}}, \widetilde{a_{i2}})$ defined in (4.2), Then the deterministic equivalent of the CC (1.4) is:

$$-\left[\frac{x_1 \exp\left\{\frac{-(x_1\lambda_{i2}-x_2\lambda_{i1})\gamma_{i2}-\lambda_{i1}(b_i-\sum_{j=3}^{m}\tilde{a}_{ij}x_j)}{(1-\rho_i)\,x_1}\right\}}{\lambda_{i1}(x_1\lambda_{i2}-x_2\lambda_{i1})} + \frac{x_2 \exp\left\{\frac{-(x_2\lambda_{i1}-x_1\lambda_{i2})\gamma_{i1}-\lambda_{i2}(b_i-\sum_{j=3}^{m}\tilde{a}_{ij}x_j)}{(1-\rho_i)x_2}\right\}}{\lambda_2(x_2\lambda_1-x_1\lambda_2)}\right] \geq \frac{\alpha_i}{\lambda_{i1}\lambda_{i2}\exp\left\{\frac{\sum_{j=1}^{2}\lambda_{ij}\gamma_{ij}}{(1-\rho_i)}\right\}} \; ; \; i = 1, \dots, n' \qquad (4.7)$$

Special cases:

1) As special case of (4.4) and (4.7), when $\rho_i = 0$; that is if independence is assumed between $\widetilde{a_1}$ and $\widetilde{a_2}$, the deterministic constraint (4.4) and (4.7) which is based on approximate Downton's bivariate two-parameter exponential distribution- will be reduced to the deterministic constraint (2.21) and (2.23); respectively which are proposed in section 2 based on the joint distribution of independent two-parameter exponential random parameters with exponential marginal. That is to say the deterministic equivalents (4.4) and (4.7) are the generalized deterministic equivalents of (2.21) and (2.23) with respect to correlation between parameters.

2) Also; setting $\gamma_j = 0; j = 1,2$ in constraints (4.4) and (4.7) reduces them to the deterministic equivalents (3.3) and (3.6) respectively proposed in section 3 which

are based on approximate bivariate Downton exponential with single-parameter. That is to say the deterministic equivalents (4.4) and (4.7) are the generalized deterministic equivalents of (3.3) and (3.6) with respect to number of parameters of exponential marginals.

5. Numerical Examples

In this section we demonestrate the two proposed approaches in this paper throught two examples. Both examples assumes LHS input coefficients are random with two-parameter exponential marginals.

Example (5.1): this example illustrates the deterministic transformation of probabilistic model assuming independence between the LHS parameters using appraoch 1. Consider the probabilistic programming model

$$Max\ Z = 5x_1 + 6\,x_2 \tag{5.1}$$

S.t.;

$$P(\tilde{a}_{11}\,x_1 + \tilde{a}_{12}x_2 \le 10) \ge 0.95 \tag{5.2}$$

$$3\,x_1 + 6x_2 \le 4 \tag{5.3}$$

$$x_1, x_2 \ge 0 \tag{5.4}$$

where $\tilde{a}_{ij};\ i = 1; j = 1,2$ are independent two-parameter exponential distribution with $\lambda_{11} = \frac{1}{5}, \lambda_{12} = \frac{1}{4}, \gamma_{11} = 1, and\ \gamma_{12} = 1$

Constraint (5.2) could be transformed into deterministic using (2.6) and the model (5.1)-(5.4) could be rewritten as:

$$Max\ Z = 5x_1 + 6\,x_2 \tag{5.5}$$

S.t.;

$$4\,x_2 \exp\left\{-\frac{0.2\,x_2 - 0.25x_1 + 2.5}{x_2}\right\} - 5\,x_1 \exp\left\{-\frac{0.25x_1 - 0.2\,x_2 + 2}{x_1}\right\} - 12.11493(0.25x_1 - 0.2\,x_2) \ge 0 \tag{5.6}$$

$$3\,x_1 + 6x_2 \le 4 \tag{5.7}$$

$$x_1, x_2 \ge 0 \tag{5.8}$$

The deterministic equivalent model (5.5)-(5.8) is a non-linear programming model which could be solved using one of the non-linear programming techniques (Hafez, 2017; Ahmed, 2014; and Mokhtar et al., 2006).

Example (5.2): This example illustrates the deterministic transformation of probabilistic model assuming dependence between LHS parameters using approach 2.

Consider the probabilistic programming model (5.1)-(5.4), where $\tilde{a}_{ij};\ i = 1; j = 1,2$ are distributed as approximate Downton bivariate two-parameter exponential distribution with $\lambda_{11} = \frac{1}{5}, \lambda_{12} = \frac{1}{4}, \gamma_{11} = 1, and\ \gamma_{12} = 1$ and correlation $\rho_1 = 0.4$.

Constraint (5.2) could be transformed into deterministic using (4.4) and the model (5.1)-(5.4) could be rewritten as:

$$Max\ Z = 5x_1 + 6\ x_2 \tag{5.9}$$

S.t.;

$$4\ x_2 \exp\left\{-\frac{0.2\ x_2 - 0.25x_1 + 2.5}{0.6\ x_2}\right\} - 5\ x_1 \exp\left\{-\frac{0.25x_1 - 0.2\ x_2 + 2}{0.6\ x_1}\right\} - 8.97(0.25x_1 - 0.2\ x_2) \geq 0 \tag{5.10}$$

$$3\ x_1 + 6x_2 \leq 4 \tag{5.11}$$

$$x_1, x_2 \geq 0 \tag{5.12}$$

Again; the deterministic equivalent model (5.9)-(5.12) is a non-linear programming model which could be solved using one of the non-linear programming techniques (Hafez, 2017; Ahmed, 2014; and Mokhtar et al., 2006).

6. Conclusions

In this paper, we proposed two approaches for transforming CCP models into deterministic; where the randomness exists in the LHS input coefficients. The first approach assumes that the input coefficients are independent and distributed as two-parameter exponential distribution, through lemma (2.1), lemma (2.2) and theorem (2.1) and corollary (2.1) for m input coefficients. It was shown that this approach is a generalization of Biswal approach for independent single-parameter exponential distribution.

The second approach assumes that the input coefficients are correlated with single parameter Downton bivariate exponential distribution, and the deterministic equivalent transformation was presented through theorem (3.1) and corollary (3.1). Then, we proposed an extension to later approach to deal with two-parameter exponential input coefficients under the assumption of dependence, and the deterministic equivalent transformation was presented in theorem (4.1) and corollary (4.1).

It was shown that the proposed extension of approach 2 for two-parameter exponential distribution is the general form of both approaches; approach 1 when the correlation coefficient is zero and $m = 2$, and approach 2 for single-parameter exponential distribution when the location parameters are zero.

References

Ackooij ,W. van ; Henrion, R. ; Moller, A. ; Zorgati, R. 2011. "On Joint Probabilistic Constraints with Gaussian Coefficient Matrix." *Operations Research Letters*, vol. 39, pp.99–102.

Ahmed, S. 2014. "Convex relaxations of chance constrained optimization problems." *Optimization Letters*, vol. 8(1), pp.1–12.

Atalay, D. Kumru; Apaydin, A. 2011. "Gamma Distribution Approach in Chance-Constrained Stochastic Programming Model." *Journal of Inequalities and Applications*, vol. 108.

Biswal, M.; Biswal, N.; Li, D. 1998. "Probabilistic Linear Programming Problems with Exponential Random Variables: A Technical Note." *European Journal of Operational Research*, vol. 111, pp. 589-597.

Charnes, A.; Cooper, W. 1959. "Chance Constrained Programming." *Management Science*, vol. 6, pp. 73 – 79.

Charnes, A. and Cooper, W. 1962. "Chance Constraints and Normal Deviates." *Journal of the American Statistical Association*, vol.57, pp. 138-148.

Charnes, A. and Cooper, W. 1963. "Deterministic Equivalents for Optimizing and Satisficing under Chance Constraints." *Operations Research*, vol.11, pp.18-39.

Downton, F. 1970. "Bivariate Exponential Distributions in Reliability Theory." *Journal of the Royal Statistical Society*. Series B (Methodological), vol. 32(3), pp. 408-417.

El-Dash, A. A. 1984. *Chance-Constrained and Nonlinear Goal Programming*. Ph.D. thesis, University of North Wales, Bangor.

Freund, John E. 1961."A Bivariate Extension of the Exponential Distribution." *Journal of the American Statistical Association*, vol. 56, no. 296, pp. 971–977.

Gouda, A., Szántai, T. 2010. "On numerical calculation of probabilities according to Dirichlet distribution." Annals of Operations Research, 177: 185–200

Gumbel, E. 1960. "Bivariate Exponential Distributions." *Journal of the American Statistical Association*,vol. 55(292), pp.698-707.

Hafez; N. Mohemmed, 2017. *A joint Chance Constrained Programming Appraoch and its Applications.* unpublished thesis; Department of Mathematics, Insurance and Applied Statistics; Faculty of commerce and Business administration, Helwan University.

Houda, M., and Lisser, A. 2015. "Archimedean Copulas in Joint Chance Constrained Programming." *Operations Research and Enterprise Systems Third International Conference*, Angers, France, March 6–8, 2014.

Jagannathan, R. 1974. "Chance-Constrained Programming with Joint Constraints." *Operations Research*, vol. 22(2), pp. 358-372.

Kotz, S.; Balakrishnan, N.; and Johnson, L. Norman 2000. *Continuous Multivariate Distributions.* Volume 1: Models and Application; 2nd ed.; John Wiley & Sons, Inc.

Lingaraj, B. and Wolfe, H. 1974. "Certainty Equivalent of a Chance Constraint if the Random Variable Follows a Gamma Distribution." *Sankhya: The Indian Journal of Statistics*, Vol. 36, Series B, Pt. 2, pp. 204-208.

Liu, B. 2002. Theory and Practice of Uncertain Programming. Physica-Verlag, Heidelberg.

Mokhtar; S._Bazaraa, Hanif; D. Sherali, C. M. Shett 2006. *Nonlinear Programming Theory and Algorithms.* John Wiley & Sons, Inc.

Prékopa, A. 1974. "Programming Under Probabilistic Constraints with a Random Technology Matrix." *Math. Operations research*, 5, pp. 109–116.

Prékopa, A. 1995. *Stochastic Programming.* Kluwer Academic Publishers, Dordrecht, The Netherlands.

Prékopa, A. 2003. *Probabilistic programming.* In: A. Ruszczy´nski, A. Shapiro (eds.) Stochastic Programming, Handbooks in Operations Research and Management Science, vol. 10, pp. 267–351.

Sengupta, K. Jati 1972. "Chance-Constrained Linear Programming with Chi-Square Type Deviates." *Management Science*, Vol. 19 (3), p.337.

Symonds, G. 1967. "Deterministic Solutions for a Class of Chance Constrained Programming Problems." *Operations Research*, vol. 15(3), pp. 495-512.