Transmuted Burr Type II Distribution

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Abstract

This paper is concerned with introducing an extended Burr Type II distribution named as transmuted Burr Type II distribution. An approximated mean and variance for this distribution are obtained. Reliability, hazard rate, reversed hazard. Quantile, mode and median are derived. Distribution of the smallest, the largest, the lower record value and the upper record value are obtained. Parameter estimation using Maximum likelihood are obtained. To illustrate the proposed model, example was given.

Key words: Lower record and Upper record values, Newton Raphson, Order Statistics, Parameter estimation, Transmuted Burr Type II distribution.

1. Introduction

Burr distributions have numerous applications. For example, in problems related to hypothesis testing, lifetime data, statistical quality control, in survival and reliability studies. In addition, it is employed in financial studies, economics, income and wage distribution as well as in environmental studies.

Aryal and Tsokos (2011) generalized the two parameters Weibull distribution to develop a transmuted Weibull distribution. They provided a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior. The usefulness of the transmuted Weibull distribution for modelling reliability data is illustrated using real data.

Ashour and Eltehiwy (2013) proposed and studied a generalization of the Lomax distribution so-called the transmuted Lomax distribution. They derived various structural properties including explicit expressions for the moments, quantile and mean deviations of the new distribution. They performed the estimation of the model parameters by Maximum likelihood method.

Asha and Raja (2014) defined the transmuted probability distribution function corresponding to a distribution function $G(x)$. They studied some general properties of the transmuted probability distribution function in relation to the base distribution. In particular, they studied the transmuted exponentiated Fréchet (TEF) distribution. The different methods of estimation of parameters such as weighted least squares and maximum likelihood estimates of this distribution are studied. They performed a real data analysis for this distribution and they found that this class is more flexible. They showed that the TEF distribution is much better fit.

Merovci, and Puka (2014) generalized the Pareto distribution to develop a transmuted Pareto distribution. They provided a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior. The usefulness of the transmuted Pareto distribution for modeling data is illustrated using real data.

Pal and Tiensuwan (2014) introduced a Beta transmuted Weibull distribution which contains a number of distributions as special cases. They discussed the properties of the distributions and derived explicit expressions for the mean deviations, Bonferroni and Lorenz curves and reliability. They studied the distribution and moments of order statistics. They introduced the log Beta transmuted Weibull model to analyze censored data. They illustrated the usefulness of the new distribution in analyzing positive data.

Abdul-Moniem (2015) introduced a new distribution called transmuted Burr Type III distribution (TBIIID). He discussed some properties of this distribution. He handled the estimation of unknown parameters for TBIIID using real data.

Das and Barman (2015) defined a generalized form of the transmuted distribution. They studied their moments and other properties of this distribution. They derived transmuted skew exponential distribution. They discussed its different distributional properties.

Bourguignon et al. (2016) derived a simple representation for the transmuted-G family density function as a linear mixture of the G and exponentiated-G densities. They investigated the asymptotes and shapes and obtain explicit expressions for the ordinary and incomplete moments, quantile and generating functions, mean deviations, Renyi and Shannon entropies and order statistics and their moments. They estimated the model parameters of the family by the method of maximum likelihood. They derived empirically the flexibility of the proposed model with an application to a real data set.

Shahzad and Asghar (2016) introduce an extended Dagum distribution named as transmuted Dagum distribution which can be used for income distribution. They pointed out that this distribution is to provide more flexible distribution to model variety of data. They derived the moments, moment generating function, quantile, reliability and hazard functions. They estimated the parameters of the new model using maximum likelihood function. They illustrated the new model using real data for the city of Islamabad, Pakistan.

This paper is divided into four sections. The first section is the introduction. The second section contains the transmuted Burr Type II distribution. The third section is devoted to the estimation of the parameters. Section four contains the application.

2.Transmuted Burr Type II distribution

A random variable X has the transmuted Burr Type II distribution if it is derived using the following equation:

$$F(x) = G(x)[(1+\lambda) - \lambda G(x)] \text{ (Shahzad 2016)} \tag{1}$$

where $G(x)$: is the (distribution function) cdf of the Burr Type II distribution and λ is additional parameter that is called transmuted parameter, which makes the distribution more flexible to the data sets.

The (probability density function) pdf of the Burr Type II distribution is:

$$g(x;\alpha) = \alpha e^{-x}(1+e^{-x})^{-\alpha-1}, \text{(Encyclopedia of Statistical Sciences 2006)} \quad -\infty \le X \le \infty;\ \alpha > 0 \tag{2}$$

and its cdf is

$$G(x;\alpha) = (1+e^{-x})^{-\alpha}, \text{(Encyclopedia of Statistical Sciences 2006)} \quad -\infty \le X \le \infty;\ \alpha > 0 \tag{3}$$

where α is the shape parameter and λ is the transmuted parameter.

Now using equation (3) and equation (1) to obtain the cdf of the transmuted Burr Type II distribution as follows:

$$F(x;\alpha,\lambda) = (1+e^{-x})^{-\alpha}[(1+\lambda) - \lambda(1+e^{-x})^{-\alpha}], \qquad -\infty < X < \infty \tag{4}$$

its respective pdf of transmuted Burr Type II distribution is given by:

$$f(x;\alpha,\lambda) = \alpha\lambda e^{-x}(1+e^{-x})^{-\alpha-1} - 2\alpha\lambda e^{-x}(1+e^{-x})^{-2\alpha-1} + \alpha e^{-x}(1+e^{-x})^{-\alpha-1}, \qquad -\infty < X < \infty \tag{5}$$

where the parameter λ is $-1 \le \lambda \le 1$ and when $\lambda = 0$ the pdf and the cdf of transmuted distribution reduces to the parent distribution.

The reliability function is given by:

$$R(x) = 1 - F(x) = 1 - [(1+e^{-x})^{-\alpha}\{1+\lambda-\lambda(1+e^{-x})^{-\alpha}\}] \tag{6}$$

The hazard rate function is as follows:

$$h(x) = \frac{f(x)}{R(x)} = \frac{\alpha\lambda e^{-x}(1+e^{-x})^{-\alpha-1} - 2\alpha\lambda e^{-x}(1+e^{-x})^{-2\alpha-1} + \alpha e^{-x}(1+e^{-x})^{-\alpha-1}}{1 - [(1+e^{-x})^{-\alpha}\{1+\lambda-\lambda(1+e^{-x})^{-\alpha}\}]} \tag{7}$$

and the reversed hazard rate function is given by:

$$Rh(x) = \frac{f(x)}{F(x)} = \frac{\alpha\lambda e^{-x}(1+e^{-x})^{-\alpha-1} - 2\alpha\lambda e^{-x}(1+e^{-x})^{-2\alpha-1} + \alpha e^{-x}(1+e^{-x})^{-\alpha-1}}{[(1+e^{-x})^{-\alpha}\{1+\lambda-\lambda(1+e^{-x})^{-\alpha}\}]} \tag{8}$$

The r^{th}non-central moment of the transmuted Burr Type II distribution can't be obtained in closed form. But this distribution can be transformed to the Beta Type II distribution. So, approximate mean and approximate variance can be obtained using both the mean and the variance of the Beta distribution.

Let $y = g(x) = e^{-x}$,then $g(\mu) = e^{-\mu}$

The first and the second derivatives for e^{-x} are:

$g'(x) = -e^{-x}$, $g'(\mu) = -e^{-\mu}$

$g''(x) = e^{-x}$, $g''(\mu) = e^{-\mu}$

The mean of the Beta Type II distribution is:

$$E(x) = \frac{\alpha}{\lambda - 1},$$

and the variance of the Beta Type II distribution is:

$$V(x) = \frac{\alpha(\alpha+\lambda-1)}{(\lambda-2)(\lambda-1)^2}$$

The approximated mean for the transmuted Burr Type II distribution is given by:

$$E(y) \approx g(\mu) + \frac{1}{2}\sigma^2 g''(\mu) = e^{-\left(\frac{\alpha}{\lambda-1}\right)} + \frac{1}{2}\frac{\alpha(\alpha+\lambda-1)}{(\lambda-2)(\lambda-1)^2} e^{-\left(\frac{\alpha}{\lambda-1}\right)} \tag{9}$$

The approximated variance for the transmuted Burr Type II distribution is:

$$V(y) = \sigma^2\left(g'(\mu)\right)^2 = \left[\frac{\alpha(\alpha+\lambda-1)}{(\lambda-2)(\lambda-1)^2}\right]\left(e^{-\left(\frac{\alpha}{\lambda-1}\right)}\right)^2 \tag{10}$$

Quantile function is:

$$Q(q) = F^{-1}(q) \tag{11}$$

Using equation (11) the q^{th} quantile function for the transmuted Burr Type II distribution is:

$$q = \frac{1}{(1+e^{-x})^{-\alpha}[(1+\lambda) - \lambda(1+e^{-x})^{-\alpha}]} \rightarrow X = -\ln\left[\left(\frac{q+\lambda q-1}{\lambda q-q}\right)^{-\frac{1}{\alpha}} - 1\right] \tag{12}$$

The Median is the 50^{th} percentile hence median of transmuted Burr Type II distribution is given by:

$$Median = -\ln\left[\left(\frac{.5+\lambda 0.5-1}{0.5\lambda-0.5}\right)^{-\frac{1}{\alpha}} - 1\right] \tag{13}$$

The mode of the random variable X that satisfies the equation $f'(x) = 0$ is the solution of:

$$-4\alpha\lambda - 2\lambda + 2\lambda e^{x}(1+e^{-x}) + \alpha(1+e^{-x})^{\alpha} + (1+e^{-x})^{\alpha} - e^{x}(1+e^{-x})^{\alpha+1}$$

$$+\alpha\lambda(1+e^{-x})^{\alpha} + \lambda(1+e^{-x})^{\alpha} - \lambda e^{x}(1+e^{-x})^{\alpha+1} = 0 \tag{14}$$

Let $X_1, X_2, \ldots, X_n$ be any real valued random variables and its ordered values denoted as $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ then the values $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics of the random variable, X.

The density of the n^{th} ordered statistics follows the transmuted Burr Type II distribution is as follows:

$$f_{(n)}\left(x_{(n)}\right) = n\left[F\left(x_{(n)}\right)\right]^{n-1} f\left(x_{(n)}\right) = n[(1+e^{-x_{(n)}})^{-\alpha}\{1+\lambda-\lambda(1+e^{-x_{(n)}})^{-\alpha}\}]^{n-1} \times$$

$$\alpha\lambda e^{-x}(1+e^{-x})^{-\alpha-1} - 2\alpha\lambda e^{-x}(1+e^{-x})^{-2\alpha-1} + \alpha e^{-x}(1+e^{-x})^{-\alpha-1} \quad (15)$$

Suppose that the smallest value follows the transmuted Burr Type II distribution then the density of the smallest order statistic is given by:

$$f_{(1)}(x_{(1)}) = n[1 - F(x_{(1)}]^{n-1} f(x_{(1)}) = n[1 - (1+e^{-x_{(1)}})^{-\alpha}\{1+\lambda-\lambda(1+e^{-x_{(1)}})^{-\alpha}\}]^{n-1} \times$$

$$\alpha\lambda e^{-x}(1+e^{-x})^{-\alpha-1} - 2\alpha\lambda e^{-x}(1+e^{-x})^{-2\alpha-1} + \alpha e^{-x}(1+e^{-x})^{-\alpha-1} \quad (16)$$

Generally, the pdf of the r^{th} order statistics is given by:

$$f_{(r)}(x_{(r)}) = \frac{n!}{(r-1)!\,(n-r)!}[F(x_{(r)}]^{r-1}[1 - F(x_{(r)}]^{n-r} f(x_{(r)})$$

$$= \frac{n!}{(r-1)!\,(n-r)!}[(1+e^{-x_{(r)}})^{-\alpha}(1+\lambda-\lambda(1+e^{-x_{(r)}})^{-\alpha})]^{r-1} \times$$

$$[1-(1+e^{-x_{(r)}})^{-\alpha}(1+\lambda-\lambda(1+e^{-x_{(r)}})^{-\alpha})]^{n-r} \times$$

$$\alpha\lambda e^{-x}(1+e^{-x})^{-\alpha-1} - 2\alpha\lambda e^{-x}(1+e^{-x})^{-2\alpha-1} + \alpha e^{-x}(1+e^{-x})^{-\alpha-1} \quad (17)$$

The upper record value is:

$$f_{U(n)}(x) = \frac{1}{\Gamma(n)}[-\log(1-F(x))]^{n-1} f(x)$$

$$= \frac{1}{\Gamma(n)}[-\log(1-(1+e^{-x_{(U)}})^{-\alpha}(1+\lambda-\lambda(1+e^{-x_{(U)}})^{-\alpha}))]^{n-1}$$

$$(\alpha\lambda e^{-x}(1+e^{-x})^{-\alpha-1} - 2\alpha\lambda e^{-x}(1+e^{-x})^{-2\alpha-1} + \alpha e^{-x}(1+e^{-x})^{-\alpha-1}) \quad (18)$$

The lower record value is:

$$f_{L(n)}(x) = \frac{1}{\Gamma(n)}[-\log(F(x))]^{n-1} f(x)$$

$$= \frac{1}{\Gamma(n)}[-\log((1+e^{-x_{(L)}})^{-\alpha}(1+\lambda-\lambda(1+e^{-x_{(L)}})^{-\alpha}))]^{n-1}$$

$$(\alpha\lambda e^{-x}(1+e^{-x})^{-\alpha-1} - 2\alpha\lambda e^{-x}(1+e^{-x})^{-2\alpha-1} + \alpha e^{-x}(1+e^{-x})^{-\alpha-1}) \quad (19)$$

Suppose that U is the standard uniform variate. Then the random variable X:

$$X = -\ln\left[\left(\frac{u-\lambda}{1-\lambda}\right)^{-\frac{1}{\alpha}} - 1\right]^{-1} \tag{20}$$

follows the transmuted Burr Type II distribution and is used to generate the random variable for the distribution assuming α, λ are known.

3.Estimation of the parameters:

The maximum likelihood method is used to estimate the parameters of the Transmuted Burr Type II distribution. Let $(x_1, x_2, \dots, x_n)$ be a random sample from Transmuted Burr Type II distribution with density function as $f(\underline{x}; \alpha, \lambda)$. The Likelihood function of the Transmuted Burr Type II distribution for the parameters α, λ is given by:

$$L(\underline{x}; \alpha, \lambda) = \prod_{i=1}^{n} f(\underline{x}; \alpha, \lambda)$$

$$= \prod_{i=1}^{n} \alpha\lambda e^{-x}(1+e^{-x})^{-\alpha-1} - 2\alpha\lambda e^{-x}(1+e^{-x})^{-2\alpha-1} + \alpha e^{-x}(1+e^{-x})^{-\alpha-1}$$

$$L(x; \alpha, \lambda) = (\lambda\alpha)^n e^{-\sum x} \prod_{i=1}^{n}(1+e^{-x})^{-\alpha-1} - (2\alpha\lambda)^n e^{-\sum x} \prod_{i=1}^{n}(1+e^{-x})^{-2\alpha-1}$$
$$+(\alpha)^n e^{-\sum x} \prod_{i=1}^{n}(1+e^{-x})^{-\alpha-1} \tag{21}$$

and

$$\text{Log } L(x; \alpha, \lambda) = \text{n}\log(\alpha\lambda) - 3\sum x - 2(\alpha+1)\Sigma\log(1+e^{-x}) - \text{n}\log(2\alpha\lambda) + n\log\alpha$$

$$-(2\alpha+1)\sum\log(1+e^{-x}) \tag{22}$$

Partially differentiating (22) with respect to the parameters α, λ respectively we have the following equations

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - 4\Sigma\log(1+e^{-x}) = 0 \tag{23}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \frac{n}{\lambda} = 0 \tag{24}$$

The maximum likelihood estimates (MLEs) of α, λ can be obtained by solving (23) and (24) simultaneously. The estimators of the parameters α and λ can be obtained in closed form as:

$$\hat{\alpha} = \frac{n}{4\sum_{i=1}^{n}\log(1+e^{-x})}$$

$$\hat{\lambda} = 0$$

3.Application

In this section, we compared the performance of the transmuted Burr Type II distribution with Burr Type II. The real data set represents the strength of 1.5 cm glass fibers measured at the National Physical laboratory, England (Shanker, *et al.* 2015) as follows in Table (1).

Table (1) real data for glass fibers

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81	2
0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01
0.77	1.11	1.28	1.42	1.5	1.54	1.6	1.62	1.66	1.69	1.76	1.84	2.24
0.81	1.13	1.29	1.48	1.5	1.55	1.61	1.62	1.66	1.7	1.77	1.84	0.84
1.24	1.3	1.48	1.51	1.55	1.61	1.63	1.67	1.7	1.78	1.89		

The descriptive statistics for these data are obtained in Table (2).

Table (2) Descriptive Statistics for the real data of glass fibers

N	min	max	range	mean	S.D.	C.V.	Q_1	Q_2	Q_3	Skewness	Kurtosis	Lower bound	Upper bound
63	0.55	2.24	1.69	1.51	0.32	0.21	1.36	1.59	1.69	−0.92	1.1	1.43	1.59

Since the value of the skewness measure is negative. So, the distribution of the data is skewed to the left. Also since the value of the kurtosis measure is less than 3, so the distribution for the data is flat.

In order to compare the transmuted Burr Type II distribution with Burr Type II we consider the Akaike information criterion (AIC), Akaike information corrected criterion (AICC), Bayesian information criteria and one-sample Kolmogorov-Smirnov (KS) goodness of fit test for the data set. The better distribution is the one with the smallest values of the previous criteria.

These criteria are:

$AIC = 2k - 2\log L,$

$AICC = AIC + 2k(k+1)/(n-k-1),$

$BIC = 2L + k\log n.$

where

k: is the number of the parameters in each distribution

n: is the sample size

$\log L$: is the log of the likelihood function.

The estimated parameters of the transmuted Burr Type II and Burr Type distributions and the values of the previous criteria are given in Table (3).

Table (3) Comparison between Transmuted Burr Type II and Burr Type II Distributions

Model	Estimates	$\log L$	AIC	AICC	BIC	KS
Transmuted Burr Type II	$\hat{\alpha} = 1.117$	-71.094	146.188	146.388	-138.59	0.254
	$\hat{\lambda} = -0.443$					
Burr Type II	$\hat{\alpha} = 1.611$	-83.15	168.301	168.366	-164.501	0.2857

The results of Table (3) indicates that the proposed transmuted Burr Type II fits well as it has the smallest values for AIC and AICC as compared to Burr Type II distribution. Using the one-sample KS test for goodness of fit the $p - value = 0.034$ which is greater than the significance level $\alpha = 0.01$ so the null hypothesis cannot be rejected for the transmuted Burr Type II distribution. Also $p - value = 0.012$ which is greater than the significance level $\alpha = 0.01$ so the null hypothesis cannot be rejected for the Burr Type II distribution.

A second data set is the data set for annual net income for a random sample of size $n = 44$ individuals from Cairo governorate (CAPMAS (2015)) as shown in Table (4):

Table (4) The data for annual net income in Cairo in 2015

26080	25465	72000	86645	44145	24180
231653	52993	30574	21853	155689	26300
12025	28779	53953	64925	27882	19785
304373	28521	24000	40483	45538	60410
45910	230060	47600	43105	43738	44559
40291	42736	49825	67681	154748	171196
20206	51677	58467	41387	58772	76292
47891	42818				

The descriptive statistics for these data are obtained in Table (5).

Table (5) Descriptive Statistics for the real data of annual net income of Cairo in 2015

N	min	max	range	mean	S.D.	C.V.	Q_1	Q_2	Q_3	Skewness	Kurtosis	Lower bound	Upper bound
44	0.12	3.04	2.92	0.66	0.63	0.95	0.29	0.45	0.64	2.39	5.5	0.47	0.85

Since the value of the skewness measure is positive so the distribution of the data is skewed to the right. Also since the value of the kurtosis measure is greater than 3, so the distribution for the data is peaked.

The estimated parameters of the transmuted Burr Type II and Burr Type distributions and the values of the criteria are given in Table (6).

Table (6) Comparison between Transmuted Burr Type II and Burr Type II Distributions

Model	Estimates	$\log L$	AIC	AICC	BIC	KS
Transmuted Burr Type II	$\hat{\alpha} = 1.531$ $\hat{\lambda} = -0.458$	-55.48	114.961	115.254	-107.674	0.179
Burr Type II	$\hat{\alpha} = 2.207$	-77.674	157.347	157.442	-153.704	0.34

The results of Table (6) indicates that the proposed transmuted Burr Type II fits well as it has the smallest values for AIC and AICC as compared to Burr Type II distribution. Using the one-sample KS test for goodness of fit, the $p - value = 0.042$ which is greater than the significance level $\alpha = 0.01$ so the null hypothesis cannot be rejected for the transmuted Burr Type II distribution. Also $p - value = 0.011$ which is greater than the significance level $\alpha = 0.01$ so the null hypothesis cannot be rejected for the Burr Type II distribution.

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