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Research article

Study on an Extension to Lindley Distribution: Statistical Properties, Estimation and Simulation

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Abstract: Recently, Nedjar and Zeghdoudi [6] proposed a new lifetime model called gamma Lindley distribution. Roozegar and Nadarajah [8] introduced some notes around gamma Lindley distribution including only some statistical properties and estimations depending on the same probability density function which proposed by Nedjar and Zeghdoudi [6]. In fact, the model proposed by Nedjar and Zeghdoudi [6] is not a probabilistic model. Further, some of its fundamental properties as well as parameter estimations are incorrect. Hence, all corrections which proposed by Roozegar and Nadarajah [8] are also incorrect. On the other hand, Messaadia and Zeghdoudi [5] proposed only one remark around the parameter space of gamma Lindley distribution which proposed by Nedjar and Zeghdoudi [6], but the mathematical properties and estimations are still wrong for both Nedjar and Zeghdoudi [6] and Roozegar and Nadarajah [8]. Because, Messaadia and Zeghdoudi [5] did not discuss any corrections around quantile function, entropies, estimation methods, simulation and data analysis. In this paper, several corrections including probability density function, quantile function, entropies, estimation the model parameters using the maximum likelihood estimation and moment estimation methods and simulation are discussed in-detail because the previous three papers not make a benefit to the readers, especially in practical field.

Keywords: Lindley distribution; Estimation methods; Simulation
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1. Introduction

Recently, some attempts have been made to define new flexible distributions to analyze various types of data in applied fields, especially, in medical, renewable energy, ecology and reliability analysis fields. See for example, Jehhan et al. [4], Eliwa et al. [2], El-Morshedy and Eliwa [3], Okorie et al. [7], Alizadeh et al. [1], among others. Lindley (Li) distribution is one of these distributions, which has some nice properties to be used in lifetime data analysis, especially in applications modeling stress-strength model. This model can be shown as a mixture of gamma (2,θ), say G(2,θ), and exponential (θ), say E(θ), with non-negative mixture weights $P_1 = \frac{1}{1+\theta}$ and $P_2 = \frac{\theta}{1+\theta}$, respectively. Thus, the PDF of Li distribution can be expressed as follows

$$\begin{aligned} v(x;\theta) &= P_1\,\theta^2 x e^{-\theta x} + P_2\,\theta e^{-\theta x} \\ &= \frac{\theta^2}{1+\theta}(x+1)e^{-\theta x} \quad ; x,\theta > 0, \end{aligned} \tag{1.1}$$

where $\sum_{i=1}^{2} P_i = 1$. Due to its wide applicability in many fields, several works aimed at extending Li distribution become very important. Among of those modifications, gamma Lindley (GLi) distribution (see Nedjar and Zeghdoudi [6]) and generalized Lindley (GNLi) distribution (see Roozegar and Nadarajah[8]).

The remainder of the paper is structured as follow: Section 2 presents the improper PDF of the GLi, section 3 presents the proper PDF of the GLi, Section 4 presents incorrect and correct statistical properties of the GLi, also the incorrect estimation methods and their corrections, section 5 presents simulations, section 6 presents applications using real data and section 7 presents the conclusion of the study.

2. Improper PDF for GLi Distribution

Nedjar and Zeghdoudi [6] proposed this model as a mixture of two components: G(2,θ) and Li(θ) with mixture weights $\frac{\beta-1}{\beta}$ and $\frac{1}{\beta}$, respectively. Or by another way, it can be shown as a mixture of three components: G(2,θ), E(θ) and G(2,θ) with mixture weights $M_1 = \frac{\beta-1}{\beta}$, $M_2 = \frac{\theta}{\beta(1+\theta)}$ and $M_3 = \frac{1}{\beta(1+\theta)}$, respectively. This means that the PDF of GLi distribution can be expressed as follows

$$\begin{aligned} f(x;\theta,\beta) &= M_1\,\theta^2 x e^{-\theta x} + M_2 \theta e^{-\theta x} + M_3\,\theta^2 x e^{-\theta x} \\ &= \frac{\beta\,(1+\theta)-\theta}{\beta\,(1+\theta)}\theta^2 x e^{-\theta x} + \frac{\theta}{\beta\,(1+\theta)}\theta e^{-\theta x} \quad ; x > 0, \end{aligned} \tag{2.1}$$

where $\theta,\beta > 0$. From Equation (2.1), it is observed that GLi distribution can be also shown as a mixture of G(2,θ) and E(θ) with mixture weights $L_1 = \frac{\beta(1+\theta)-\theta}{\beta(1+\theta)}$ and $L_2 = \frac{\theta}{\beta(1+\theta)}$, respectively. Unfortunately, the PDF $f(x;\theta)$ in Equation (2.1) is not a probabilistic model, because the mixture weights must be non-negative. But in this case, it is found that the mixture weight $L_1 = \frac{\beta(1+\theta)-\theta}{\beta(1+\theta)} < 0$ for $\beta < \frac{\theta}{1+\theta}$. Hence, Equation (2.1) is not a suitable PDF for GLi distribution, because it can be negative for some values of the parameters $\theta > 0$ and $\beta > 0$. It's a contradiction for the PDF properties. Figure 1 shows the plots of incorrect PDF and its corresponding CDF for various values of the model parameters when $\beta < \frac{\theta}{1+\theta}$.

From Figure 1, it is clear also that the CDF can be take values greater than 1. It's a contradiction for the CDF properties.

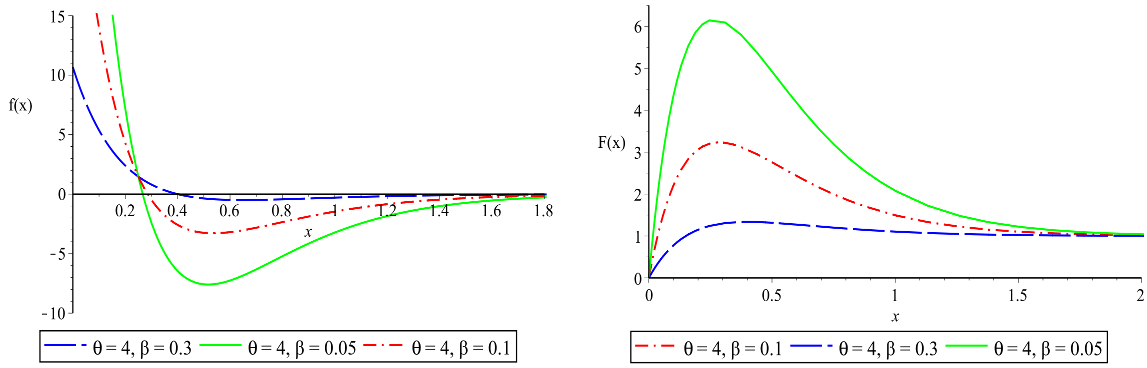

Figure 1. The plots of improper PDF (left panel) and improper CDF (right panel for GLi distribution when $\beta < \frac{\theta}{1+\theta}$.

3. Proper PDFs for GLi Distribution

Nedjar and Zeghdoudi [6], Roozegar and Nadarajah [8] and Messaadia and Zeghdoudi [5] didn't propose a proper PDF for GLi distribution. To obtain a suitable PDF for GLi distribution, we have two propositions.

The first proposition: Modify the parameter space to be $\theta > 0$ and $\beta \geq \frac{\theta}{1+\theta}$. So, the random variable X is said to have GLi distribution if its PDF can be expressed as follows

$$f(x;\theta,\beta) = \frac{\theta^2}{\beta(1+\theta)}\left([\beta + \theta\beta - \theta]\, x + 1\right) e^{-\theta x} \quad ; x, \theta > 0, \beta \geq \frac{\theta}{1+\theta}. \tag{3.1}$$

Figure 2 shows the plots of correct PDF and its corresponding CDF for various values of the model parameters when $\beta \geq \frac{\theta}{1+\theta}$.

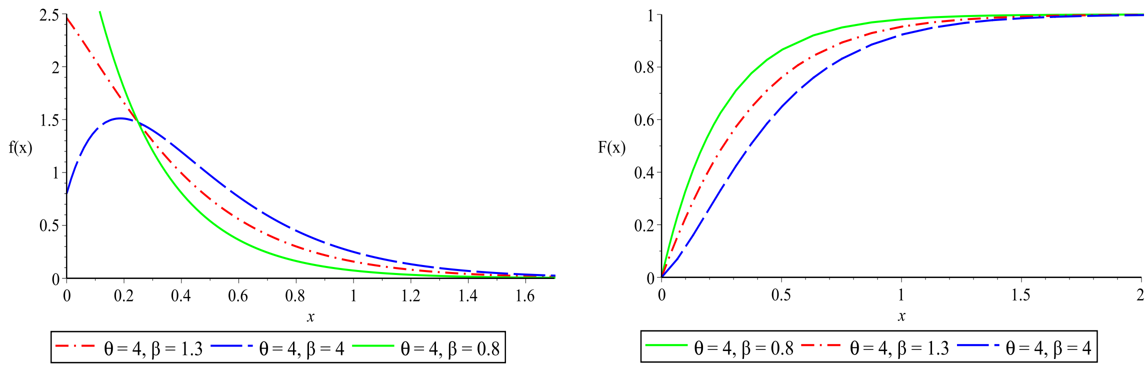

Figure 2. The plots of proper PDF (left panel) and proper CDF (right panel) for GLi distribution when $\beta \geq \frac{\theta}{1+\theta}$.

Note 1.

1. If $\beta = \frac{\theta}{1+\theta}$, then GLi distribution tends to E distribution.
2. If $\beta = 1$, then GLi distribution tends to Li distribution with one parameter.
3. If $\lambda = \beta + \theta\beta - \theta$, then GLi distribution tends to Li distribution with two-parameter.

The second proposition: Generate a new GLi (NGLi) distribution using a new two mixing parameters. The NGLi distribution can be shown as a mixture of G(2,θ) and Li(θ) with mixture weights $H_1 = \frac{\beta}{1+\beta}$ and $H_2 = \frac{1}{1+\beta}$, respectively. So, the PDF and CDF of NGLi distribution can be expressed as follows

$$\begin{aligned} f(x;\theta,\beta) &= H_1\theta^2 x e^{-\theta x} + H_2\frac{\theta^2}{1+\theta}(x+1)e^{-\theta x} \\ &= \frac{\theta^2}{(1+\beta)(1+\theta)}\left([\beta+\theta\beta+1]\,x+1\right)e^{-\theta x} \quad ; x>0 \end{aligned} \tag{3.2}$$

and

$$F(x;\theta,\beta) = \frac{1}{(1+\beta)(1+\theta)}\left\{[\beta+\theta\beta+1]\left[1-(\theta x+1)\,e^{-\theta x}\right]-\theta e^{-\theta x}+\theta\right\} \quad ; x>0, \tag{3.3}$$

respectively, where $\theta, \beta > 0$. Figure 3 shows the PDF and the corresponding CDF of NGLi distribution for various values of the model parameters.

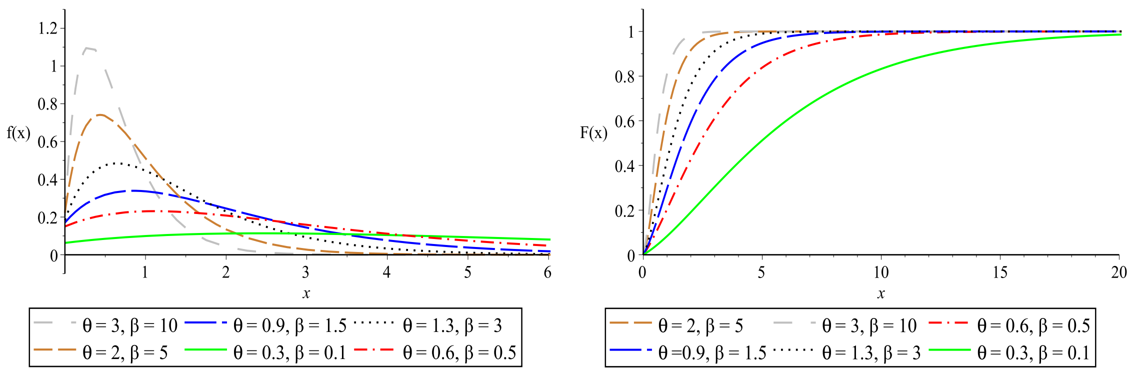

Figure 3. The plots of PDF (left panel) and CDF (right panel) for NGLi distribution.

The NGLi distribution is better than GLi distribution in practical application because of its new parameter space where there is no constraint on θ and β except they are positive.

4. Incorrect Statistical Properties and Its Corrections

4.1. Incorrect quantile function (QF) and its correction

Nedjar and Zeghdoudi [6] and Roozegar and Nadarajah [8] didn't propose a correct QF for GLi distribution. Messaadia and Zeghdoudi [5] didn't study the QF. Nedjar and Zeghdoudi [6] proposed the QF as follows

$$Q_X(u) = -\frac{\beta(1+\theta)}{\theta\left[\beta(1+\theta)-\theta\right]} - \frac{1}{\theta}W_{-1}\left(\frac{\beta(1+\theta)(y-1)}{\beta(1+\theta)-\theta}e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}}\right); 0<u<1, \tag{4.1}$$

where $\theta, \beta > 0$, W_{-1} denotes negative branch of Lambert W function (see Theorem 1, Page 169) and y is considered one of typos. Actually, the proof of this theorem is incorrect in case of $\beta \leq \frac{\theta}{1+\theta}$ for the following reasons:

1. The authors started its proof using a correct formula $F(x) = u$ to get the QF. But, the authors multiplied the both sides of equation $[\theta(\beta(1+\theta)-\theta)x+\beta(1+\theta)]e^{-\theta x} = \beta(1+\theta)(1-u)$ by $\frac{1}{\beta(1+\theta)-\theta}\ e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}}$. It is observed that, if $\beta = \frac{\theta}{1+\theta}$, then $\frac{1}{\beta(1+\theta)-\theta}\ e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}} = \frac{0}{0}$ (undefined). Thus, we wonder how the authors can multiply the both sides of equation by undefined term?.
2. The authors prove that $\frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta}\ e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}} \in \left(\frac{-1}{e},0\right)$. It's incorrect for the following reasons:
   - If $\beta = \frac{\theta}{1+\theta}$, then $\frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta}\ e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}} = \infty \notin \left(\frac{-1}{e},0\right)$.
   - If $\beta < \frac{\theta}{1+\theta}$, then $\frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta}\ e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}} = \frac{c(u-1)}{c-1}e^{-\frac{c}{c-1}} \notin \left(\frac{-1}{e},0\right)$, where $\beta = \frac{c\theta}{1+\theta}, 0 < c < 1$.

From the previous two reasons, we can conclude that $\frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta}e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}} \notin \left(\frac{-1}{e},0\right)$ if $\beta \le \frac{\theta}{1+\theta}$. It's a contradiction with Lambert W function. Thus, the correct QF can be expressed as follows

$$Q_X(u) = \begin{cases} -\frac{\beta(1+\theta)}{\theta[\beta(1+\theta)-\theta]} - \frac{1}{\theta}W_{-1}\left(\frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta}e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}}\right); & \beta > \frac{\theta}{1+\theta} \\ \frac{-\ln(1-u)}{\theta} & ; \quad \beta = \frac{\theta}{1+\theta}, \end{cases} \tag{4.2}$$

where $\frac{-\ln(1-u)}{\theta}$ is the QF for the E distribution.

On the other hand, Roozegar and Nadarajah [8] proposed the QF depending on improper PDF for GLi distribution. Hence, its QF is incorrect when $\beta < \frac{\theta}{1+\theta}$.

4.2. *Incorrect entropies and its corrections*

In statistics, entropy is a measure of variation of the uncertainty. Two popular entropy measures are the Rényi and Shannon entropies. Nedjar and Zeghdoudi [6] and Roozegar and Nadarajah [8] didn't propose correct entropies for GLi distribution. Messaadia and Zeghdoudi [5] didn't study the entropies. The Rényi entropy of the random variable X is discussed by Nedjar and Zeghdoudi [6] as follows

$$I_\gamma(X) = \frac{1}{1-\gamma}\log\left[\frac{\theta^{2\gamma}e^{\frac{\gamma}{\beta+\beta\theta-\theta}}}{\beta^\gamma(1+\theta)^\gamma}\int_0^\infty u^\gamma e^{\frac{-\gamma}{\beta+\beta\theta-\theta}u}du\right];\ \gamma > 0 \text{ and } \gamma = 1. \tag{4.3}$$

Moreover, the Shannon entropy can be derived from Equation (4.3) when $\gamma \longrightarrow \infty$. Actually, Equation (4.3) is incorrect, in addition the Shannon entropy for the following reasons:

1. $\gamma = 1$ is considered one of typos where γ can be take all positive values except 1.
2. For $\gamma > 0$ and $\gamma \ne 1$, the Rényi entropy for the non-negative variable X can be expressed as follows

$$\begin{aligned} I_\gamma(X) &= \frac{1}{1-\gamma}\log\int_0^\infty f^\gamma(x)\,dx \\ &= \frac{1}{1-\gamma}\log\left[\frac{\theta^{2\gamma}}{\beta^\gamma(1+\theta)^\gamma}\int_0^\infty([\beta+\theta\beta-\theta]x+1)^\gamma e^{-\theta\gamma x}\,dx\right] \\ &= \frac{1}{1-\gamma}\log\left[\frac{\theta^{2\gamma}}{\beta^\gamma(1+\theta)^\gamma}\sum_{i=0}^{\gamma}\binom{\gamma}{i}[\beta+\theta\beta-\theta]^i\int_0^\infty x^i e^{-\theta\gamma x}\,dx\right] \\ &= \frac{1}{1-\gamma}\log\left[\frac{\theta^{2\gamma}}{\beta^\gamma(1+\theta)^\gamma}\sum_{i=0}^{\gamma}\binom{\gamma}{i}\frac{[\beta+\theta\beta-\theta]^i}{[\theta\gamma]^{i+1}}\Gamma(i+1)\right], \end{aligned} \tag{4.4}$$

where $\gamma = 1, 2, 3, ...$

3. The Shannon entropy can be derived from Equation (4.4) when $\gamma \longrightarrow 1$, not when $\gamma \longrightarrow \infty$.

On the other hand, Roozegar and Nadarajah [8] proposed the entropies depending on improper PDF for GLi distribution. Hence, its entropies are incorrect when $\beta < \frac{\theta}{1+\theta}$.

4.3. *Incorrect maximum likelihood (ML) estimation and its correction*

Nedjar and Zeghdoudi [6] and Roozegar and Nadarajah [8] didn't propose correct estimation for GLi distribution. Messaadia and Zeghdoudi [5] didn't study the ML estimation. Nedjar and Zeghdoudi [6] derived the log-likelihood equation of single observation as follows

$$L(x;\theta,\beta) = 2\ln\theta - \ln\beta - \ln(\theta+1) + \ln\left[(\beta+\theta\beta-\theta)\,x+1\right] - \theta x. \tag{4.5}$$

The derivatives of Equation (4.5) with respect to θ and β are:

$$\frac{\partial L(x;\theta,\beta)}{\partial\theta} = \frac{2}{\theta} - \frac{1}{\theta+1} - x + \frac{(\beta-1)\,x}{(\beta+\theta\beta-\theta)\,x+1} \tag{4.6}$$

and

$$\frac{\partial L(x;\theta,\beta)}{\partial\beta} = \frac{-1}{\beta} + \frac{(1+\theta)\,x}{(\beta+\theta\beta-\theta)\,x+1}, \tag{4.7}$$

respectively. The ML estimators can be obtained by solving the non-linear Equations (4.6) and (4.7). Therefore, the ML estimator $\widehat{\theta}$ of θ and $\widehat{\beta}$ of β can be expressed as follows

$$\widehat{\beta} = \frac{1}{1+x} \text{ and } \widehat{\theta} = \frac{1}{x}. \tag{4.8}$$

In fact, the estimation in Equation (4.8) is inaccurate for the following reasons:

1. When the non-linear Equations (4.6) and (4.7) are solved, we get

$$\widehat{\beta} = \frac{\widehat{\theta}}{\left(1+\widehat{\theta}\right)\left(2-\widehat{\theta}x\right)} \text{ where } \frac{1}{x} \le \widehat{\theta} < \frac{2}{x}. \tag{4.9}$$

   Equation (4.9) shows the range of $\widehat{\theta}$ and therefore $\widehat{\beta}$. For n observations the estimation in Equation (4.9) can be expressed as follows

$$\widehat{\beta} = \frac{\widehat{\theta}}{\left(1+\widehat{\theta}\right)\left(2-\widehat{\theta}\overline{x}\right)} \text{ where } \frac{1}{\overline{x}} \le \widehat{\theta} < \frac{2}{\overline{x}}, \tag{4.10}$$

   where $\overline{x}$ represents the mean of data.

2. Equation (4.8) represents only the ML estimators for E distribution where $\widehat{\beta} = \frac{1}{1+x} = \frac{\widehat{\theta}}{1+\widehat{\theta}}$. Recall, **Note 1** in the proposed paper.

On the other hand, Roozegar and Nadarajah [8] proposed the ML estimator $\widehat{\theta}$ of θ and $\widehat{\beta}$ of β as follows

$$\widehat{\beta} = \frac{1}{1+\overline{x}} \text{ and } \widehat{\theta} = \frac{1}{\overline{x}}. \tag{4.11}$$

Actually, the estimation in Equation (4.11) is inaccurate. Recall, Equation (4.10). Messaadia and Zeghdoudi [5] didn't propose any corrections on ML estimation.

4.4. Incorrect moment (MO) estimation and its correction

Nedjar and Zeghdoudi [6] derived the MO estimators using the first moment (m) and second moment (m_2) for GLi distribution where

$$m = \frac{2\beta(1+\theta)-\theta}{\theta\beta(1+\theta)} \text{ and } m_2 = \frac{6\theta\beta+6\beta-4\theta}{\theta^2\beta+\theta^3\beta}. \tag{4.12}$$

The solving of the previous non-linear system gives

$$\widehat{\beta} = \frac{\widehat{\theta}}{\left(1+\widehat{\theta}\right)\left(2-\widehat{\theta}m\right)} \text{ and } m_2\widehat{\theta}^2 - 4m\widehat{\theta} + 2 = 0, \tag{4.13}$$

where $m_2 = s^2 + m^2$ and s^2 is the variance. The authors discussed Equation (4.13) as follows:

1. If $m = 0$ and $m_2 = 0$, then $\widehat{\beta} = \widehat{\theta} = \phi$.
2. If $m \neq 0$ and $m_2 = 0$, then $\widehat{\theta} = \frac{1}{2m}$.
3. If $m_2 \neq 0$, then $\widehat{\theta} = \frac{1}{m_2}\left(2m + \sqrt{2}\sqrt{-s^2+m^2}\right)$ or $\widehat{\theta} = \frac{1}{m_2}\left(2m - \sqrt{2}\sqrt{-s^2+m^2}\right)$.

In fact, the authors must write $\widehat{\theta} = \frac{1}{m_2}\left(2m + \sqrt{2}\sqrt{-s^2+m^2}\right)$ for $m \geq s$ and $\frac{1}{m} \leq \widehat{\theta} < \frac{2}{m}$. It is considered a necessary condition to get a positive value for each $\widehat{\theta}$ and $\widehat{\beta}$. Because for some data, we may find that $m < s$ such as actuarial data sets. Moreover, we wonder why the authors didn't discuss the case of $\widehat{\theta} = \frac{1}{m_2}\left(2m - \sqrt{2}\sqrt{-s^2+m^2}\right)$, $m_2 \neq 0$?. Although, this case may be give also a positive value for $\widehat{\theta}$ if $m \geq s$ and therefore $\widehat{\beta}$ if $\frac{1}{m} \leq \widehat{\theta} < \frac{2}{m}$.

On the other hand, Roozegar and Nadarajah [8] and Messaadia and Zeghdoudi [5] didn't propose any corrections on MO estimation.

5. Simulation

Nedjar and Zeghdoudi [6] and Roozegar and Nadarajah [8] didn't propose correct simulation for some values of the parameters. Messaadia and Zeghdoudi [5] didn't propose any simulation for the model parameters. In this section, we assess the performance of the MLE with respect to sample size n for the following two reasons:

1. Nedjar and Zeghdoudi [6] and Roozegar and Nadarajah [8] studied the performance of this method using incorrect initial values like $(\theta, \beta) = (1, 0.1)$ and $(3, 0.5)$ where $\beta < \frac{\theta}{1+\theta}$.
2. Nedjar and Zeghdoudi [6] and Roozegar and Nadarajah [8] didn't propose the number of samples (N) of size n.

For more information about simulation study see [10], [11], and [12].

Hence, we assess the performance of the MLE using correct initial values and an obvious algorithm as follows:

1. Generate $N = 500$ samples of size $n = 10, 11, 12, ...,50$ from GLi$(0.5, 0.5)$ and GLi$(1.0, 0.5)$.
2. Compute the MLEs for N samples, say $\widehat{\theta}_j$ and $\widehat{\beta}_j$ for $j = 1, 2, ..., N$.

3. Compute the biases and mean-squared errors (MSEs), where

$$\text{bias} = \frac{1}{N}\sum_{j=1}^{N}\left(\widehat{\varpi}_j - \varpi\right) \text{ and } \text{MSE} = \frac{1}{N}\sum_{j=1}^{N}\left(\widehat{\varpi}_j - \varpi\right)^2.$$

4. The empirical results are given in Figures 4 and 5.

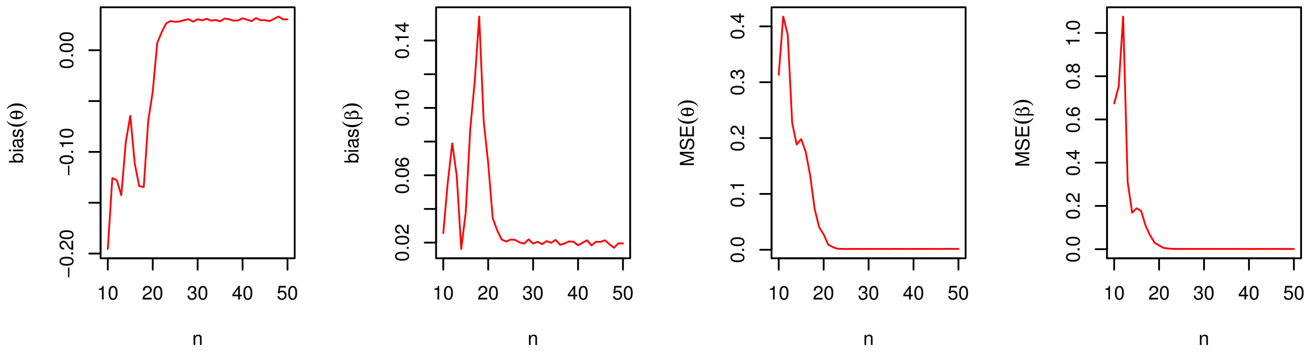

Figure 4. The bias and MSE of $\widehat{\theta}$ and $\widehat{\beta}$ versus for GLi model when $(\theta, \beta) = (0.5, 0.5)$.

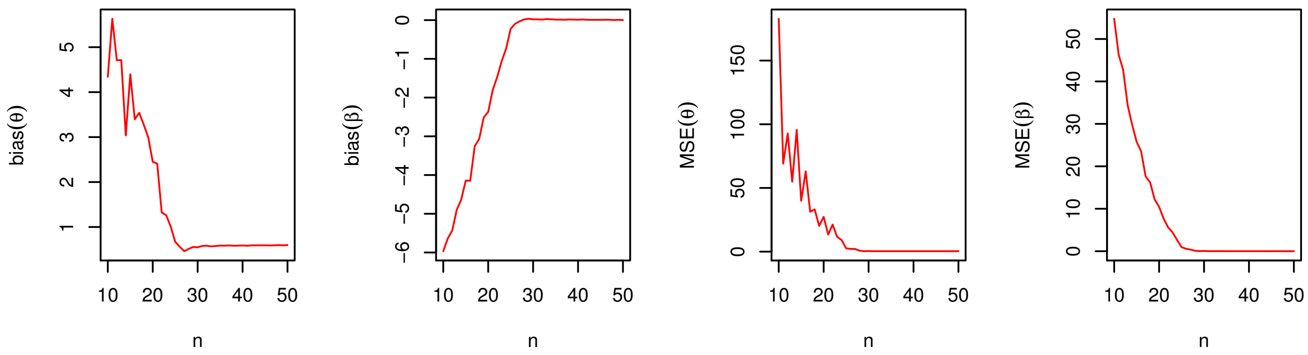

Figure 5. The bias and MSE of $\widehat{\theta}$ and $\widehat{\beta}$ versus for GLi model when $(\theta, \beta) = (1.0, 0.5)$.

From Figures 4 and 5 the following observations can be made:

1. The magnitude of bias always decreases to zero as $n \to \infty$.
2. The MSEs always decrease to zero as $n \to \infty$. This shows the consistency of the estimators.
3. The MLE method can be used quite effectively for data analysis purposes.

We have presented results only for $(\theta, \beta) = (0.5, 0.5)$ and $(1.0, 0.5)$. But, the results are similar for other choices for θ and β.

6. Real Data with Incorrect Estimators and Its Corrections

In this section, we show at first that Nedjar and Zeghdoudi [6] and Roozegar and Nadarajah [8] proposed incorrect estimators for the first real data set considered. After that, we derive the correct estimators for this data. The authors used some goodness-of-fit measures, namely, $-L$, Kolmogorov-Smirnov (KS) statistic as well as Akaike information criterion (AIC) and Bayesian information criterion (BIC).

The first data set (I): This data represents the failure times for a sample of 15 electronic components in an acceleration life test (see Lawless [9]). The mean (m) and standard deviation (s) of this data are 27.546 and 20.059, respectively. Roozegar and Nadarajah [8] proposed incorrect estimation for the model parameters depending on inaccurate Equation (4.11) as follows $\widehat{\theta} = 0.036, \widehat{\beta} = 0.035$ and $L = -64.738$. Messaadia and Zeghdoudi [5] didn't propose any estimation for the model parameters. Nedjar and Zeghdoudi [6] estimated the model parameters and goodness-of-fit measures as follows: $\widehat{\theta} = 0.684$ and $\widehat{\beta} = 1.129$ with $L = -64.015$, $AIC = 132.03$, $BIC = 133.45$ and $KS = 0.094$. Actually, the estimation results of Nedjar and Zeghdoudi [6] are incorrect for the following reasons:

1. The value of $\widehat{\theta}$ is incorrect, because $\widehat{\theta}$ must be located in the interval $\left[\frac{1}{27.546}, \frac{2}{27.546}\right[$ according to Equation (4.10).
2. If $\widehat{\theta} = 0.684$ and $\widehat{\beta} = 1.129$, then $L = -255.617$, $AIC = 515.234$, $BIC = 516.650$, $KS = 0.835$ and p-value = 0.00.
3. If $\widehat{\theta} = 0.684$, then $\widehat{\beta} = -0.0241 < 0$ according to Equation (4.10). It is a fatal mistake.

Table 1 shows the correct MLEs with their corresponding standard errors (in parentheses) as well as the correct goodness-of-fit measures for this data.

Table 1. The correct estimation for data set I using the ML method.

Model ↓ Estimation⟶	$\widehat{\theta}$(SE)	$\widehat{\beta}$(SE)	$-L$	KS (p-value)	AIC	BIC
GLi	0.062(0.017)	0.205(0.292)	64.108	0.095(0.997)	132.215	133.632

The approximate 95% two sided confidence interval of the parameters $\widehat{\theta}$ and $\widehat{\beta}$ are given respectively as $[0.036, 0.095]$ and $[0, 0.776]$ where the variance-covariance matrix can be expressed as follows

$$I_0^{-1} = \begin{pmatrix} 0.000296 & 0.003809 \\ 0.003809 & 0.085075 \end{pmatrix}. \tag{6.1}$$

Figure 6 shows the fitted densities and fitted CDF as well as QQ plot for correct and incorrect estimators which support our results.

Moreover, Figure 6 shows the profiles of the log-likelihood function for incorrect estimators using data set I.

From Figure 7, it is observed that the estimator $\widehat{\beta}$ can be take different values. This means that this estimator not a unique. Figure 8 shows the profiles of the log-likelihood function for correct estimators.

From Figure 8, it is clear that the parameters are unimodal functions. So, these two estimators are the best for data set I.

On the other hand, we wonder why the authors didn't discuss the MO estimators in data analysis section?. Table 2 shows the MO estimators as well as the KS statistic and its p-value for this data.

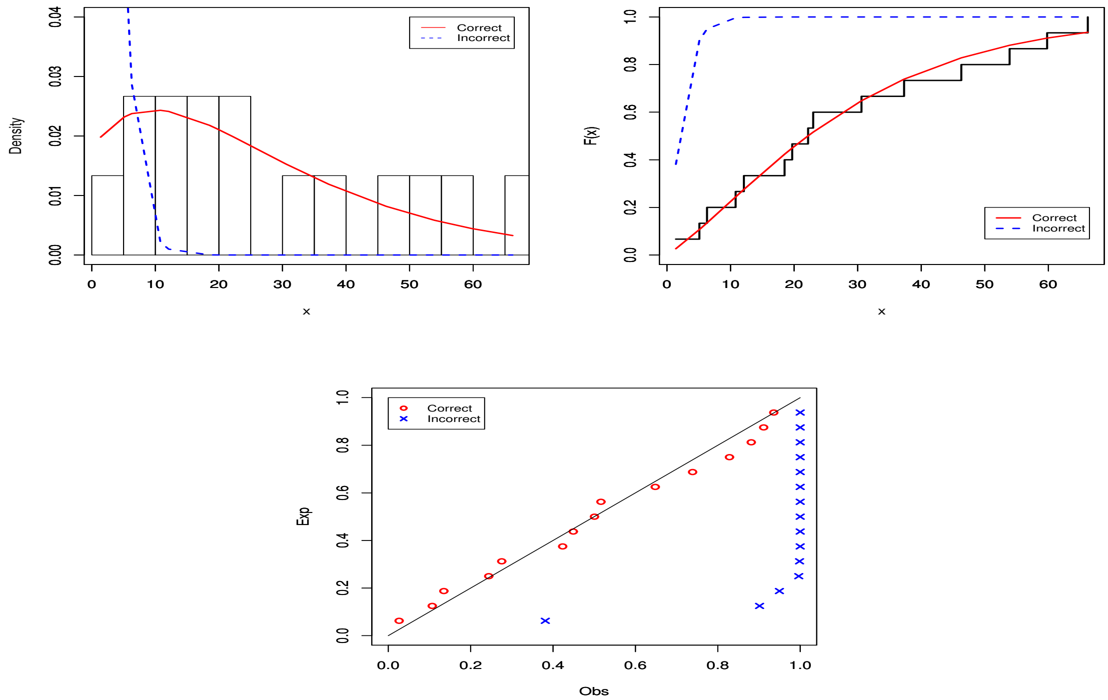

Figure 6. The fitted PDF (left panel), the estimated CDF (right panel) and PP plots (bottom panel) of GLi distribution for data set I.

Table 2. Estimation for data set I using the MO method.

Model ↓ Estimation⟶	$\widehat{\theta}$	$\widehat{\beta}$	KS (p-value)
GLi	0.0704	1.1029	0.1107(0.9923)

According to Equation (4.13), it is found that $\widehat{\theta}$ can take either 0.0244 or 0.0704. Depending on the condition $\frac{1}{m} \leq \widehat{\theta} < \frac{2}{m}$, the value of $\widehat{\theta} = 0.0244$ must be rejected.

7. Conclusions

In this study, we have shown that the PDF of the gamma Lindely (GLi) distribution developed by Nedjar and Zeghdoudi [6] is improper. The corrected PDF of the GLi distribution is therefore developed. The incorrect statistical properties of the distribution are corrected and the correct estimators for estimating the parameters of the distribution are developed. Simulation experiments are performed to examine the performance of the correct estimators. The results revealed that the correct estimators are consistent. Finally we illustrated the applications of the distribution using the incorrect estimators and their corrections. The findings revealed that the correct distribution provides better fit to the data sets compared to the incorrect distribution.

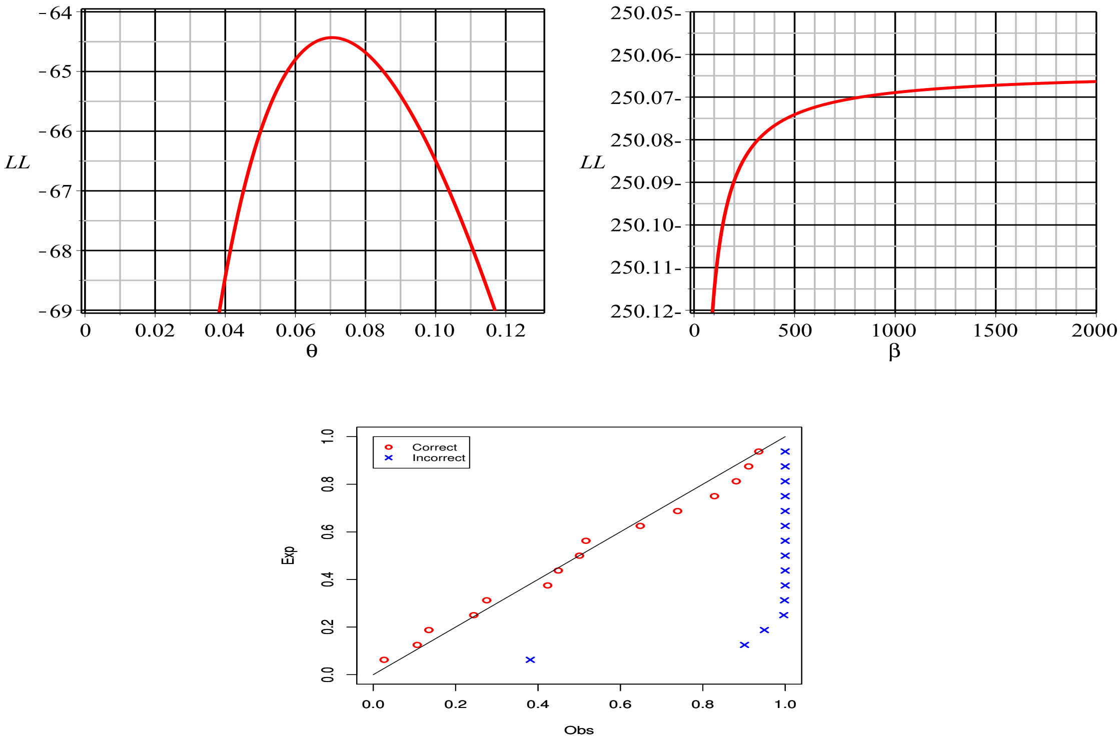

Figure 7. The profiles of the log-likelihood function for incorrect estimators using data set I.

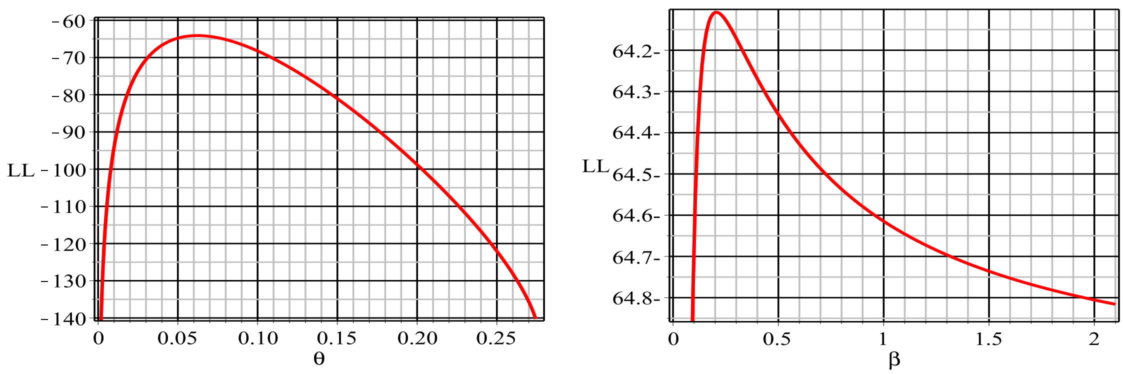

Figure 8. The profiles of the log-likelihood function for correct estimators using data set I.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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