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Thermodynamic Functions of the Universe in the Presence of Viscosity in the Relativity Theory and Lyra Geometry

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Abstract: This paper investigates the thermodynamic functions of the universe in the presence of viscous fluid in general relativity and Lyra geometry. The space-time is modeled by Bianchi type VI_0 cosmological model. The field equations' solution is obtained considering that the scalar expansion θ of the cosmological model is proportional to the eigenvalue σ_1^1 of the shear tensor σ_i^j. The thermodynamic functions are studied for the universe and their values are compared in general relativity and Lyra geometry. For all obtained models, the entropy is consistent with the second law of thermodynamics. The physical and geometrical properties of the obtained models are discussed. Also, we show that the Lyra term β plays the role of the variable cosmological term in the relativity theory and can not be identified as a viscosity term. Without the viscosity term, we obtain the entropy as a constant $\boldsymbol{S}$= constant. That is the universe is in an adiabatic state. The Bianchi type VI_0 cosmological model presented in this paper explains a stage of evolution with the positive deceleration parameter.
Keywords: Lyra geometry; Bianchi type VI_0 cosmological model; Viscosity, Energy momentum tensor; Thermodynamic functions.

1. Introduction

After Einstein [1] introduced his general relativity theory, which is a geometrizing of gravitation by identifying the metric tensor,g_{ij}, with the gravitational potentials. In 1918, Weyl [2] introduced a generalization of Riemannian geometry by introducing a scalar field Φ to be identified with the electromagnetic field in a trial to geometrize both electromagnetism and gravitation. This generalization was criticized by Einstein [3] due to the concept of the non-integrability of length transfer. In 1952, Lyra [4] proposed a modification of Riemannian geometry. He introduced the notation of a gauge function,$x^0(x^i)$, in his manifold. In Lyra geometry, the transfer of length is integrable, and the connection is preserved as in Riemannian geometry. When $x^0(x^i) = 1$, the curvature scalar of Lyra and Weyl is identical.

Bianchi type VI_0 cosmological model is suitable for describing the universe since it is inhomogeneous, anisotropic and a generalization of the FRW cosmological model, in addition, it plays an important role with the Bianchi family, in understanding and describing the early and present stages of the evolution of the universe

In relativity theory, Bianchi type VI_0 space-time with different forms of matter distribution was investigated. Dark energy Bianchi type VI_0 model with the equation of state as a variable was studied by Amirhashchi et al. [5]. Sharma et al. [6] investigated inhomogeneous Bianchi type VI_0 space-time in the case of stiff matter $(p = \rho)$. In the existence of dark energy fluid and an attractive massive scalar fluid, Aditya et al. [7] investigated the dynamical aspects of the Bianchi type VI_0 model. In the presence of perfect fluid, Mishra and Biswal [8] presented a self-consistent system of a five-dimensional Bianchi type VI_0 model with dark energy. Bali and Poonia [9] studied Bianchi type VI_0 inflationary model with flat potential. Bianchi type VI_0 cosmological model in the existence of the electromagnetic field and variable deceleration parameter was studied by Hegazy and Rahaman [10]. Roy and Narain [11] investigated the inhomogeneous Bianchi type VI_0 model. Bianchi type VI_0 in the theory of self-creation and Lyra geometry was studied by Hegazy and Rahaman [12]. Priyanka et al. [13] presented dark energy Bianchi type I_0 , cosmological models, with the equation of state parameters as a constant and time-dependent. For a cosmological term Λ as a function of t,

Tripathi et al. [14] investigated an inhomogeneous Bianchi type VI_0 cosmological model. Kate et al. [15] examined some solutions in Bianchi type VI_0 models with modification of chaplain gas. Adhav et al. [16] discussed the nature of the anisotropic dark energy Bianchi type VI_0 model.

In the existence of viscosity, some cosmological models in different theories of relativity were studied. Bianchi type I string cosmological model with negative constant deceleration parameter and bulk viscous fluid in the general relativity theory studied by Singh and Baro [17]. Bianchi type VI_0 holographic Ricci dark energy model in the Brans-Dicke theory [18] was studied by Santhi et al. [19]. In $f(R,T)$ theory: Prasad et al. [20] studied the bulk viscous accelerating universe, Sahoo and Reddy [21] studied LRS Bianchi type I bulk viscous models, Ram and Kumari [22] discussed Bianchi types V and I cosmological models, Samanta et al. [23] studied bulk viscous Kaluza-Klein models and validity of the second law of thermodynamics, Mahanta [24] studied bulk viscous cosmological models and Satish and Venkateswarlu [25] investigated Kaluza Klein cosmological models with bulk viscous fluid. By using the technique of Letelier and Stachel in the Bimetric theory of gravitation, the Bianchi type I bulk viscous fluid string dust cosmological model with the magnetic field was investigated by Borkar and Charjan [26]. With a new proposed form of time-dependent deceleration parameter, Singh and Bishi [27] presented FRW cosmological model in the Brans Dicke theory. An inhomogeneous plane-symmetric bulk viscous model with variable Λ studied by Pandey and Pradhan [28]. Bali and Pradhan [29] presented Bianchi type III string cosmological models with variable bulk viscosity. In Lyra geometry: Accelerating Bianchi type VI_0 model with bulk viscosity was studied by Asgar and Ansari [30], Kandalkar and Samdurkar [31] investigated LRS Bianchi type I in the presence of the viscosity and Five-dimensional homogeneous cosmological models with variable bulk viscosity and G were studied by Singh et al. [32]. Singh and Kale [33] dealt with anisotropic bulk viscous models with variables G and Λ. In Saez-Ballester's theory, Mishra and Dua [34] constructed a new class of bulk viscous string cosmological models. In the presence of viscosity and variable cosmological term Λ, Bianchi type I was studied by Bali and Singh [35]. In Lyra geometry and relativity theory, Hegazy [36] introduced Bianchi type I space-time. Mohanty and Samanta [37] studied five-dimensional cosmological models in the absence of bulk viscosity, in the existence of constant bulk viscosity, and in the case of variable bulk viscosity.

As a consequence of the previous studies, we investigate the Bianchi type VI_0 cosmological model in the presence of viscous fluid. In section 2 we present the results and discussion of the study. The conclusion is indicated in section 3.

2. Results and discussion

For the Bianchi type VI_0 model, ds^2 (Line element) takes the form:

$$ds^2 = [A(t)]^2dx^2 + [B(t)]^2e^{2mx}dy^2 + [C(t)]^2e^{-2mx}dz^2 - dt^2, \quad (1)$$

In the normal gauge $(x^0(x^i) = 1)$, the field equations as obtained by Sen [38] on Lyra geometry reads as.

$$G_{ij} + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k = -\chi T_{ij}, \quad (2)$$

where G_{ij} is the Einstein tensor and ϕ_i is a displacement vector that has only a time component and is given by: $\phi_i = (0,0,0,[\beta(t)])$. (3)

T_{ij} is the momentum tensor in the form [39], [40]:

$$T_i^j = (\rho + p)u_iu^j + pg_i^j - \xi\theta[g_i^j + u_iu^j], \quad (4)$$

where ξ is the coefficient of the bulk viscosity, θ the scalar expansion, ρ the density, and p the pressure. In co-moving coordinates $(u^1 = 0 = u^2 = u^3, u^4 = 1, u_4 = -1, u_iu^j = -1)$ with (4) we get:

$$T_1^1 = [p - \xi\theta] = T_2^2 = T_3^3, \quad T_4^4 = -\rho, \quad (5)$$

Form(1) and (2), we get:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{m^2}{A^2} + \frac{3}{4}\beta^2 = -\chi[p - \xi\theta], \quad (6)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m^2}{A^2} + \frac{3}{4}\beta^2 = -\chi[p - \xi\theta], \quad (7)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m^2}{A^2} + \frac{3}{4}\beta^2 = -\chi[p - \xi\theta], \quad (8)$$

$$m[\frac{\dot{B}}{B} - \frac{\dot{C}}{C}] = 0, \quad (9)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{m^2}{A^2} - \frac{3}{4}\beta^2 = \chi\rho. \quad (10)$$

Dot means the ordinary differentiation according to the

coordinate t.

The conservation T^i_j ($T^i_{j;i} = 0$) gives:

$$\dot{\rho} + (p + \rho - \xi\theta)(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) = 0. \quad (11)$$

Conservation of the L. H. S. of (2) gives:

$$\frac{3}{2}\beta(\dot{\beta} + \beta[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}]) = 0. \quad (12)$$

Solutions of (12) are given by:

$$\beta = 0, \quad and \quad \beta = \frac{N_1}{ABC}, \quad (13)$$

where N_1 is a constant of integration.

In the case $\beta = 0$, we reduce to the general theory of relativity which will be studied later.

Equation (13) gives the value of β by A, B, C, and equation (9) gives

$$C = k_1 B. \quad (14)$$

By using (9) in (7) and (8) we see that the two equations are equal. Hence, we have five equations (6), (7) or (8), (10)-(12) with five unknowns A, C or B, ρ, p and ξ. Logically, five equations with five unknowns so solutions are possible. From (6) and (7) we get:

$$\frac{\ddot{B}}{B} + \frac{\dot{B}\dot{C}}{BC} - \frac{\ddot{A}}{A} - \frac{\dot{A}\dot{C}}{AC} + \frac{2m^2}{A^2} = 0. \quad (15)$$

To find the solution of (15), we impose the following constraints (physical condition) on space-time: considering that the scalar expansion θ of the cosmological model is proportional to the eigenvalue σ^1_1 of the shear tensor σ^j_i, that is

$$A = (BC)^L, \quad (16)$$

where L is a constant.

From (14) and (16), equation (15) becomes:

$$\frac{\ddot{B}}{B} + (2L + 1)\frac{\dot{B}^2}{B^2} = bB^{-4L}, \quad (17)$$

where $b = \frac{2m^2}{k_1^{2L}(2L-1)}$.

From (17), $B(t)$ reads as:

$$B(t) = \left(\sqrt{\frac{m^2 k_1^{-2L}}{(2L-1)}}\ 2Lt + k_4\right)^{\frac{1}{2L}}, \quad (18)$$

where k_4 is a constant.

From (14) and (16) we get:

$$A(t) = k_1^L\left(\sqrt{\frac{m^2 k_1^{-2L}}{(2L-1)}}\ 2Lt + k_4\right), \quad C(t) = k_1\left(\sqrt{\frac{m^2 k_1^{-2L}}{(2L-1)}}\ 2Lt + k_4\right)^{\frac{1}{2L}}. \quad (19)$$

The line element (1) reads as:

$$ds^2 = k_1^{2L}T^2dx^2 + T^{\frac{1}{L}}e^{2mx}dy^2 + k_1^2T^{\frac{1}{L}}e^{-2mx}dz^2 - dt^2, \quad (20)$$

where

$$T = \left(\sqrt{\frac{m^2 k_1^{-2L}}{(2L-1)}}\ 2Lt + k_4\right).$$

From (13) the additional term β introduced by Lyra reads as:

$$\beta = N_1 k_1^{-L-1} T^{-\frac{L+1}{L}}. \quad (21)$$

For the model (20), the spatial volume $V = k_1^{L+1}(T)^{\frac{1}{L}+1}$. The non-vanishing components of the (σ^j_i) read as $\sigma^1_1 = -2\sigma^2_2 = -2\sigma^3_3 = \frac{2(2L-1)}{3T}$. The shear $\sigma^2 = \frac{(2L-1)m^2k_1^{-2L}}{3(T)^2}$. the scalar expansion $\theta = \frac{2(L+1)\sqrt{\frac{m^2k_1^{-2L}}{2L-1}}}{T}$.

From (10), we get:

$$\rho = \frac{k_1^{-2(L+1)}(T)^{-\frac{2(L+1)}{L}}(8k_1^2(L+1)m^2(T)^{2/L} + 3(1-2L){N_1}^2)}{4(2L-1)\chi}. \quad (22)$$

From (6)-(8), the term $(p - \xi\theta)$ reads as:

$$p - \xi\theta = \frac{2(L-1)m^2k_1^{-2L}}{(2L-1)\chi(T)^2} - \frac{3{N_1}^2k_1^{-2(L+1)}(T)^{-\frac{2(L+1)}{L}}}{4\chi}, \quad (23)$$

We can split (23) to gives:

$$p = \frac{k_1^{-2L}\left(\frac{8Lm^2}{2L-1} - \frac{3{N_1}^2(T)^{-2/L}}{k_1^2}\right)}{4\chi(T)^2}, \quad (24)$$

and

$$\xi\theta = \frac{2m^2k_1^{-2L}}{(2L-1)\chi(T)^2}. \quad (25)$$

Hence :

$$\xi = \frac{m^2k_1^{-2L}}{(L+1)(2L-1)\chi\sqrt{\frac{m^2k_1^{-2L}}{2L-1}}(T)}. \quad (26)$$

At the beginning of evolution, p, ρ, and ξ begin with large values as $t = 0$ and at $t \to \infty$, p, ρ, and ξ tend to zero.

2.1. Effect of the viscosity on thermodynamic functions

The second law of thermodynamics state that the entropy of the universe will increase over time and can never be negative. Entropy defines as a measure of randomness in the system. Due to internal and external dissipative effects, the entropy changes and these changes is always non-negative i.e. $d\boldsymbol{S} > 0$. The enthalpy ($\boldsymbol{H}$) is a measure of the heat content of the physical system (universe). The Gibbs energy ($\boldsymbol{G}$) can be

defined as the maximum amount of work that can be extracted from a universe. Helmholtz's energy ($\boldsymbol{F}$) is a measure of the useful work in the universe.

In the second self-creation theory, Hegazy [41] studied Bianchi's type VI model in the existence of perfect fluid and obtained a mathematical expression for deriving entropy. Hegazy and Rahman [12] proved that the additional component introduced by Lyra has no effect on the entropy because its arises from geometry and is not a part of the momentum tensor. In the relativity theory, Hegazy and Rahman [42] studied the thermodynamic functions of the universe in the existence of the electromagnetic field. Hegazy [43] explained the thermodynamic functions of the universe with help of the scalar field ϕ. Hegazy [44] gave the effect of viscosity on the thermodynamic functions of the universe. Hegazy [36] identified the Lyra term as viscosity and explains its effect on thermodynamic functions.

For the entropy problem $\boldsymbol{S}$, we have $d\boldsymbol{S} > 0$ (second law of thermodynamics). The change in the entropy with time, reads as [36], [44]:

$$\frac{d\boldsymbol{S}_{Lyra}}{dt} = \frac{2\pi V[\xi\theta^2]}{H} = -\frac{12\pi m^2 k_1^{1-L}(T)^{\frac{1}{L}-1}}{\chi - 2L\chi}, \tag{27}$$

by integration we get:

$$\boldsymbol{S}_{Lyra} = -\frac{6\pi m^2 k_1^{1-L}(T)^{1/L}}{(\chi-2L\chi)\sqrt{\frac{m^2 k_1^{-2L}}{2L-1}}}. \tag{28}$$

If the internal energy U is given by ρV and the temperature $T = \frac{H\ (mean\ \ Hubble\ \ parameter)}{2\pi}$ [45] we get: The enthalpy ($\boldsymbol{H_{Lyra}}$), the Helmholtz energy ($\boldsymbol{F_{Lyra}}$) and the Gibbs energy ($\boldsymbol{G_{Lyra}}$) read as:

$$\boldsymbol{H}_{Lyra} = U + PV = (\rho + P)V = \frac{k_1^{-L-1}(T)^{-\frac{L+1}{L}}\left(4k_1^2(2L+1)m^2(T)^{2/L} + 3(1-2L){N_1}^2\right)}{2(2L-1)\chi.}, \tag{29}$$

$$\boldsymbol{F}_{Lyra} = U - T\boldsymbol{S}_{Lyra} = -\frac{3{N_1}^2 k_1^{-L-1}(T)^{-\frac{L+1}{L}}}{4\chi}, \tag{30}$$

$$\boldsymbol{G}_{Lyra} = \boldsymbol{H}_{Lyra} - T\boldsymbol{S}_{Lyra} = \frac{k_1^{-L-1}(T)^{-\frac{L+1}{L}}\left(4k_1^2 L m^2(T)^{2/L} + 3(1-2L){N_1}^2\right)}{2(2L-1)\chi}. \tag{31}$$

2.2. Bianchi type VI_0 cosmological model in the relativity theory only.

In the relativity theory (Lyra term $\beta = 0$), there are no changes in the metric coefficients since they are not dependent on β. The physical quantities: ρ_{GR} , p_{GR} , and ξ_{GR} are changed and read as:

$$\rho_{GR} = \frac{2(L+1)m^2 k_1^{-2L}}{(2L-1)\chi(T)^2}, \tag{32}$$

$$p_{GR} - \xi_{GR}\theta = \frac{2(L-1)m^2 k_1^{-2L}}{(2L-1)\chi(T)^2}. \tag{33}$$

Equation (33) can be divided as:

$$\xi_{GR}\theta = \frac{m^2 k_1^{-2L}}{\chi(T)^2}, \tag{34}$$

and

$$p_{GR} = \frac{(4L-3)m^2 k_1^{-2L}}{(2L-1)\chi(T)^2}. \tag{35}$$

From (27), we get:

$$\frac{d\boldsymbol{S}_{GR}}{dt} = \frac{2\pi V[\xi_{GR}\theta^2]}{H} = \frac{6\pi m^2 k_1^{1-L}(T)^{\frac{1}{L}-1}}{\chi}, \tag{36}$$

by integration we get:

$$\boldsymbol{S}_{GR} = \frac{3\pi m^2 k_1^{1-L}(T)^{1/L}}{\chi\sqrt{\frac{m^2 k_1^{-2L}}{2L-1}}}. \tag{37}$$

The thermodynamic functions read as:

$$\boldsymbol{H}_{GR} = \frac{(6L-1)m^2 k_1^{1-L}(T)^{\frac{1}{L}-1}}{(2L-1)\chi}, \tag{38}$$

$$\boldsymbol{G}_{GR} = -\frac{L(2L-5)m^2 k_1^{1-L}(T)^{\frac{1}{L}-1}}{(2L-1)\chi}, \tag{39}$$

$$\boldsymbol{F}_{GR} = -\frac{(L+1)(2L-3)m^2 k_1^{1-L}(T)^{\frac{1}{L}-1}}{(2L-1)\chi}. \tag{40}$$

Now, we make a comparative study between the results obtained in relativity theory and Lyra geometry. The constants are taken as $m = 1$, $k_1 = 1$, $N_1 = 5$, $\xi = 8\pi$, $K_4 = 1$ and $L = 2$.

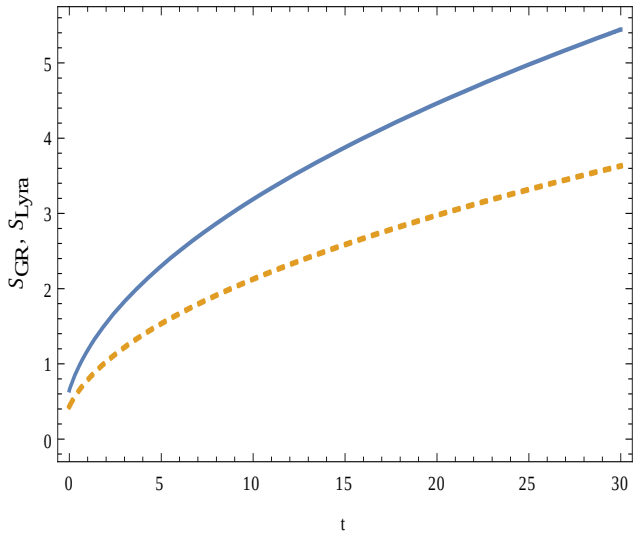

Figure 1. The entropy $\boldsymbol{S}_{GR}$ (Thick line) and the entropy $\boldsymbol{S}_{Lyra}$ (Dashed line) vs. time t, $0 < t < 30$.

The behavior of the entropy in relativity theory and Lyra geometry is consistent with the fundamental of the second law of thermodynamics, $d\boldsymbol{S}_{GR} > 0$, $d\boldsymbol{S}_{Lyra} > 0$. The entropy $\boldsymbol{S}_{Lyra}$ and $\boldsymbol{S}_{GR}$ begin with small values at the beginning of evolution and increase as time is released and reach infinities at the end of evolution.

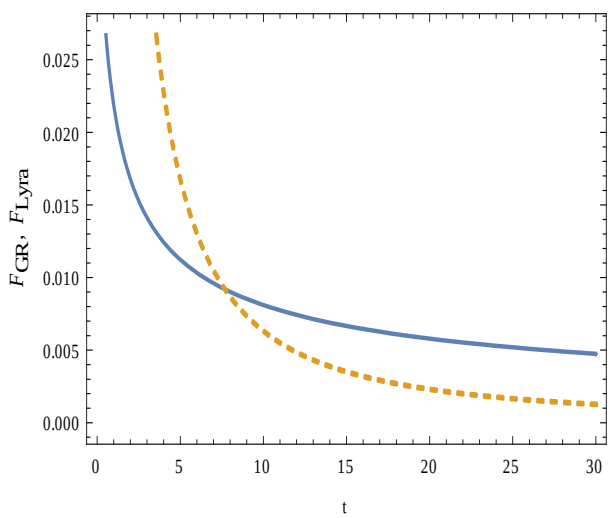

Figure 2. The absolute values of $\boldsymbol{F}_{GR}$ (Thick line) and $\boldsymbol{F}_{Lyra}$ (Dashed line) vs. time t, $0 < t < 30$.

For the interval $0 < t \leq 4$, F_{Lyra} has no value. For the interval $4 < t < 8$, the value of $\boldsymbol{F}_{Lyra} > \boldsymbol{F}_{GR}$. For $t > 8$, the value of $\boldsymbol{F}_{GR} > \boldsymbol{F}_{Lyra}$. We can say that. For the interval $4 < t < 30$, the value of $\boldsymbol{F}_{GR}$ and $\boldsymbol{F}_{Lyra}$ are closer to each other. Hence, we can conclude that the two theories (Lyra, General relativity) are complete with each other.

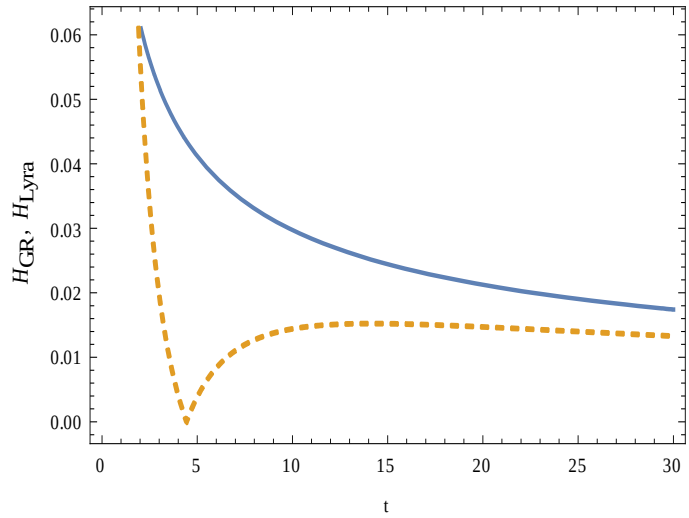

Figure 3. The absolute values of $\boldsymbol{H}_{GR}$ (Thick line) and $\boldsymbol{H}_{Lyra}$ (Dashed line) vs. time t, $0 < t < 30$.

At $t = 0$, $\boldsymbol{H}_{GR}$ has a large value. $\boldsymbol{H}_{GR}$ reduces to a small value at the end of evolution. At $t = 3$, from the beginning of evolution $\boldsymbol{H}_{GR} = \boldsymbol{H}_{Lyra}$. For the interval $3 < t < 4$, $\boldsymbol{H}_{Lyra}$ reduces to reach zero, and as $t > 4$, $\boldsymbol{H}_{Lyra}$ is increasing to reach nearly the same value as $\boldsymbol{H}_{GR}$. It is noticed that the Lyra geometry gave different behaviors for the enthalpy of general relativity compared with Lyra geometry.

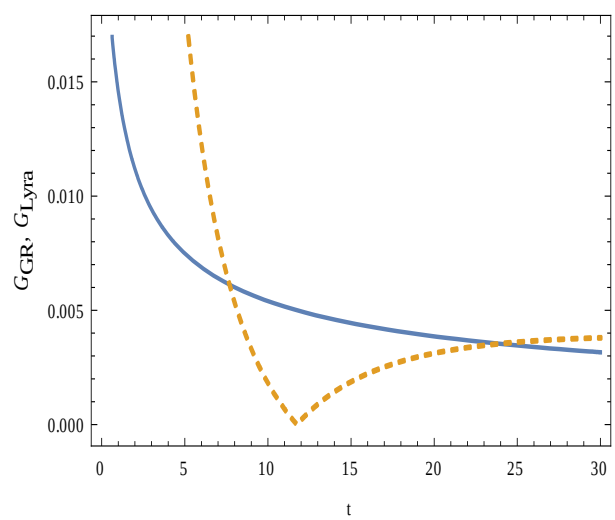

Figure 4. The absolute values of $\boldsymbol{G}_{GR}$ (Thick line) and $\boldsymbol{G}_{Lyra}$ (Dashed line) vs. time t, $0 < t < 30$.

$\boldsymbol{G}_{GR}$ has large value at the beginning of evolution and decreases as time elapsed. For the interval $0 \leq t \leq 5$, $\boldsymbol{G}_{Lyra} = zero$. For the interval $5 < t < 12$, $\boldsymbol{G}_{Lyra}$ decreases to reach zero and as $t > 12$, $\boldsymbol{G}_{Lyra}$ increases. We can say that Lyra's

geometry is not enough to explain the behavior of $\boldsymbol{G}$ from the beginning of evolution. So, we need the two theories to complete each other. For ($t < 6$, $\boldsymbol{G} = zero$) missing stage in Lyra geometry can be described from general relativity.

2.3. Bianchi Type VI_0 Cosmological Model in the Absence of Viscosity.

In the absence of the viscosity term $\xi = 0$, from (6) - (10) we obtain the same value for the metric coefficients $A(t), B(t)$, and $C(t)$ as given in equations (18) and (19). The pressure, density, and thermodynamics function of the universe will be different from the previous cases and read as:

Case 1: In Lyra geometry

$$p = \frac{k_1^{-2(1+L)} T^{-\frac{2(1+L)}{L}} \left(3(1-2L){N_1}^2 + 8(-1+L)m^2 k_1^2 T^{2/L}\right)}{4(-1+2L)\chi}, \tag{41}$$

$$\rho = \frac{k_1^{-2(1+L)} T^{-\frac{2(1+L)}{L}} \left(3(1-2L){N_1}^2 + 8(1+L)m^2 k_1^2 T^{2/L}\right)}{4(-1+2L)\chi}. \tag{42}$$

From (27), the change in the entropy with time becomes: $\frac{d\boldsymbol{S}_{Lyra}}{dt} = 0$, by integration we get $\boldsymbol{S}_{Lyra} = C_1$. The thermodynamic functions read as:

$$\boldsymbol{H}_{Lyra} = \frac{k_1^{-1-L} T^{-\frac{1+L}{L}} \left(3(1-2L){N_1}^2 + 8Lm^2 k_1^2 T^{2/L}\right)}{2(-1+2L)\chi}, \tag{43}$$

$$\boldsymbol{F}_{Lyra} = \frac{-\frac{4(1+L)C_1\sqrt{\frac{m^2 k_1^{-2L}}{-1+2L}}}{\pi} + \frac{k_1^{-1-L} T^{-1/L}\left(9(1-2L){N_1}^2 + 24(1+L)m^2 k_1^2 T^{2/L}\right)}{(-1+2L)\chi}}{12T}, \tag{44}$$

$$\boldsymbol{G}_{Lyra} = \frac{-\frac{2(1+L)C_1\sqrt{\frac{m^2 k_1^{-2L}}{-1+2L}}}{\pi} + \frac{k_1^{-1-L} T^{-1/L}\left(9(1-2L){N_1}^2 + 24Lm^2 k_1^2 T^{2/L}\right)}{(-1+2L)\chi}}{6T}. \tag{45}$$

Case 2: In general relativity

$$\rho_{GR} = \frac{2(1+L)m^2 k_1^{-2L}}{(-1+2L)\chi T^2}, \quad p_{GR} = \frac{2(-1+L)m^2 k_1^{-2L}}{(-1+2L)\chi T^2}. \tag{46}$$

The change in the entropy $\frac{d\boldsymbol{S}_{GR}}{dt} = 0$ that is $\boldsymbol{S}_{GR} = C_1$ (constant). The thermodynamics functions read as:

$$\boldsymbol{H}_{GR} = -\frac{4Lm^2 k_1^{1-L} T^{-1+\frac{1}{L}}}{\chi - 2L\chi}, \tag{47}$$

$$\boldsymbol{G}_{GR} = -\frac{(1+L)C_1\sqrt{\frac{m^2 k_1^{-2L}}{-1+2L}}}{3\pi T} - \frac{4Lm^2 k_1^{1-L} T^{-1+\frac{1}{L}}}{\chi - 2L\chi}, \tag{48}$$

$$\boldsymbol{F}_{GR} = \frac{(1+L)\left(-\frac{C_1\sqrt{\frac{m^2 k_1^{-2L}}{-1+2L}}}{\pi} - \frac{6m^2 k_1^{1-L} T^{\frac{1}{L}}}{\chi - 2L\chi}\right)}{3T}. \tag{49}$$

In the following, we make a comparative study between the results obtained in Lyra geometry and general relativity.

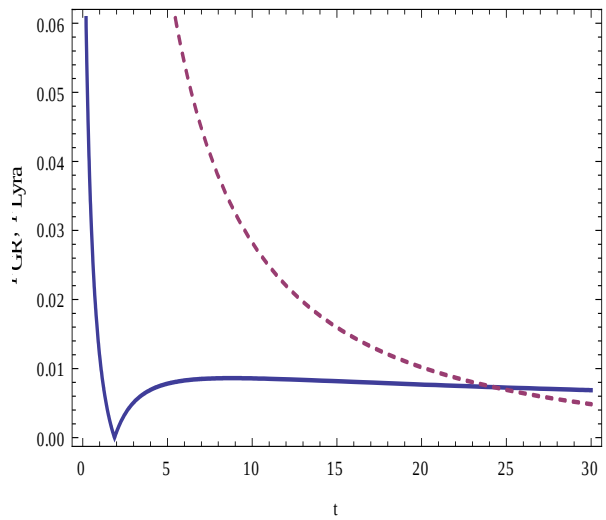

Figure.5 The absolute values of the Helmholtz $\boldsymbol{F}_{GR}$ (Thick line) and the Helmholtz $\boldsymbol{F}_{Lyra}$ (Dashed line) vs. time t, $0 < t < 30$.

For $t \leq 5$, $\boldsymbol{F}_{Lyra} = 0$ and as $t > 5$ $\boldsymbol{F}_{Lyra}$ decreases to reach a small value at the end of the present evolution. For $0 < t < 2$, $\boldsymbol{F}_{GR}$ is reduced to reach zero at $t = 2$ and increases again to reach nearly the same value of $\boldsymbol{F}_{Lyra}$ at the end of the present stage of evolution.

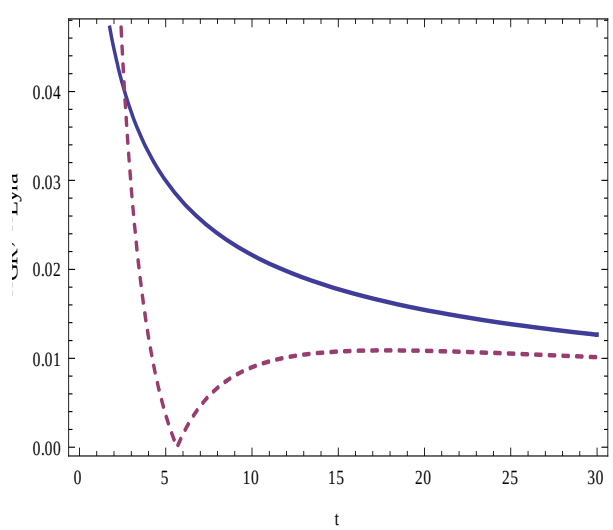

Figure.6 The absolute values of the Enthalpy $\boldsymbol{H}_{GR}$ (Thick line) and the Enthalpy $\boldsymbol{H}_{Lyra}$ (Dashed line) vs. time t, $0 < t < 30$.

At the beginning of evolution, the values of H_{GR} and H_{Lyra} are closer to each other. In the interval $0 < t < 6$, H_{Lyra} decreases to reach a small value, and as $t > 6$ they increase and reach nearly the same value of H_{GR} at the end of the evolution.

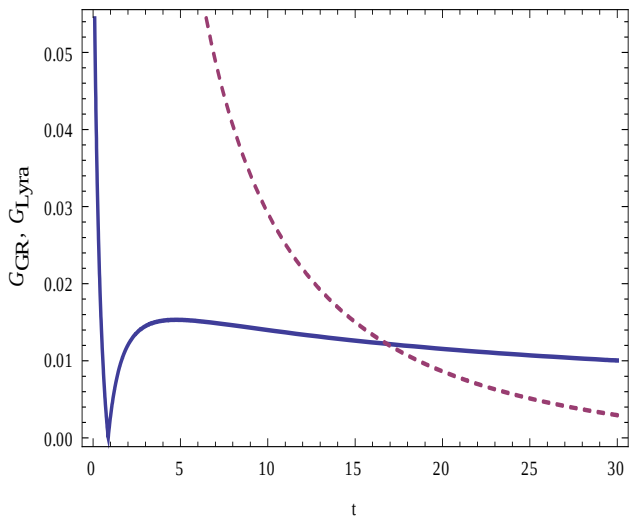

Figure. 7 The absolute values of the Gibbs function $\boldsymbol{G}_{GR}$ (Thick line) and the Gibbs function $\boldsymbol{G}_{Lyra}$ (Dashed line) vs. time t, $0 < t < 30$.

For the interval $t < 6$, $\boldsymbol{G}_{Lyra} = 0$. $\boldsymbol{G}_{GR}$ decreases to reach zero at $t = 1$ and increases again until the end of evolution. At $t = 17$ $\boldsymbol{G}_{Lyra} = \boldsymbol{G}_{GR}$ and for > 17 $\boldsymbol{G}_{Lyra} < \boldsymbol{G}_{GR}$.

3. Conclutions

In the present paper, we have studied Bianchi's type VI_0 model in Lyra geometry and the relativity theory. In this study, the Lyra term can not be defined as a viscosity term as in [36] since we obtain a nonintegrable equation for the change in the entropy (not a useful choice). The suitable description of the additional term introduced by Lyra is it plays the role of a variable cosmological term in general relativity. The additional term introduced by Lyra affects the behavior of the pressure and the density which causes a change in the thermodynamic functions of the universe. The model does not explain an accelerating universe as $q > 0$. If we tried to deal with an accelerating universe, we need $L < 0.5$ which makes the universe imaginary (not accepted). So, the presented model represents a stage of evolution in which $q > 0$. The pressure and the density have large values at the beginning of evolution, reduce to constant values as $t = t_0$ and reach small values at the end of the present evolution. The behavior of entropy in general relativity and Lyra geometry is consistence with the second law of thermodynamics.

In the absence of the viscosity term $\xi = 0$, the entropy S is obtained as a constant in Lyra geometry and in general relativity. But, It is known that the values of the pressure and density change with time which means that the entropy must change also with time which means S is not a constant and must be increased (the second law of thermodynamics). Hence, in the presence of the perfect fluid, the gravitational theory based on Lyra geometry and the general relativity theory is not suitable for explaining the entropy as an increasing function of t. An alternative gravitational theory that can give a good explanation for the entropy as an increasing function of t in the case of a perfect fluid is the second self-creation theory [12], [43], and [44]. Due to the additional term β introduced by Lyra, the behaviors of the thermodynamic functions obtained in Lyra geometry are differs from those obtained in the general relativity theory as indicated in figures (5), (6), and (7).

4. References

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