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EXACT SOLUTION OF ARBITRARILY LAMINATED COMPOSITE BEAMS USING A HIGHER-ORDER THEORY

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ABSTRACT

Analysis of arbitrarily laminated composite beams is presented based on a higher-order shear deformation theory. The governing equations are derived by minimizing the total potential energy of arbitrarily laminated beams undergoing axial and transverse shear strains under laterally distributed load. The exact solution of the governing equations is presented for hinged-hinged beam. The displacement and stresses of several laminated beams are calculated and compared with published results. The results of a parametric study showing the nature of axial and interlaminar shear for various ply-stacking patterns, beam aspect ratios and transverse shear are demonstrated.

KEYWORDS Higher-order theory. Composite beams. Exact solution.

NOMENCLATURE

E	Young's modulus.
E_{11}, E_{22}	Young's moduli in 1 and 2 directions respectively.
G_{12}, G_{13}, G_{23}	shear moduli in 1-2, 1-3 and 2-3 planes respectively.
L, b, h	Beam length, width and thickness respectively.
u, w	Axial and lateral beam displacements respectively.
q	Lateral distributed load per unit length
α	The angle between the fiber axis and the x axis.
θ	Beam rotation about y-axis
ν	Poisson's ratio.
ν_{12}	Poisson ratio for transverse strain in the 2-direction when stressed in the 1-direction.
σ_1, σ_2, τ_{12}	In-plane stresses in 1-2 coordinate.
σ_x, τ_{xz}	Axial and transverse shear stresses.

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INTRODUCTION

Beam structures are among the most important structures in aerospace applications. Multilayered composites have wide applications in aerospace industry due to their high strength-to-weight and stiffness-to-weight ratios. Conventional analysis of beams uses the classical beam theory based on Bernoulli-Euler hypothesis [1], and neglects shear deformation. This theory adequately describes the behavior of slender beams, but is less adequate for thick beams in which shear deformations are important.

Timoshenko [2] extended the classical theory to produce a first-order shear deformation theory. This is an improvement on the classical theory which reduces to it as the beam becomes thinner. A defect of Timoshenko theory is that the assumed displacement approximation violateas the "no-shear" boundary condition at the top and bottom of the beam. Levinson [3] introduced a higher-order theory to correct the drawback of Timoshenko theory. It is based on a cubic in-pane displacement approximation that satisfies the no-shear condition.

Bickford [4] noted that the derivation used by Levison was variationally inconsistent, and derived a corrected version from Hamilton's principle. In addition, he presented some representative solutions for simple beams.

Heyligher and Reddy [5] presented a finite element solution for Bickford's theory using polynomial shape functions. J. Petrolito [6] presented a finite element for isotropic beams based on a higher-order shear deformation theory. Solutions of the governing differential equations are derived and used as element shape functions.

For laminated beams, the classical lamination theory [7, 8, 9] is adequate to predict the global response of laminates with relatively small thickness. Because of the low shear modulus to in-plane stiffness ratio, the important role of transverse shear deformation, which is not contained in classical lamination theory, cannot be neglected. S. Gopalakrishnan et al [10] derived a refined 2-node, 4-DOF beam element based on a higher-order shear deformation theory in asymmetrically stacked laminates. V. G. Mokos and E. J. Saountzakis [11] developed a boundary element method for the solution of the general transverse shear loading of composite beams of arbitrary constant cross section. Exact solution for the bending of thin and thick cross-ply laminated beams was presented by Khedir and Reddy [12 and 13] using the state space concept.

In the present work analysis of arbitrarily laminated composite beams is presented based on a higher-order shear deformation theory. The governing equations are derived by minimizing the total potential energy of arbitrarily laminated beams undergoing axial and transverse shear strains under laterally distributed load. The exact solution of the governing equations is presented for hinged-hinged beam. The displacement and stresses of several laminated beams are calculated and compared with published results. A parametric study showing the nature of axial and interlaminar shear for various ply stacking, beam aspect ratios and transverse shear is discussed.

MATHEMATICAL FORMULATION

Kinematics Relations

Assuming that the beam is subjected to lateral load only as shown in Fig. (1), the deformation of the beam is described by two displacements u and w, and a rotation, θ. These displacements are assumed to be of the form [6]:

$$u = u(x,z) = z\theta - \frac{4}{3}\frac{z^3}{h^2}(\theta + \frac{\partial w}{\partial x})$$
$$\theta = \theta(x) \qquad (1)$$
$$w = w(x)$$

where h is the depth of the beam.

Strain-Displacement Relations

The beam is considered as a wide beam. So, The only non-zero strains are [6]

$$\varepsilon_x = \frac{\partial u}{\partial x} = z\frac{\partial \theta}{\partial x} - \frac{4}{3}\frac{z^3}{h^2}(\frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2})$$
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = (1 - 4\frac{z^2}{h^2})(\theta + \frac{\partial w}{\partial x}) \qquad (2)$$

Stress-Strain Relations

The laminate stresses are

$$\sigma_x = \overline{Q}_{11}\,\varepsilon_x$$
$$\tau_{xz} = \overline{Q}_{55}\,\gamma_{xz} \qquad (3)$$

where $\overline{Q}_{11}$ and $\overline{Q}_{55}$ are given in the Appendix.

GOVERNING EQUATIONS

Differential Equations

Minimizing the total potential energy of the beam can derive the governing equations for static analysis of the beam. In the present case, the total potential energy, Π, is

$$\Pi = \frac{1}{2}\int_{-h/2}^{h/2}\int_0^b\int_0^L(\sigma_x\,\varepsilon_x + \tau_{xz}\,\gamma_{xz})\,dx\,dy\,dz - \int_0^L q\,w\,dx \qquad (4)$$

where q is the applied transverse load per unit length of the beam, b is the width and L is the length of the beam. Taking into consideration that the variation in the potential energy is due to the variation in the displacement and strain, the first variation of the potential energy, $\delta\Pi$, can be written as:

$$\delta\Pi = \int_{-h/2}^{h/2}\int_0^b\int_0^L (\sigma_x\,\delta\varepsilon_x + \tau_{xz}\,\delta\gamma_{xz})\,dx\,dy\,dz - \int_0^L q\,\delta w\,dx \tag{5}$$

Substituting equations (1)-(3) into equation (5) and integrating over the width and depth of the beam equation (5) becomes

$$\begin{aligned}\delta\Pi = \int_0^L [EI_\theta \frac{\partial\theta}{\partial x}\frac{\partial\delta\theta}{\partial x} + EI_{\theta w}\left(\frac{\partial\theta}{\partial x}\frac{\partial^2\delta w}{\partial x^2} + \frac{\partial\delta\theta}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) + EI_w \frac{\partial^2 w}{\partial x^2}\frac{\partial^2\delta w}{\partial x^2} \\ + GA^*\left(\theta\delta\theta + \theta\frac{\partial\delta w}{\partial x} + \delta\theta\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}\right)]dx - \int_0^L q\,\delta w\,dx\end{aligned} \tag{6}$$

where $EI_\theta, EI_{\theta w}, EI_w$ and GA^* are the bending and shear stiffness of the laminated composite beam, and are defined in the Appendix. Integration by parts and equating to zero gives the equilibrium equations of arbitrarily laminated beam

$$\begin{aligned}&[EI_{\theta w}\theta' + EI_w w'']'' - [GA^*(\theta + w')]' - q = 0 \\ &[EI_\theta\theta' + EI_{\theta w} w'']' - GA^*(\theta + w') = 0\end{aligned} \tag{7}$$

where a prime denotes $\frac{d}{dx}$. The procedure also leads to the definition of the generalized forces used in expressing the beam boundary conditions

$$\begin{aligned}F_1 &= -[EI_{\theta w}\theta' + EI_w w'']' + GA^*(\theta + w') \\ F_2 &= EI_\theta\theta' + EI_{\theta w} w'' \\ F_3 &= EI_{\theta w}\theta' + EI_w w''\end{aligned} \tag{8}$$

The force F_1 can be interpreted as a generalized shear force, while F_2 and F_3 are generalized moments. With these definitions, the appropriate boundary conditions for the beam are as follows:

1) either w or F_1 is specified;
2) either θ or F_2 is specified;
3) either w' or F_3 is specified;

For most practical problems the properties of the beam are constant along the length of the beam. In this case, equations (7) and (8) reduce to

$$EI_w w'''' - GA^* w'' + EI_{\theta w}\theta''' - GA^*\theta' = q \tag{9a}$$

$$EI_{\theta w} w''' - GA^* w' + EI_\theta \theta'' - GA^*\theta = 0 \tag{9b}$$

and

$$F_1 = -EI_w w''' + GA^* w' - EI_{\theta w}\theta'' + GA^*\theta \tag{10a}$$

$$F_2 = EI_{\theta w} w'' + EI_\theta \theta' \tag{10b}$$

$$F_3 = EI_w w'' + EI_{\theta w}\theta' \tag{10c}$$

Therefore, the higher-order beam theory is represented by a system of ordinary differential equations of order six.

Boundary Conditions

Fixed end

$$w = 0; \qquad \theta = 0; \qquad w' = 0 \tag{11}$$

Hinged end

$$w = 0; \qquad F_2 = 0; \qquad F_3 = 0 \tag{12}$$

Free end

$$F_1 = 0; \qquad F_2 = 0; \qquad F_3 = 0 \tag{13}$$

GENERAL SOLUTION OF THE GOVERNING EQUATIONS

Uncoupled Differential Equations

To obtain the exact solution of equation (9) the uncoupled differential equations are derived. Differentiating equation (9b) and subtracting from (9a) lead to

$$(EI_{\theta w} - EI_\theta)\theta''' + (EI_w - EI_{\theta w})w^{iv} = q \tag{14}$$

Also from equation (9b):

$$\theta = \frac{EI_\theta}{GA^*}\theta'' + \frac{EI_{\theta w}}{GA^*}w''' - w' \tag{15}$$

Differentiating equation (15) and substituting for θ''' from equation (14), equation (15) becomes:

$$\theta' = \frac{EI^2_{\theta w} - EI_\theta EI_w}{GA^*(EI_{\theta w} - EI_\theta)} w^{iv} - w'' + \frac{EI_\theta}{GA^*(EI_{\theta w} - EI_\theta)} q \tag{16}$$

Substituting for θ' and θ''' from equation (16) in equation (9a), the resulting uncoupled equation for w is

$$w^{vi} - \lambda^2 w^{iv} = -\frac{EI_\theta}{EI^2_{\theta w} - EI_\theta EI_w} q'' - \frac{\lambda^2}{EI} q \tag{17a}$$

In a similar way the uncoupled equation for θ is

$$\theta^{v} - \lambda^2 \theta''' = \frac{EI_{\theta w}}{EI^2_{\theta w} - EI_\theta EI_w} q'' + \frac{\lambda^2}{EI} q \tag{17b}$$

where;

$$\lambda^2 = \frac{GA^*(2EI_{\theta w} - EI_\theta - EI_w)}{EI^2_{\theta w} - EI_\theta EI_w} \tag{18}$$

$$EI = EI_\theta + EI_w + -2EI_{\theta w} \tag{19}$$

General Solution

The general solution of differential equation (17) comprises homogeneous and particular solutions.

Homogeneous solution

The homogeneous solution of equation (17) [6] is

$$\begin{aligned} w_h &= C_1 \sinh(\lambda x) + C_2 \cosh(\lambda x) + C_3 x^3 + C_4 x^2 + C_5 x + C_6 \\ \theta_h &= C_7 \sinh(\lambda x) + C_8 \cosh(\lambda x) + C_9 x^2 + C_{10} x + C_{11} \end{aligned} \tag{20}$$

where $C_1, C_2, \ldots, C_{11}$ are unknown coefficients to be determined from the boundary conditions. Since the theory is of sixth-order, only six of these coefficients are independent and the remaining five are dependent. For convenience, $C_1, C_2, \ldots, C_6$ are chosen as the independent coefficients. To find the dependencies, equation (20) is substituted into equation (9b), giving the following relationships between the coefficients:

$$C_7 = R C_2 \quad ; \quad C_8 = R C_1$$
$$C_9 = -3C_3 \quad ; \quad C_{10} = -2C_4 \tag{21}$$
$$C_{11} = Q C_3 - C_5$$

where $R = \dfrac{R_2 \lambda^3 - R_1 \lambda}{R_1 - \lambda^2}$; $Q = \dfrac{6(R_2 - 1)}{R_1}$

$$R_2 = \frac{EI_{\theta w}}{EI_\theta} \quad ; \quad R_1 = \frac{GA^*}{EI_\theta}$$

Particular solutions

Particular solutions can be derived once the loading function is specified. Particular solutions are derived for three different loading cases.

Load case 1

Assuming the load $q(x)$ is given by

$$q(x) = \frac{x}{L} q_1 \tag{22}$$

where q_1 is the magnitude of the load at the beam end and L is the beam length. A particular solution of equation (22) is

$$w_1 = \frac{x^5}{120 EI\, L} q_1$$
$$\theta_1 = \left(-\frac{x^4}{24 EI\, L} + \frac{EI_{\theta w} - EI_\theta}{2GA^* EI\, L} x^2 + \frac{EI_\theta (EI_{\theta w} - EI_\theta)}{(GA^*)^2 EI\, L}\right) q_1 \tag{23}$$

It can be seen from equation (23) that the solution reduces to the classical beam theory with $\theta = -w'$ in the limit as GA^* tends to infinity.

Load case 2

Assuming the load $q(x)$ is given by

$$q(x) = \frac{x}{L} q_1 + (1 - \frac{x}{L}) q_2 \tag{24}$$

where q_1 and q_2 are the magnitude of the load at the beam ends. For this case the solution of equation (24) is

$$w_2 = [1 + \frac{q_2}{q_1} (\frac{5L}{x} - 1)] w_1$$
$$\theta_1 = (1 - \frac{q_2}{q_1}) \theta_1 + [-\frac{x^3}{6EI} + \frac{EI_{\theta w} - EI_\theta}{GA^* EI} x] q_2 \tag{25}$$

Load case 3

The solution in case of uniform distributed load q can be obtained from equation (25) by putting $q_1 = q_2 = q$

$$w_3 = \frac{x^4}{24EI} q \quad ; \quad \theta_3 = (-\frac{x^3}{6EI} + \frac{EI_{\theta w} - EI_\theta}{GA^* EI} x) q \tag{26}$$

General solution

The general solution is obtained for Hinged-Hinged beam under load case 3. Application of boundary conditions of the type (12) at the beam-ends (x=0 and x=L) gives the integration coefficients $C_1, C_2, ..., C_6$. It was found that for small aspect ratios (even for $L/h \geq 1$) the solution is unstable since sinh and cosh terms go to infinity as the aspect ratio increase. So, without loss of generality the coefficients C_1 and C_2 are set equal to zero. Setting w and F_2 equal to zero at x=0 and x=L the remaining four coefficients are

$$\begin{aligned} C_3 &= -\frac{L}{12EI} q \quad ; \quad C_4 = -\frac{EI_\theta}{2GA^* EI} q \\ C_5 &= \frac{L^3}{24EI} q + \frac{EI_\theta}{2GA^* EI} q \quad ; \quad C_6 = 0 \end{aligned} \tag{27}$$

The general solution is

$$\begin{aligned} w &= C_3 x^3 + C_4 x^2 + C_5 x + \frac{x^4}{24EI} q \\ \theta &= C_9 x^2 + C_{10} x + C_{11} - \frac{x^3}{6EI} q + \frac{(EI_{\theta w} - EI_\theta) x}{GA^* EI} q \end{aligned} \tag{28}$$

where the coefficients C_3, C_4 and C_6 are given by equation (27) and coefficients C_9, C_{10} and C_{11} are given by equation (21).

RESULTS AND DISCUSSIONS

The general solution (28) is applied to isotropic as well as arbitrarily laminated beams.

Problem 1

In this problem a simply supported isotropic beam of length L and depth h under uniform load is analyzed. The Poisson's ratio of the material is 0.3. Table (1) shows the results for the maximum displacement and normal stress for various values of the aspect ratio, L/h. The results are compared with those in Ref. [6]. In all cases the results have been normalized by dividing by those of the classical isotropic beam theory.

The results show that, even for small aspect ratio $L/h \leq 5$, ignoring sinh and cosh terms does not affect the solution. Also, it is clear from the results that the higher-order beam theory reduces to the classical isotropic beam theory as the beam becomes thinner. In Table 1 the difference in the displacement is attributed to the difference in stiffness, so use presented results with replacing E in beam theory with $E = \frac{E}{1-\nu^2}$ in plate theory.

Problem 2

In this problem a hinged-hinged laminated composite beam under uniform load as shown in Fig. 2 is solved. The material of the beam is carbon/epoxy with a $[0_4/45_4/-45_4]_s$ layup. The material properties are

E_{11}= 131 GPa; E_{22}= 131 GPa; G_{12}=6.55 GPa, ν_{12} =0.28
Since in the presented solution G_{13} and G_{23} are included, they are assumed equal to G_{12}. The data for the problem are summarized as follows:

Length L= 25.4 cm	Width b= 1.27 cm
Thickness h= 0.315 cm	Distributed load q = 380 N/m

The axial strain and in-plane stresses are calculated at mid span and compared with those calculated according to the classical lamination theory of Ref. [9], and are presented in Table 2. The results show good agreement because $L/h = 80$, which gives a thin beam.

Problem 3

In this problem the beam of Problem 2 is reconsidered with the following changes in width and load:

Width b= 5.08 cm	Distributed load q = 3500 N/m

The interlaminar shear stresses are calculated at the beam-ends and compared with those in Ref. [9] and presented in Table 3. The interlaminar shear stress in Ref. [9] is calculated from equilibrium of forces in x-direction, and the classical lamination theory is applied. The results show that the axial stresses are nearly coincident because the beam is sufficiently slender. The shear stress, however, has a discrepancy. The axial and interlaminar shear stress distributions through the beam thickness are shown in Figs. 3 and 4.

Parametric Study

A parametric study is conducted to understand the behavior of laminated beams under uniform distributed load. The material properties of Problems 2 and 3 are considered. To show the limits of the higher-order shear deformation theory in laminated beams with different ply stacking pattern, numerous aspect ratios are considered. Table 4 presents the central displacement normalized to the well-known central displacement of hinged-hinged beam ($5qL^4/384EI$). It can be seen from the Table that the Euler-Beroulli beam theory underestimates the deflection of laminated beams by a factor depending on the ply-stacking.

In order to bring out the effect of shear deformation in case of laminated beams, three different shear moduli are considered:

1) $G_{13}=G_{23}=G_{12}$ 2) $G_{13}=G_{23}=10^3\ G_{12}$ 3) $G_{13}=G_{23}=10^5\ G_{12}$

The interlaminar shear at the mid-plane of the beam-ends and the axial stress on the top surface at the mid-span are calculated for different ply-stacking, beam aspect ratios and transverse shear moduli. In all cases the length to width ratio is 10. The results are normalized to the classical lamination theory presented in Ref. [9], and are presented in Tables 5 and 6. It is clear from the results that the higher-order theory for laminated beam reduces to the classical lamination theory as the beam becomes thinner and the transverse shear moduli increase. Also, the values of the stresses as predicted by the classical lamination theory show a discrepancy with those calculated by the higher- order theory. The shear stress results are found not to be significantly affected by changes in the values of the shear moduli G_{13} and G_{23}.

CONCLUSION

A higher order shear deformation theory is proposed for the exact analysis of arbitrarily laminated composite beams. Several case studies are considered. Results show superiority of the proposed theory over the classical lamination theory, particularly for thick stubby beams.

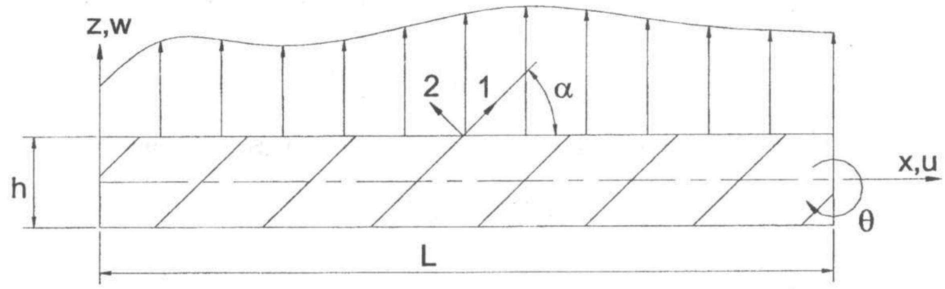

Fig. 1. Beam Geometry

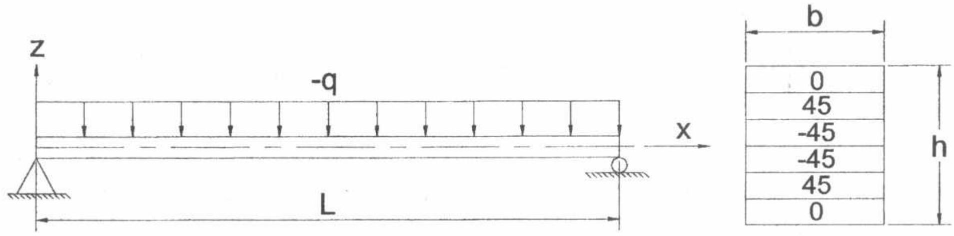

Fig. 2. Hinged-hinged laminated beam under distributed load

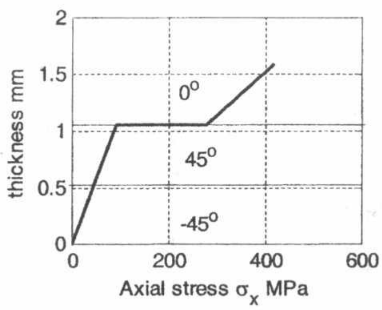

Fig. 3 Variation of axial stress through the beam thickness

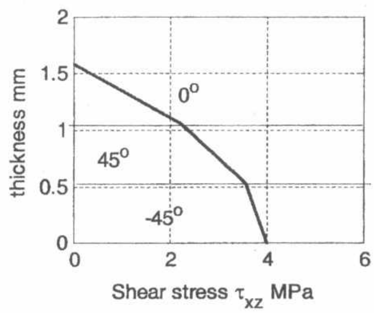

Fig. 4 Variation of shear stress through the beam thickness

Table 1. Normalized central displacement, w^*, and normalized central stress, σ^*_x of isotropic beam, $\nu = 0.3$

L/h	Displacement			Stress	
	Present	Present/(1-ν^2)	Ref. [6]	Ref. [6]	Present
1	3.436	3.776	3.434	1.408	1.347
2	1.541	1.694	1.620	1.102	1.087
3	1.191	1.309	1.277	1.045	1.039
4	1.068	1.174	1.156	1.016	1.022
5	1.011	1.111	1.100	1.004	1.014
10	0.935	1.027	1.025	1.001	1.003
25	0.914	1.004	1.004	1.000	1.001
50	0.911	1.001	1.001	1.000	1.000
100	0.910	1.000	1.000	1.000	1.000
1000	0.910	1.000	1.000	1.000	1.000

Table 2. Comparison of axial strain and in-plane stresses with those in Ref. [9]

Ply Group	z mm	ε_x %		σ_1 [MPa]		σ_2 [MPa]		τ_{12} [MPa]	
		Present	Ref. [9]	Present	Ref. [9]	Present	Ref. [9]	Present	Ref. [9]
0	1.575	0.138	0.138	181.9	181.5	4.356	4.353	0	0
45	1.050	0.092	0.092	62.05	62.07	6.631	6.634	-6.019	-6.021
-45	0.525	0.046	0.046	31.01	31.03	3.314	3.3178	3.008	3.011

Table 3. Comparison of axial and shear stresses with those in Ref. [9]

Ply Group	z mm	σ_x [MPa]		z mm	τ_{xz} [MPa]	
		Present	Ref. [9]		Present	Ref. [9]
0	1.575	418.84	418.73	1.050	2.236	2.885
45	1.050	92.94	92.96	0.525	3.557	3.461
-45	0.525	46.46	46.48	0.000	4.024	3.654

Table 4. Normalized central displacement of hinged-hinged composite beam

L/h Ply group	1	2	3	4	5	10	100	10^3
$[0_4/45_4/-45_4/45_4/-45_4/0_4]$	15.56	4.65	2.62	1.912	1.584	1.146	1.002	1.0
$[45_4/-45_4/0_4]_s$	8.23	2.81	1.80	1.452	1.289	1.072	1.001	1.0
$[15_4/30_4/0_4]_s$	16.72	4.93	2.75	1.983	1.629	1.157	1.002	1.0
$[15_4/30_4/45_4/-45_4/-30_4/-15_4]$	16.01	4.75	2.67	1.938	1.600	1.150	1.002	1.00
$[0_4/15_4/30_4]_s$	19.42	5.61	3.05	2.151	1.737	1.184	1.002	1.0
$[0_{12}/90_{12}]$	11.62	3.65	2.18	1.664	1.425	1.106	1.001	1.0
$[0_8/90_4]_s$	19.58	5.65	3.06	2.161	1.743	1.186	1.002	1.0
$[90_4/0_8]_s$	9.48	3.12	1.94	1.530	1.339	1.085	1.001	1.0

Table 5. Normalized shear stress τ_{xz} at the middle plane in different beams with different laminates

L/h	<10	100	10^3
$[0_4/45_4/-45_4/45_4/-45_4/0_4]$	1.101	1.101	1.101
$[45_4/-45_4/0_4]_s$	0.892	0.892	0.892
$[15_4/30_4/0_4]_s$	1.005	1.005	1.005
$[15_4/30_4/45_4/-45_4/-30_4/-15_4]$	1.065	1.065	1.065
$[0_4/15_4/30_4]_s$	1.033	1.033	1.033
$[0_{12}/90_{12}]$	0.543	0.543	0.543
$[0_8/90_4]_s$	1.067	1.067	1.067
$[90_4/0_8]_s$	0.801	0.801	0.801

Table 6. Normalized axial stress in different beams with different laminates and transverse shear moduli G_{13} and G_{23}

L/h	$[0_4/45_4/-45_4/45_4/-45_4/0_4]$			$[45_4/-45_4/0_4]_s$			$[15_4/30_4/45_4/-45_4/-30_4/-15_4]$			$[15_4/30_4/0_4]_s$		
	G_{12}	$10^3\ G_{12}$	$10^5\ G_{12}$	G_{12}	$10^3\ G_{12}$	$10^5\ G_{12}$	G_{12}	$10^3\ G_{12}$	$10^5\ G_{12}$	G_{12}	$10^3\ G_{12}$	$10^5\ G_{12}$
1	2.7611	1.0018	1.000	2.1471	1.0011	1.000	3.0145	1.0020	1.0000	3.2037	1.0022	1.0000
2	1.4403	1.0004	1.000	1.2868	1.0003	1.000	1.5036	1.0005	1.0000	1.5509	1.0006	1.0000
3	1.1957	1.0002	1.000	1.1275	1.0001	1.000	1.2238	1.0002	1.0000	1.2449	1.0002	1.0000
4	1.1101	1.0001	1.000	1.0717	1.0001	1.000	1.1259	1.0001	1.0000	1.1377	1.0001	1.0000
5	1.0704	1.0001	1.000	1.0459	1.000	1.000	1.0806	1.0001	1.0000	1.0881	1.0001	1.0000
10	1.0176	1.000	1.000	1.0115	1.000	1.000	1.0201	1.0000	1.0000	1.0220	1.0000	1.0000
100	1.0002	1.000	1.000	1.0001	1.000	1.000	1.0002	1.0000	1.0000	1.0002	1.0000	1.0000
10^3	1.000	1.000	1.000	1.000	1.000	1.000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 6. cont. Normalized axial stress in different beams with different laminates and transverse shear moduli G_{13} and G_{23}

L/h	$[0_4/15_4/30_4]_s$			$[0_{12}/90_{12}]$			$[0_8/90_4]_s$			$[90_4/0_8]_s$		
	G_{12}	$10^3\ G_{12}$	$10^5\ G_{12}$	G_{12}	$10^3\ G_{12}$	$10^5\ G_{12}$	G_{12}	$10^3\ G_{12}$	$10^5\ G_{12}$	G_{12}	$10^3\ G_{12}$	$10^5\ G_{12}$
1	3.6091	1.0026	1.000	2.5612	1.0016	1.000	3.6170	1.0026	1.000	2.7751	1.0018	1.000
2	1.6523	1.0007	1.000	1.3903	1.0004	1.000	1.6543	1.0007	1.000	1.4438	1.0004	1.000
3	1.2899	1.0003	1.000	1.1735	1.0002	1.000	1.2908	1.0003	1.000	1.1972	1.0002	1.000
4	1.1631	1.0002	1.000	1.0976	1.0001	1.000	1.1636	1.0002	1.000	1.1109	1.0001	1.000
5	1.1044	1.0001	1.000	1.0624	1.0001	1.000	1.1047	1.0001	1.000	1.0710	1.0001	1.000
10	1.0261	1.0000	1.000	1.0156	1.000	1.000	1.0262	1.000	1.000	1.0178	1.000	1.000
100	1.0003	1.0000	1.000	1.0002	1.000	1.000	1.0003	1.000	1.000	1.0002	1.000	1.000
10^3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

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APPENDIX

Bending and Torsion Stiffness of Laminated Beam
According to Higher-order theory

The stress-strain constants appearing in equation (3) are

$$\overline{Q}_{11} = C^4 Q_{11} + S^4 Q_{22} + 2S^2C^2 (Q_{12} + 2Q_{33})$$

$$\overline{Q}_{55} = C^2\ Q_{55} + S^2\ Q_{44}$$

$$Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}} \quad ; \quad Q_{12} = \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}} \quad ; \quad Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}}$$

$$Q_{33} = G_{12} \quad ; \quad Q_{44} = G_{23} \quad ; \quad Q_{55} = G_{13}$$

$$C = \cos\alpha \quad ; \quad S = \sin\alpha$$

α is the angle between the fiber axis and the x axis.

The bending stiffness appearing in equation (6) are

$$EI_\theta = EI_e + 2EI_{s1} + EI_{s2}$$
$$EI_{\theta w} = EI_{s1} + EI_{s2} \qquad \text{(A1)}$$
$$EI_w = EI_{s2}$$

where

$$(EI_e, EI_{s1}, EI_{s2}) = b\int_{-h/2}^{h/2} \overline{Q}_{11}(z^2, z^4, z^6)dz \qquad \text{(A2)}$$

$$EI_e = b\sum_{k=1}^{N} \overline{Q}_{11}{}^k[(\overline{Z}_k)^2 t_k + \frac{(t_k)^3}{12}]$$
$$EI_{s1} = -\frac{4b}{15h^2}\sum_{k=1}^{N} \overline{Q}_{11}{}^k[5(\overline{Z}_k)^4 t_k + 2.5(\overline{Z}_k)^2(t_k)^3 + \frac{(t_k)^5}{16}] \qquad \text{(A3)}$$
$$EI_{s2} = \frac{16b}{63h^4}\sum_{k=1}^{N} \overline{Q}_{11}{}^k[7(\overline{Z}_k)^6 t_k + \frac{35}{4}(\overline{Z}_k)^4(t_k)^3 + \frac{21}{16}(\overline{Z}_k)^2(t_k)^5 + \frac{(t_k)^7}{64}]$$

The shear stiffness GA^* appearing in equation (6) is

$$GA^* = GA_1 + GA_2 + GA_3 \qquad \text{(A4)}$$

$$(GA_1, GA_2, GA_3) = b\int_{-h/2}^{h/2} \overline{Q}_{55}(1, z^2, z^4)dz \qquad \text{(A5)}$$

$$GA_1 = b\sum_{k=1}^{N} \overline{Q}_{55}{}^k t_k$$
$$GA_2 = -\frac{8b}{h^2}\sum_{k=1}^{N} \overline{Q}_{55}{}^k[\sum_{k=1}^{N} \overline{Q}_{11}{}^k[(\overline{Z}_k)^2 t_k + \frac{(t_k)^3}{12}] \qquad \text{(A6)}$$
$$GA_3 = \frac{16b}{5h^4}\sum_{k=1}^{N} \overline{Q}_{55}{}^k[\sum_{k=1}^{N} \overline{Q}_{11}{}^k[5(\overline{Z}_k)^4 t_k + 2.5(\overline{Z}_k)^2(t_k)^3 + \frac{(t_k)^5}{16}]$$