Constraint Mixture Multi-Item Single-Source Continuous Review Inventory Model with Varying Holding Cost, Crisp and Fuzzy Units

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Abstract:

This paper is going to present two different methods of solving the multi item single source (MISS) problem considering two different sets of assumptions and treatments. So the paper consists of two parts, the first part is going to derive the probabilistic continuous review inventory, with mixture shortage, when the holding cost is a function of the ordered quantity. The objective is to minimize the expected total cost using Lagrange multipliers. The mathematical optimal solutions for the order quantity, Q^*, and the reorder point, r^*, can be obtained when the lead time is a constant and the demand is a random variable (when demand follows Normal, Exponential and Chi-Square distributions). The second part is devoted to study the same model when the demand, the order and the holding unit costs are triangular fuzzy numbers. An actual application is added to illustrate the models.

Keywords: Continuous Review, Inventory, Mixture, Multi-Item Single-Source, Fuzzy.

1- Introduction

The occurrence of shortage in an inventory system is a phenomenon in real situations. Generally the demand during stockout period is either regarded as completely backordered or lost forever. In some cases while a few customers are ready to wait till the next arrival of stock (backorder case), the remaining may be impatient and would persist on satisfying their demand immediately from some other sources (lost sales). So, in this study it is assumed that, the backorder fraction, γ, where $0< \gamma <1$, is dependent of time. Inventory models which involve both backorders and lost sales are known as models with a mixture shortage.

Hadley and Whitin (1961) and (1963) discussed probabilistic continuous review inventory models with constant units of cost and the lead time demand is a random variable. Tersin (1994) examined unconstrained inventory model with constant units of cost and demand follows normal distribution. Hariri and Abou-EI-Ata (1997) considered a multi-item inventory model with varying order cost under one restriction. Abou-EI-Ata et al (2003) studied a probabilistic multi-item inventory model with varying order cost and zero lead-time under two restrictions. Fergany and El-Wakeel (2006) treated a constrained continuous review lost sales inventory system with varying order cost and lead time demand distributed normally. Yao and Wu (2000) presented the signed distance method of defuzzification, after this, Yao and Chiang (2003) compared the defuzzification of triangular fuzzy numbers using centroid method and signed distance method. They established that, defuzzifying of a fuzzy number using the signed distance is better than the centroid and more sensible. Panda and Kar (2005) developed multi item stochastic purchase EOQ models and fuzzy random environments considering demand to be dependent on the unit price cost which is a decision variable, lead time is zero and there is no shortage. Chiang et al (2005) considered fuzzy inventory with backorder. They used the signed distance method to defuzzify the fuzzy total cost and obtain an estimate of the total cost in fuzzy sense. Also, Vijayan and Kumaran (2008) considered continuous review and periodic review mixture inventory models under fuzzy environment using the signed distance method.

This article presents a probabilistic multi-item single-source (MISS) continuous review inventory system (in two parts in this study). The first part to investigate a probabilistic MISS continuous review inventory model with mixture shortage, varying holding cost, under the expected shortage cost restriction when the demand is random variable and the lead time is constant. The optimal order quantity, Q^*, the optimal reorder point, r^*, and the minimum expected total cost, $E(TC(Q, r))$, are obtained mathematically. The second part devoted to study the same above model when the demand, the order unit cost and the holding unit cost are triangular fuzzy numbers. Then, it will be defuzzified using the signed distance. Also, a numerical example is added with the results graphs for each part. An application for the two models to compare them results numerically and graphically.

Glossary of Notations

Here some of the essential notations are adopted for developing the study:

MISS	The multi-item single-source,
i	The number of items, and i=1,2,...,m,
$E(TC(Q, r))$	The expected total cost function of the MISS inventory model,
$\overline{D_i}$	The average of demand rate,
Q_i^*	The optimal order quantity,
r_i^*	The optimal reorder point,
L_i	The average value of the lead time,
x_i	The random variable represents the lead time demand,
$f(x_i)$	The probability density function of the lead time demand,
$r_i - x_i$	The random variable represents the net inventory when the procurement quantity arrives if the lead- time demand $x_i < r_i$,
$R(r)_i$	The probability of shortage or the reliability function and, $R(r)_i = p(x_i > r_i) = \int_{r_i}^{\infty} f(x_i)\, dx_i = 1 - F(x_i)$,
$\overline{S}(r)_i$	The expected shortage quantity per cycle and, $\overline{S}(r)_i = E(x_i - r_i) = \int_{r_i}^{\infty} (x_i - r_i) f(x_i)\, dx_i$,
c_{oi}	The order cost of the unit per cycle,
c_{hi}	The holding cost of the unit per cycle,
c_{bi}	The backorder cost of the unit per cycle,
c_{li}	The lost sales cost of the unit per cycle,
γ_i	The backorder fraction as such dependent of time, $0< \gamma_i <1$,
K_{si}	The limitation on the expected shortage cost,
λ_{si}	The Lagrange multiplier of the shortage cost constraint,
$L(Q_i, r_i, \lambda_{si})$	Lagrange equation, which denoted by G1,
β	A constant real number selected to provide the best fit of estimated expected cost function, and here is $0< \beta <1$,
$\widetilde{D_i}$	The fuzzy set of demand rate,

$\tilde{c}_{oi}$	The fuzzy set of order unit cost,
$\tilde{c}_{hi}$	The fuzzy set of holding unit cost,
$M_{\tilde{D}_i}(x_i)$	The membership function of fuzzy $\overline{D_i}$,
$M_{\tilde{c}_{o_i}}(x_i)$	The membership function of fuzzy c_{oi},
$M_{\tilde{c}_{h_i}}(x_i)$	The membership function of fuzzy c_{hi}.

2- The Probabilistic Model

The system is a continuous review, which means that the demands are recorded as they occur and the stock level is know at all times. An order quantity of size Q per cycle is placed every time the stock level reaches a certain reorder point r (where Q and r are two independent decision variables). This section introduces a probabilistic inventory system in which there are the following assumptions are considered:

(i) Under continuous review,
(ii) With mixture shortage cost,
(iii) Has varying holding cost,
(iv) Subject to shortage restriction,
(v) The lead time is a constant and the demand is a random variable.

2-1 **Formulation and analysis**

A continuous review, $<Q, r>$, model with varying holding cost when the demand (D_i) is continuous random variable, the lead time (L_i) is constant and the distribution of the lead time demand (x_i) depends on the distribution of the demand will be studied.

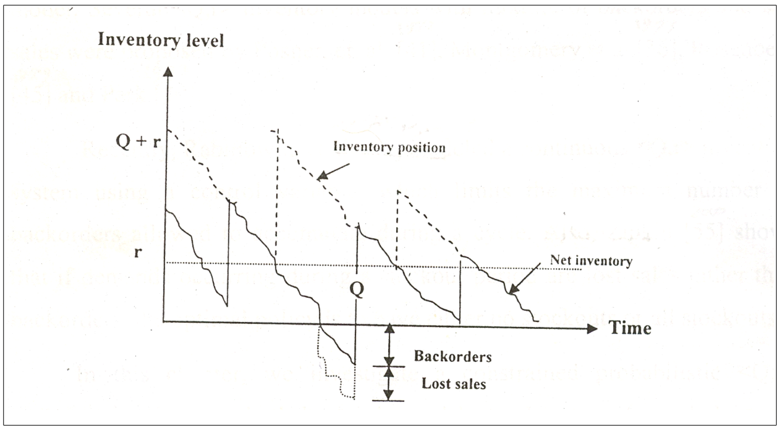

Figure 1: A continuous review system with mixture shortage.

It is possible to develop the expected total cost as the sum of the expected order cost, $E(OC)_i$, the expected holding cost, $E(HC)_i$, and the expected shortage cost, $E(SC)_i = E(BC)_i + E(LC)_i$, as follows:

$$E1 = \sum_i E(TC(Q_i, r_i)) = \sum_i \{E(OC)_i + E(HC)_i + E(SC)_i\}$$

where $i=1,2,3,\ldots,m$ representing i^{th} item,

$$E(OC)_i = c_{oi}\frac{\bar{D}_i}{Q_i}.$$

And

$$E(HC)_i = c_{hi}Q_i^{\beta}\left(\frac{Q_i}{2} + r_i - E(X_i) + (1-\gamma_i)\bar{S}(r_i)\right),$$

where, $c_h(Q)$ is the varying holding unit cost per cycle, it is a function of ordered quantity, and defined as: $c_h(Q) = c_{hi}Q_i^{\beta}$ for each item, and β is a constant real number selected to provide the best fit of estimated expected cost function. $E(BC)_i$ is the expected backorder cost per cycle and given as:

$$E(BC)_i = c_{bi}\,\gamma_i\frac{\bar{D}_i}{Q_i}\int_{r_i}^{\infty}(x_i - r_i)f(x_i)\,dx_i = c_{bi}\,\gamma_i\frac{\bar{D}_i}{Q_i}\bar{S}(r_i)$$

$E(LC)_i$ is the expected lost sales cost per cycle and given as:

$$E(LC)_i = c_{li}(1-\gamma_i)\frac{\bar{D}_i}{Q_i}\int_{r_i}^{\infty}(x_i - r_i)f(x_i)\,dx_i = c_{li}(1-\gamma_i)\frac{\bar{D}_i}{Q_i}\bar{S}(r_i)$$

Then, the expected total cost, E(TC(Q, r)), is given as:

$$E(TC(Q,r)) = \sum_{i=1}^{m}\left[\begin{array}{c} c_{oi}\frac{\bar{D}_i}{Q_i} + c_{hi}Q_i^{\beta}\left(\frac{Q_i}{2} + r_i - E(X_i) + (1-\gamma_i)\bar{S}(r_i)\right) \\ + c_{bi}\frac{\gamma_i\bar{D}_i}{Q_i}\bar{S}(r_i) + c_{li}\frac{\bar{D}_i}{Q_i}(1-\gamma_i)\bar{S}(r_i)\end{array}\right] \quad (1)$$

Subject to the following expected shortage constraint:

$$\sum_{i=1}^{m} \frac{\bar{D}_i}{Q_i}\left(\beta c_{bi} + (1-\beta) c_{li}\right)\bar{S}(r_i) \le K_{si}$$

The main objective is to minimize the expected total cost *E(TC(Q, r))*, which is a convex programming problem. Then to solve this primal function, under the above constraint, the Lagrange multipliers technique should be used as follows:

$$G1 = \sum_{i=1}^{m} L(Q_i, r_i, \lambda_{si}) = \sum_{i=1}^{m} \left[\begin{array}{c} c_{oi}\frac{\bar{D}_i}{Q_i} + c_{hi}Q_i^{\gamma}\left(\frac{Q_i}{2} + r_i - E(x_i)\right) + c_{bi}\frac{\beta \bar{D}_i}{Q_i}\bar{S}(r_i) \\ + \left(c_{hi}Q_i^{\gamma} + c_{li}\frac{\bar{D}_i}{Q_i}\right)(1-\beta)\bar{S}(r_i) \\ + \lambda_{si}\left[\frac{\bar{D}_i}{Q_i}\left(\beta c_{bi} + (1-\beta)c_{li}\right)\bar{S}(r_i) - K_{si}\right] \end{array} \right] \quad (2)$$

The optimal values of the order quantity (Q_i^*) and reorder quantity (r_i^*), which are minimizing the expected total cost, can be calculated by setting each of the corresponding first partial derivatives of Equation (1) equal to zero, then the following is obtained:

$$\frac{\partial G1}{\partial Q_i} = -\frac{c_{oi}\bar{D}_i}{Q_i^2} + \frac{c_{hi}(\gamma+1)}{2}Q_i^{\gamma} + c_{hi}\gamma\left[r_i - E(x_i) + (1-\beta)\bar{S}(r_i)\right]Q_i^{\gamma-1}$$
$$-\frac{(1+\lambda_{si})}{Q_i^2}\left[\beta c_{bi}\bar{D}_i + (1-\beta)c_{li}\bar{D}_i\right]\bar{S}(r_i) = 0$$

And,

$$\frac{\partial G1}{\partial r_i} = c_{hi}Q_i^{\gamma} + \frac{\beta c_{bi}\bar{D}_i}{Q_i}\left(-R(r_i)\right) + \left(\frac{c_{li}\bar{D}_i}{Q_i} + c_{hi}Q_i^{\gamma}\right)(1-\beta)\left(-R(r_i)\right)$$
$$+\frac{\lambda_{si}}{Q_i}\left(\beta c_{bi}\bar{D}_i + (1-\beta)c_{li}\bar{D}_i\right)\left(-R(r_i)\right) = 0$$

where,

$$\frac{\partial \bar{S}(r_i)}{\partial r_i} = \frac{\partial}{\partial r_i}\int_{r_i}^{\infty}(x_i - r_i)f(x_i)dx_i = -\int_{r_i}^{\infty} f(x_i)dx_i = -\left[1 - \int_{0}^{r_i} f(x_i)dx_i\right]$$

$$\frac{\partial \bar{S}(r_i)}{\partial r_i} = -\left[1 - F(x_i)\right] = -R(r_i)$$

[see Hogg and Craig (1978)], then it is found that:

$$c_{hi}(\gamma+1)Q_i^{\gamma+2} + 2c_{hi}\gamma\left[r_i - E(x_i) + (1-\beta)\bar{S}(r_i)\right]Q_i^{\gamma+1}$$
$$-2\left[A + BW\bar{S}(r_i)\right] = 0 \quad (3)$$

and,

$$R(r_i^*) = \frac{c_{hi}\, Q_i^{*}}{BW + (1-\beta_i)\, c_{hi}\, Q_i^{*}}\,, \tag{4}$$

where,

$$A = c_{oi}\,\overline{D}_i,\; B = (1+\lambda_{si}),\; W = \beta_i c_{bi}\overline{D}_i + (1-\beta_i)\, c_{li}\, \overline{D}_i . \tag{5}$$

It is clearly, there is no closed form solution of Equations (3) and (4) to obtain Q_i^* and r_i^*, then an iterative method must be used. It is clear that the constraint must always be active.

2-2 Special cases

Two special cases of this model are deduced as follows:

●Case 1: For the Equations (3) and (4), let
$\beta = 0$, $\gamma = 0 \Rightarrow c_h(Q) = c_{hi}$, $\lambda_{si} = 0$ and $K_{si} \to \infty$ then, the optimal order quantity and reorder point can be obtained by:

$$Q_i^* = \sqrt{\frac{2\,\overline{D}_i\,(c_{oi} + c_{li}\,\overline{S}(r^*))}{c_{hi}}}\,, \quad R(r^*) = \frac{c_{hi}\, Q_i^*}{c_{li}\,\overline{D}_i + c_{hi}\, Q_i^*} \tag{6}$$

This is unconstrained lost sales continuous review inventory model with constant units of cost and lost sales for the i^{th} item, see Hadley and Whitin (1963).

●Case 2: For the Equations (3) and (4), let
$\beta = 1$, $\gamma = 0 \Rightarrow c_h(Q) = c_{hi}$, $\lambda_{si} = 0$ and $K_{si} \to \infty$ then, the optimal order quantity and reorder point can be obtained by:

$$Q_i^* = \sqrt{\frac{2\,\overline{D}_i\,(c_{oi} + c_{bi}\,\overline{S}(r^*))}{c_{hi}}}\,, \quad R(r^*) = \frac{c_{hi}}{c_{bi}\,\overline{D}_i}\, Q_i^* \tag{7}$$

This is unconstrained backorders continuous review inventory model with constant unit costs and back order, see Hadley and Whitin (1963).

Now, the distribution of lead time demand x when the lead time L is constant and the demand D is random variable is studied.

$$x = \sum_{\ell=1}^{L} D_\ell$$

where, $\ell=1,...,L$, and L is not a random variable, but is a constant number of periods. Now, it is possible to obtain the distribution of lead time demand, $f(x)$, in a direct manner by the characteristic functions of D and x, [See Fabrycky and Banks (1967)], which are related as:

$$\phi_x(t) = \prod_{\ell=1}^{L} \phi_D(t) = [\phi_D(t)]^L$$

Hence, the corresponding distribution of the lead time demand x can be deduced when the demand is as follows: The normal distribution, or the exponential distribution, or the chi-square distribution, or unknown distribution.

(I) The demand distributed normally

If the demand, D_i for item i, distributed as normal distribution with mean μ_i, variance σ_i^2, and then, the characteristic function of D_i is given by:

$$\phi_{D_i}(t) = e^{-\frac{1}{2} t^2 \sigma_i^2 + \sqrt{-1}\, t\, \mu_i}$$

thus the characteristic function of x_i is given as:

$$\phi_{x_i}(t) = \left[\phi_{D_i}(t)\right]^{L_i} = e^{-\frac{1}{2} t^2 L_i \sigma_i^2 + \sqrt{-1}\, t\, L_i\, \mu_i}$$

This means that, the lead time demand x_i follows normal distribution with mean $L_i\, \mu_i$ and variance $L_i\, \sigma_i^2$. The reliability function and the expected shortage can be obtained as follows:

$$R(r_i) = \int_{r_i}^{\infty} f(x_i)\, dx_i = \frac{1}{\sigma_i \sqrt{2\pi L_i}} \int_{r_i}^{\infty} e^{-\frac{1}{2}\left(\frac{x_i - L_i \mu_i}{\sigma_i \sqrt{L_i}}\right)^2} dx_i .$$

Thus,

$$R(r_i) = \left[1 - \Phi\left(\frac{r_i - L_i \mu_i}{\sigma_i \sqrt{L_i}}\right)\right]. \qquad (8)$$

Also:

$$\bar{S}(r_i) = \int_{r_i}^{\infty} (x_i - r_i) f(x_i)\, dx_i ,$$

then,

$$\bar{S}(r_i) = \sigma_i \sqrt{L_i}\; \varphi\left(\frac{r_i - L_i \mu_i}{\sigma_i \sqrt{L_i}}\right) + (L_i \mu_i - r_i)\left[1 - \Phi\left(\frac{r_i - L_i \mu_i}{\sigma_i \sqrt{L_i}}\right)\right], \qquad (9)$$

where

$$\varphi\left(\frac{r_i - L_i \mu_i}{\sigma_i \sqrt{L_i}}\right) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2L_i}\left(\frac{r_i - L_i \mu_i}{\sigma_i \sqrt{L_i}}\right)^2} ,$$

and

$\Phi\left(\frac{r_i - L_i \mu_i}{\sigma_i \sqrt{L_i}}\right)$ its cumulative function.

The expected total cost can be minimized mathematically by substituting from Equations (9) and (8) into (3) and (4) respectively, for any i^{th} item, it is found:

$$c_{hi}(\gamma-1)Q_i^{\gamma-2} - 2c_{hi}\left[r_i - L_i\mu_i + (1-\gamma)\left[\begin{array}{c}\sigma\sqrt{L_i}\,\phi\left(\frac{r_i-L_i\mu_i}{\sigma\sqrt{L_i}}\right) \\ -(L_i\mu_i - r_i)R(r_i)\end{array}\right]\right]Q_i^{\gamma-3}$$

$$-2\left[A + B\,W\left[\sigma\sqrt{L_i}\,\phi\left(\frac{r_i-L_i\mu_i}{\sigma\sqrt{L_i}}\right) - (L_i\mu_i - r_i)R(r_i)\right]\right] = 0 \quad , \qquad (10)$$

and

$$R(r_i) = \frac{c_{hi}Q_i^{\gamma-1}}{BW + (1-\gamma)c_{hi}Q_i^{\gamma-1}} = 1 - \Phi\left(\frac{r_i-L_i\mu_i}{\sigma\sqrt{L_i}}\right) \quad , \qquad (11)$$

where, A, B and W are defined in (5).

(II) The demand distributed exponentially

If the demand D_i follows exponential distribution with parameter θ_i, and then, the characteristic function of D_i is given by:

$$\phi_{D_i}(t) = \left[1 - \frac{i\,t}{\theta_i}\right]^{-1}$$

Then the characteristic function of x_i will be given as:

$$\phi_{x_i}(t) = \left[\phi_{D_i}(t)\right]^{L_i} = \left[1 - \frac{i\,t}{\theta_i}\right]^{-L_i}$$

This means that, the lead time demand x_i follows Gamma distribution with parameters θ_i and L_i. The reliability function and the expected shortage quantity can be obtained as follows:

$$R(r_i) = \int_{r_i}^{\infty} f(x_i)\,dx_i = \frac{\theta_i^{L_i}}{\Gamma(L_i)}\int_{r_i}^{\infty} x_i^{L_i-1}\,e^{-\theta_i x_i}\,dx_i = \sum_{j=0}^{L_i-1}\frac{(\theta_i r_i)^j}{j!}e^{-\theta_i r_i} \quad . \qquad (12)$$

Also:

$$\bar{S}(r_i) = \int_{r_i}^{\infty}(x_i - r_i)f(x_i)\,dx_i = \frac{L_i}{\theta_i}\sum_{j=0}^{L_i}\frac{(\theta_i r_i)^j}{j!}e^{-\theta_i r_i} - r_i\,R(r_i) \quad . \qquad (13)$$

So the expected total cost can be minimized mathematically by substituting from Equations (13) and (12) into (3) and (4), respectively, it is found:

$$c_{hi}(\gamma-1)Q_i^{\gamma-2} - 2c_{hi}\left[r_i - \frac{L_i}{\theta_i} + (1-\gamma)\left(\begin{array}{c}\frac{L_i}{\theta_i}\sum_{j=0}^{L_i}\frac{(\theta_i r_i)^j}{j!}e^{-\theta_i r_i} \\ - r_i\,R(r_i)\end{array}\right)\right]Q_i^{\gamma-3}$$

$$-2\left[A + B\,W\left(\frac{L_i}{\theta_i}\sum_{j=0}^{L_i}\frac{(\theta_i r_i)^j}{j!}e^{-\theta_i r_i} - r_i\,R(r_i)\right)\right] = 0 \qquad (14)$$

and,

$$R(r_i^*) = \frac{c_{hi}Q_i^*}{BW + (1-\theta_i)c_{hi}Q_i^*} = \sum_{k=0}^{L_i\omega_i - 1} \frac{(\lambda_i r_i)^k}{k!} e^{-\lambda_i r_i} \quad , \qquad (15)$$

where, A, B and W are defined in (5).

(III) The demand follows Chi-square distribution

If the demand D_i follows the Chi-square distribution with parameter η_i, and then the characteristic function of D_i is given by:

$$\phi_{D_i}(t) = [1 - 2\,it]^{-\frac{\eta_i}{2}} .$$

Thus, the characteristic function of x_i will be given as:

$$\phi_{x_i}(t) = [\phi_{D_i}(t)]^{L_i} = [1 - 2\,it]^{-\frac{\eta_i L_i}{2}}$$

This means that, the lead time demand x_i follows Chi-square distribution with parameter $(L_i\,\eta_i)$. The reliability function and the expected shortage quantity can be obtained as follows:

$$R(r_i) = \int_{r_i}^{\infty} f(x_i)\,dx_i = \frac{1}{2^{\frac{L_i\eta_i}{2}}\Gamma\left(\frac{L_i\eta_i}{2}\right)} \int_{r_i}^{\infty} x_i^{\frac{L_i\eta_i}{2}-1} e^{-\frac{x_i}{2}}\,dx_i = \sum_{k=0}^{\frac{L_i\eta_i}{2}-1} \frac{\left(\frac{r_i}{2}\right)^k}{k!} e^{-\frac{r_i}{2}} \qquad (16)$$

and

$$\bar{S}(r_i) = \int_{r_i}^{\infty} (x_i - r_i) f(x_i)\,dx_i = L_i\eta_i \sum_{k=0}^{\frac{L_i\eta_i}{2}} \frac{\left(\frac{r_i}{2}\right)^k}{k!} e^{-\frac{r_i}{2}} - r_i\, R(r_i) \qquad (17)$$

The expected total cost can be minimized mathematically by substituting from Equations (17) and (16) into (3) and (4), respectively, that gives:

$$c_{hi}(\theta_i+1)Q_i^{*2} + 2c_{hi}\theta_i\left[r_i - L_i\eta_i + (1-\theta_i)\left(L_i\eta_i \sum_{k=0}^{\frac{L_i\eta_i}{2}} \frac{\left(\frac{r_i}{2}\right)^k}{k!} e^{-\frac{r_i}{2}} - r_i\, R(r_i)\right)\right]Q_i^*$$
$$- 2\left[A + BW\left(L_i\eta_i \sum_{k=0}^{\frac{L_i\eta_i}{2}} \frac{\left(\frac{r_i}{2}\right)^k}{k!} e^{-\frac{r_i}{2}} - r_i\, R(r_i)\right)\right] = 0 \qquad (18)$$

and,

$$R(r_i^*) = \frac{c_{hi}Q_i^*}{BW + (1-\theta_i)c_{hi}Q_i^*} = \sum_{k=0}^{\frac{L_i\eta_i}{2}-1} \frac{\left(\frac{r_i}{2}\right)^k}{k!} e^{-\frac{r_i}{2}} \qquad (19)$$

Where, *A, B* and *W* are defined in (5).

(III) The demand distribution is Un-known

When the demand D_i has Un-known distribution, but its parameters are known [its mean μ_i and variance σ_i^2], then the lead time demand x_i will be distributed as normal distribution with parameters mean $L_i\ \mu_i$ and variance $L_i\ \sigma_i^2$ [see Fabrycky and Banks (1967)]. In the following section, an actual data example will be discussed.

2-3 An Illustrative Application

This example was taken from actual real world inventory system. A private industry produces three items of clothes, which are seasonally in demand, to EGYPT AIR and other buyers. An observation of the behavior of demands and order quantities from May-2004 to April-2008 is done, as shown in Tables (I) and (II) in Appendix A. This interval includes eight cycles, two cycles per year (summer and winter). After investigating the pure data of the three items, using the SPSS program "One-Sample Kolmogorov-Smirnov Test", it is found that, the demand of each item has normal distribution, as shown in Table (III) in Appendix A, Table(5). The main calculus results can be summarized in Table (1), which represents the average of demand rate, order quantity and unit costs for each item. Table(2) shows the actual, backorder and lost sales, fractions and the upper limit of the shortage constraint for each item.

Table 1: The average of demand rate, order quantity and units cost.

Item	$\bar{D}$	$\bar{Q}$	c_o	c_h	c_b	c_l
1	10703.750	10744.17	2.234	7.898	0.878	9.35
2	11181.875	11215.33	2.1399	7.567	1.1026	13.254
3	7375.000	7370.833	9.768	34.542	3.2804	68.46

Table 2: Back order and lost sales fractions and restriction upper limit.

Item	γ	$1-\gamma$	K_s
1	0.56	0.44	14.000
2	0.70	0.30	15.000
3	0.67	0.33	67.000

Solution

If β is a constant real number selected to provide the best fit of estimated expected total cost function and using the mathematica program V.5 procedure as shown in Appendix B. Then, applying Equations (10) and (11) into Tables (1) and (2) to obtain the optimal solutions, λ_i^*, Q_i^*, r_i^* and the minimum expected total cost (per thousand pound) for each item, $E(TC(Q_i^*, r_i^*))$, which denoted by ETC_i in the graphs and tables), at some different values of β, where $0< \beta <1$, as shown in Table (3).

Table 3: The optimal solutions of the model for all items.

item	β	λ^*	Q^*	r^*	ESC	ETC_i
1	0.1	0.16	4272.22	15886.6	13.9684	**42.3828**
	0.2	0.28	3710.33	16224.6	13.8346	45.7499
	0.3	0.35	3215.92	16506.2	13.9866	48.9911
	0.4	0.395	2744.16	16844.2	13.8775	52.1495
	0.5	0.38	2330.55	17154	13.9123	54.9361
2	0.1	0.15	4604.97	16401.5	14.9092	**40.4441**
	0.2	0.258	4018.04	16717.8	14.8667	43.7231
	0.3	0.35	3460.51	17075.3	14.6105	46.9855
	0.4	0.37	3009.51	17309.1	14.9924	49.9391
	0.5	0.365	2562.79	17625.4	14.9619	52.7743
3	0.1	0.125	4277.26	10084	66.9185	**150.74**
	0.2	0.225	3805.96	1036	66.882	160.719
	0.3	0.309	3370.6	10636	66.6763	170.831
	0.4	0.36	2960.58	10912	66.5798	180.79
	0.5	0.36	2555.19	11213	66.3408	190.281

3- The Model with Fuzzy Units

In the real situation the demand probably will have some changes due to various un certainties. Hence, to match more realistic situation, we attempt to modify the above model by considering the fuzzy demand, order and holding costs.

3-1 Formulation and analysis

The probabilistic inventory model in Section (2) reformulated using the fuzziness in this subsection. we attempt to modify the same probabilistic model in Equation (2) by considering the fuzzy demand, also, some costs. Thus, it is assumed that the demand, the order and holding unit cost are fuzzy, which are represented by triangular fuzzy numbers, for any i^{th} item, as following:

$$\widetilde{D_i} = \left(\overline{D_i} - \delta_{1i}, \overline{D_i}, \overline{D_i} + \delta_{2i}\right),$$
$$\widetilde{c_{oi}} = \left(c_{oi} - \delta_{3i}, c_{oi}, c_{oi} + \delta_{4i}\right) \text{and}$$
$$\widetilde{c_{hi}} = \left(c_{hi} - \delta_{5i}, c_{hi}, c_{hi} + \delta_{6i}\right)$$

where, δ_{1i}, δ_{2i}, δ_{3i}, δ_{4i}, δ_{5i} and δ_{6i} are determined by the decision makers under these following restrictions:

$0 < \delta_{1i} < \overline{D_i}$, $\delta_{2i} > 0$, $0 < \delta_{3i} < c_{oi}$, $\delta_{4i} > 0$, $0 < \delta_{5i} < c_{hi}$ and $\delta_{6i} > 0$

The membership function of $\widetilde{D_i}$, $\widetilde{c_{oi}}$ and $\widetilde{c_{hi}}$ are respectively:

$$M_{\widetilde{D_i}}(x_i) = \begin{cases} \dfrac{x_i - (\overline{D_i} - \delta_{1i})}{\delta_{1i}}, & \overline{D_i} - \delta_{1i} \le x_i \le \overline{D_i} \\ \dfrac{\overline{D_i} + \delta_{2i} - x_i}{\delta_{2i}}, & \overline{D_i} \le x_i \le \overline{D_i} + \delta_{2i} \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

$$M_{\widetilde{c_{o_i}}}(x_i) = \begin{cases} \dfrac{x_i - (\widetilde{c_{o_i}} - \delta_{3i})}{\delta_{3i}}, & \widetilde{c_{o_i}} - \delta_{3i} \le x_i \le \widetilde{c_{o_i}} \\ \dfrac{\widetilde{c_{o_i}} + \delta_{4i} - x_i}{\delta_{4i}}, & \widetilde{c_{o_i}} \le x_i \le \widetilde{c_{o_i}} + \delta_{4i} \\ 0, & \text{otherwise.} \end{cases} \quad (21)$$

$$M_{\widetilde{c_{h_i}}}(x_i) = \begin{cases} \dfrac{x_i - (\widetilde{c_{h_i}} - \delta_{5i})}{\delta_{5i}}, & \widetilde{c_{h_i}} - \delta_{5i} \le x_i \le \widetilde{c_{h_i}} \\ \dfrac{\widetilde{c_{h_i}} + \delta_{6i} - x_i}{\delta_{6i}}, & \widetilde{c_{h_i}} \le x_i \le \widetilde{c_{h_i}} + \delta_{6i} \\ 0, & \text{otherwise.} \end{cases} \quad (22)$$

From Equations (20, 21 and 22), the left and right limits of α cuts of $\widetilde{D_i}$, $\widetilde{c_{oi}}$ and $\widetilde{c_{hi}}$ are respectively given as below:

$$\widetilde{D_{vi}}(\alpha) = \overline{D_i} - (1 - \alpha)\delta_{1i}, \quad \widetilde{D_{ui}}(\alpha) = \overline{D_i} + (1 - \alpha)\delta_{2i},$$
$$\widetilde{c_{ovi}}(\alpha) = c_{oi} - (1 - \alpha)\delta_{3i}, \quad \widetilde{c_{oui}}(\alpha) = c_{oi} + (1 - \alpha)\delta_{4i}, \text{ and}$$
$$\widetilde{c_{hvi}}(\alpha) = c_{hi} - (1 - \alpha)\delta_{5i}, \quad \widetilde{c_{hui}}(\alpha) = c_{hi} + (1 - \alpha)\delta_{6i}.$$

Then, the Lagrange equation in (2) when the order, the holding costs and the demand are fuzzy numbers can be represented as:

$$G1(\tilde{D}_i, \tilde{c}_{oi}, \tilde{c}_{hi}) = \frac{\widetilde{c_{oi}D_i}}{Q_i} + \tilde{c}_{hi} Q_i \begin{pmatrix} \frac{Q_i}{2} + r_i - E(x_i) \\ + (1 - \beta) \bar{S}(r_i) \end{pmatrix} + \frac{\tilde{D}_i}{Q_i} (1 + \lambda_{si}) W1 \bar{S}(r_i) - \lambda_{si} K_{si} \tag{23}$$

where, $W1 = [\gamma_i c_{bi} + (1 - \gamma_i) c_{li}]$, $\widetilde{c_{oi}D_i}$ is the product between $\tilde{D}_i$ and $\tilde{c}_{oi}$ and it can be defined as:

$$\widetilde{c_{oi}D_i} = \tilde{D}_i \otimes \tilde{c}_{oi} = \{(D_{vi}(\alpha) \cdot c_{ovi}(\alpha)), (D_{ui}(\alpha) \cdot c_{oui}(\alpha))\}.$$

It can be represented the left and right hand sides of $G1(\tilde{D}_i, \tilde{c}_{oi}, \tilde{c}_{hi})$, which are referred respectively by $G1(\tilde{D}_i, \tilde{c}_{oi}, \tilde{c}_{hi})_v(\alpha)$ and $G1(\tilde{D}_i, \tilde{c}_{oi}, \tilde{c}_{hi})_u(\alpha)$, as follows:

$$G1(\tilde{D}_i, \tilde{c}_{oi}, \tilde{c}_{hi})_v(\alpha) = \frac{(c_{oi}D_{iv})(\alpha)}{Q_i} + \frac{(D_{iv})(\alpha)}{Q_i}(1 + \lambda_{si}) W1 \bar{S}(r_i)$$
$$+ \tilde{c}_{hiv}(\alpha) Q_i \left(\frac{Q_i}{2} + r_i - E(x_i) + (1 - \beta) \bar{S}(r_i) \right) - \lambda_{si} K_{si}$$

and,

$$G1(\tilde{D}_i, \tilde{c}_{oi}, \tilde{c}_{hi})_u(\alpha) = \frac{(c_{oi}D_{iu})(\alpha)}{Q_i} + \frac{(D_{iu})(\alpha)}{Q_i}(1 + \lambda_{si}) W1 \bar{S}(r_i)$$
$$+ \tilde{c}_{hiu}(\alpha) Q_i \left(\frac{Q_i}{2} + r_i - E(x_i) + (1 - \beta) \bar{S}(r_i) \right) - \lambda_{si} K_{si}$$

where, $\widetilde{c_{oi}D_{vi}}(\alpha) = \widetilde{D_{vi}}(\alpha) \cdot \widetilde{c_{ovi}}(\alpha) = (\bar{D}_i - (1 - \alpha)\delta_{1i})(c_{oi} - (1 - \alpha)\delta_{3i})$ and,

$$\widetilde{c_{oi}D_{ui}}(\alpha) = \widetilde{D_{ui}}(\alpha) \cdot \widetilde{c_{oui}}(\alpha) = (\bar{D}_i + (1 - \alpha)\delta_{1i})(c_{oi} + (1 - \alpha)\delta_{3i}).$$

Thus, the signed distance from $G1(\tilde{D}_i, \tilde{c}_{oi}, \tilde{c}_{hi})$ to $\tilde{0}$, $\widetilde{G1}$, [see Yao and Wu (2000)] is defined as:

$$\widetilde{G1} = \frac{1}{2} \int_0^1 [G1(\tilde{D}_i, \tilde{c}_{oi}, \tilde{c}_{hi})_v(\alpha) + G1(\tilde{D}_i, \tilde{c}_{oi}, \tilde{c}_{hi})_u(\alpha)] d\alpha \tag{24}$$

Thus,

$$\widetilde{G1} = \frac{m_3}{Q_i} + \frac{(1 + \lambda_{si}) m_1}{Q_i} W1 \bar{S}(r_i) + m_2 Q_i \begin{pmatrix} \frac{Q_i}{2} + r_i - E(x_i) \\ + (1 - \beta) \bar{S}(r_i) \end{pmatrix} - \lambda_{si} K_{si} \tag{25}$$

where, $m_1 = \bar{D}_i + \frac{1}{4}(\delta_{2i} - \delta_{1i})$, $m_2 = c_{hi} + \frac{1}{4}(\delta_{6i} - \delta_{5i})$,
$W1 = [\gamma_i c_{bi} + (1 - \gamma_i) c_{li}]$, and
$m_3 = \bar{D}_i c_{oi} + \frac{1}{4}[c_{oi}(\delta_{2i} - \delta_{1i}) + \bar{D}_i(\delta_{4i} - \delta_{3i})] + \frac{1}{6}[\delta_{1i}\delta_{3i} + (\delta_{2i}\delta_{4i})]$.

The optimal values of fuzzy Lagrange function, which denoted by G1, can be calculated by setting each of the corresponding first partial derivatives of Equation (25) equal to zero, then we obtain:

$$\frac{\partial \widetilde{G}1}{\partial Q_i} = -\frac{m_3}{Q_i^2} - \frac{m_1}{Q_i^2}\left(c_{si} + W1\,\bar{S}(R_i)\right) + \frac{m_2(\beta-1)}{2}Q_i^{\beta}$$

$$+\, m_2\beta Q_i^{\beta-1}\left[r_i - E(x_i) - (1-\theta_i)\bar{S}(R_i)\right] = 0$$

And,

$$\frac{\partial \widetilde{G}1}{\partial R_i} = \frac{m_1}{Q_i}\left(c_{si} + W1\right)\left[-F(R_i)\right] + c_{hi}Q_i^{\beta} + m_2(1-\theta_i)Q_i^{\beta}\left[-F(R_i)\right] = 0$$

Thus,

$$m_2(\beta-1)Q_i^{\beta+2} + 2m_2\beta Q_i^{\beta+1}\left[r_i - E(x_i) - (1-\theta_i)\bar{S}(R_i)\right]$$

$$-\,2\left[m_3 + m_1\left(c_{si} + W1\,\bar{S}(R_i)\right)\right] = 0 \qquad (26)$$

and,

$$F(R_i) = \frac{m_2 Q_i^{\beta+1}}{m_1\left(c_{si}+W1\right) + m_2(1-\theta_i)Q_i^{\beta+1}} \qquad (27)$$

where, $m_1 = \bar{D}_i + \frac{1}{4}(\delta_{2i} - \delta_{1i})$, $m_2 = c_{hi} + \frac{1}{4}(\delta_{6i} - \delta_{5i})$,
$W1 = [\gamma_i c_{bi} + (1-\gamma_i)c_{li}]$, and
$m_3 = \bar{D}_i c_{oi} + \frac{1}{4}[c_{oi}(\delta_{2i} - \delta_{1i}) + \bar{D}_i(\delta_{4i} - \delta_{3i})] + \frac{1}{6}[\delta_{1i}\delta_{3i} + (\delta_{2i}\delta_{4i})]$.

3-2 Special Cases

There are three special cases of the model with fuzzy units such as: the model with only fuzzy demand, or with only fuzzy order unit cost or with only fuzzy holding unit cost.

(I) The model with fuzzy demand

If the crisp demand, $\bar{D}_i$ will be replaced in the total cost, by the triangular fuzzy number $\widetilde{D}_i = (\bar{D}_i - \delta_{1i}, \bar{D}_i, \bar{D}_i + \delta_{2i})$, where δ_{1i} and δ_{2i} are determined by the decision makers and $0 < \delta_{1i} < \bar{D}_i$, $\delta_{2i} > 0$. In general, the fuzzy triangular demand and -cut can be showed as in Figure (2).

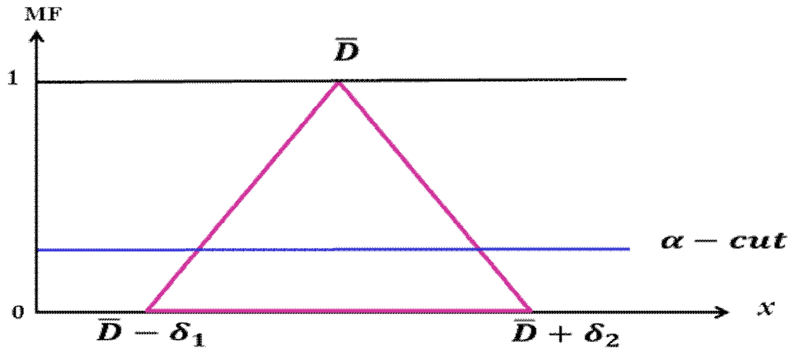

Figure 2: Triangular fuzzy number of demand.

Now, by substituting the crisp demand, $\bar{D}_i$, of Equation (2) into the triangular fuzzy number, $\tilde{D}_i$, the left and right sides (using Equation (20) of the membership function of $\tilde{D}_i$, which denoted by MF in Figure 2) are obtained. Then the signed distance of the left and right sides, is given as:

$$\widetilde{G1}(\tilde{D}_i) = \frac{m_1}{Q_i}\left[c_{oi} + \pi_{si} W1 \bar{S}(r_i)\right] + c_{hi} Q_i^{\theta} \left(\frac{Q_i}{2} + r_i - E(X_i) + (1-\beta_i)\bar{S}(r_i) \right) + \pi_{si} K_{si} \tag{28}$$

The above equation, $\widetilde{G1}(\tilde{D}_i)$, can be minimized by setting each of the corresponding first partial derivatives equal to zero, then it is found:

$$c_{hi}(\theta + 1)Q_i^{\theta} + 2c_{hi}\theta Q_i^{\theta - 1}\{r_i - E(X_i) + (1-\beta_i)\bar{S}(r_i)\} - 2m_1\{c_{oi} + \pi_{si} W1 \bar{S}(r_i)\} = 0 \tag{29}$$

And

$$R(r_i) = \frac{c_{hi} Q_i^{\theta + 1}}{m_1(\pi_{si} W1 + c_{hi}(1-\beta_i) Q_i^{\theta + 1})} \tag{30}$$

where, $m_1 = \bar{D}_i + \frac{1}{4}(\delta_{2i} - \delta_{1i})$ and $W1 = [\gamma_i c_{bi} + (1 - \gamma_i)c_{li}]$,

(II) The model with fuzzy order cost

The crisp order unit cost, c_{oi}, will be replaced by the triangular fuzzy number, $\tilde{c}_{oi} = (c_{oi} - \delta_{3i}, c_{oi}, c_{oi} + \delta_{4i})$, where δ_{3i} and δ_{4i} are determined by the decision makers and $0 < \delta_{3i} < c_{oi}$, $\delta_{4i} > 0$. The left and right sides are obtained using Equation(21). Then the signed distance, can be given as:

$$\widetilde{G1}(\tilde{c}_{oi}) = \frac{\bar{D}_i}{Q_i} m_4 + BW\bar{S}(r_i) + c_{hi} Q_i^{\theta} \left(\frac{Q_i}{2} + r_i - E(X_i) + (1-\beta_i)\bar{S}(r_i) \right) + \pi_{si} K_{si} \tag{31}$$

where, m$_4$ = $c_{oi} + \frac{1}{4}(\delta_{4i} - \delta_{3i})$, B= (1+ λ_{si}) and
$W = [\gamma_i c_{bi}\bar{D}_i + (1-\gamma_i)c_{li}\bar{D}_i]$,
Then, to find the minimum of Equation (31) can be calculated by setting each of its corresponding first partial derivatives equal to zero, then it is obtained:

$$c_{hi}(\ldots - 1)Q_i^{*2} - 2c_{hi}\ldots Q_i^{*}(r_i - E(X_i)) - (1-\ldots)\bar{S}(r_i)\ldots 2\ldots m_4 - BW\bar{S}(r_i) = 0 \quad (32)$$

$$R(r_i) = \frac{c_{hi}Q_i^{*}}{BW - c_{hi}(1-\ldots)Q_i^{*}} \quad (33)$$

where, m$_4$ = $c_{oi} + \frac{1}{4}(\delta_{4i} - \delta_{3i})$, B= (1+ λ_{si}) and
$W = [\gamma_i c_{bi}\bar{D}_i + (1-\gamma_i)c_{li}\bar{D}_i]$,

(III) The model with fuzzy holding cost

It is assumed that the crisp holding unit cost, c_{hi}, will be replaced by the triangular fuzzy number, $\widetilde{c_{hi}} = (c_{hi} - \delta_{5i}, c_{hi}, c_{hi} + \delta_{6i})$, where δ_{5i} and δ_{6i} are determined by the decision makers and $0 < \delta_{5i} < c_{hi}$, $\delta_{6i} > 0$. The left and right sides are obtained using Equation(22). Then the signed distance and their optimal solutions for Q_i^* and r_i^*, can be given respectively as:

$$\widetilde{G1}(\widetilde{c}_{hi}) = \frac{A}{Q_i} + \frac{B}{Q_i}W\bar{S}(r_i) + m_2 Q_i^{\ldots}\left(\frac{Q_i}{2} + r_i - E(X_i) + (1-\ldots)\bar{S}(r_i)\right) + \ldots_{si}K_{si} \quad (34)$$

$$m_2(\ldots - 1)Q_i^{*2} - 2m_2\ldots Q_i^{*}(r_i - E(X_i)) - (1-\ldots)\bar{S}(r_i)\ldots 2A - BW\bar{S}(r_i) = 0, \quad (35)$$

and

$$R(r_i) = \frac{m_2 Q_i^{*}}{BW - m_2(1-\ldots)Q_i^{*}}, \quad (36)$$

where, m$_2$ = $c_{hi} + \frac{1}{4}(\delta_{6i} - \delta_{5i})$, A= $c_{oi}\bar{D}_i$, B= (1+ λ_{si}) and
$W = [\gamma_i c_{bi}\bar{D}_i + (1-\gamma_i)c_{li}\bar{D}_i]$.

3-3 The Practical Application

Using the same practical example data in section (2-3) and the fuzzy options which are given in Table (4) in the studying model with fuzzy units. Then apply Equations (26 and 27) by using the mathematica program. Table (5) shows the optimal solutions and the minimum expected total cost for each item, at different values of β. It can be draw the optimal values of minimum expected total cost (of the probabilistic model, *ETC*, and the model with fuzzy units, *Fuz ETC*) for every item against β as shown in Figures (3,4 and 5).

Table 4: The options of fuzzy changes for each item.

Item	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6
1	9	1	1.82	0.1	6	2
2	7	1	1.5	0.1	4	2
3	5	1	8	2	20	4

Table 5: The optimal values of the model with fuzzy units for all items.

item	β	λ^*	Q^*	r^*	ESC	ETC_i
1	0.1	0.6	4298.97	15379.7	13.9366	**35.0277**
	0.2	0.17	3776.56	15717.6	13.8143	37.5629
	0.3	0.248	3318.58	15999.3	13.9192	40.0663
	0.4	0.283	2870.44	16309.1	13.9798	42.487
	0.5	0.285	2450.43	16647.0	13.9271	44.7502
2	0.1	0.118	4612.17	16057.7	14.8847	**35.701**
	0.2	0.223	4085.17	16332.7	14.9884	38.2826
	0.3	0.31	3563.78	16662.8	14.8772	40.9672
	0.4	0.35	3088.41	16965.3	14.9355	43.5575
	0.5	0.345	2599.33	17350.44	14.7055	46.082
3	0.1	0.09	4314.18	9908.35	66.5017	**139.83**
	0.2	0.19	3856.87	10184.3	66.4056	148.679
	0.3	0.27	3459.42	10410.2	66.9728	157.279
	0.4	0.327	3042.45	10711.2	66.490.9	166.542
	0.5	0.33	2653.01	10987.2	66.7432	175.026

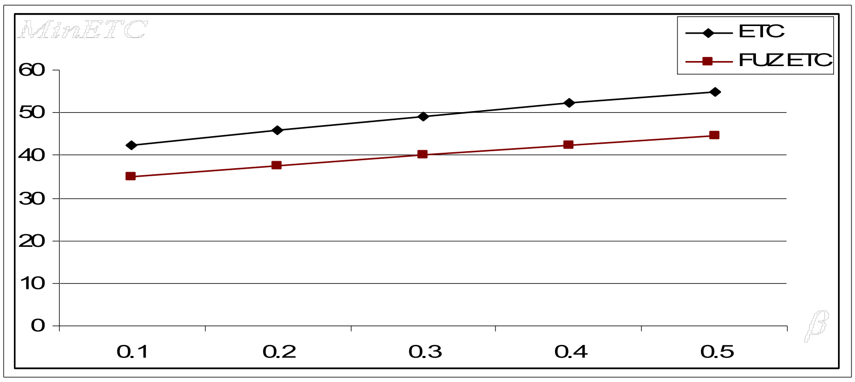

Figure 3: The minimum expected total cost in crisp and fuzzy cases, for item1.

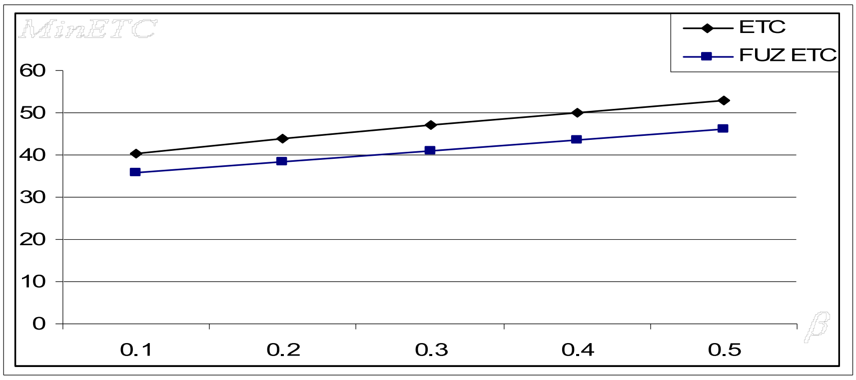

Figure 4: The minimum expected total cost in crisp and fuzzy cases, for item2.

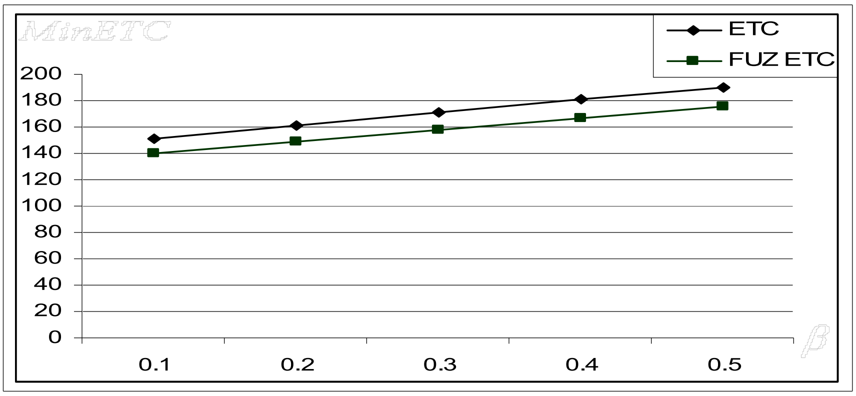

Figure 5: The minimum expected total cost in crisp and fuzzy cases, for item3.

4- Conclusion

This study suggested two parts for MISS inventory system. The first part is to derive a constraint probabilistic continuous review inventory model, with mixture shortage and varying holding. This model studied when the demand is random variable and the lead time is constant using the characteristic functions, which shows the distribution of lead time demand, x, as the relation between D and L. The objective is to minimize the expected total cost. The mathematically optimal solution for the order quantity Q^* and the reorder point r^* can be obtained (in general and when demand follows Normal, Exponential and Chi-Square distributions). The second part treats the probabilistic model using fuzziness. The signed distance method is used when the demand, order and holding unit costs are triangular fuzzy numbers. Some special cases are deduced. There is an actual application of three items for the crisp and fuzzy cases. The computational results and the figures showed that, in general, the expected total cost is directly proportional with β value. So, in general, for any manufactory or enterprises, that apply the studying inventory system, must select the value of β between 0, 1, where $0< \beta <1$, which satisfies their maximum inventory capacity, or buffer stock and market needs. Also, it is observed that from the numerical results, the fuzzy case is better than the crisp one because the minimum expected total cost is less than its crisp.

REFERENCES

[1] A bou El-Ata, M. O., Fergany, H. A., and El-wakeel, M .F. (2003): "*Probabilistic multi-item inventory model with varying order cost under two restrictions: A Geometric programming approach*". International Journal of Production Economics, Vol. 83, 223-231.

[2] Chiang, J., Yao, J. S., and Lee, H. M. (2005): "*Fuzzy inventory with backorder defuzzification by signed distance method*". Journal of Information Science and Engineering, 21, 673-694.

[3] Fabrycky,W. J. and Banks, J. (1967): "Procurement and Inventory Systems: Theory And Analysis". Reinhold Publishing Corporation, A Subsidiary of Chapman-Reinhold, Ine. United States of America.

[4] Fergany, H. A. and El-Wakeel, M. F.(2006): "*Constrained probabilistic lost sales inventory system with normal distribution and varying order cost*". Journal of Mathematics and Statistics, 2(1), pp. 363-366.

[5] Hadley,G. and Whitin,T. M. (1961): "*A Family of inventory models*". Management Science, VoL 7, No. 4, pp. 351-371.

[6] Hadley,G. and Whitin, .T.M. (1963): "Analysis of Inventory Systems". Prentice Hall, Inc., Englewood Cliffs, New Jersy.

[7] Hariri, A. M. A. and Abou El-Ata, M. O. (1997): "*Multi-item production lot size inventory model with varying order cost under a restriction: A Geometric programming approach*". Production Planning & Control, VoL 8 No. 2, pp.179-182.

[8] Hogg, R. V. and Craig, A. (1978): "*Introduction to Mathematical Statistics*". Macmillan Publishing Co., Inc. New York.

[9] Tersine, R. J. (1994): "Principles of Inventory and Materials Management". 4th Edition. Prentice- Hall, Englewood Cliffs, New Jersy.

[10] Vijayan, T. and Kumaran M. (2008): "*Inventory models with a mixture of backorders and lost sales under fuzzy cost*". European Journal of Operational Research, VoL 189, pp. 105-119.

[11] Yao, J. S., and Wu, K., (2000): "*Ranking fuzzy numbers based on decomposition principle and signed distance*". Fuzzy sets and Systems, VoL 116, pp. 275-288.

[12] Yao, J. S., and Chiang, J., (2003): "*Inventory without backorder with fuzzy total cost and fuzzy storing cost defuzzified by centroid and signed distance*". European Journal of

Operational Research, VoL 148, pp. 401-409.

APPENDIX A: THE ACTUAL DATA

The actual data in the interval from May-2004 to April-2008 and its calculations are tabulated as follows: Tables (I-a) and (I-b) shown the real inventory quantity (Q) and demand rate (D) for the three items. After investigating the pure data, using the SPSS program, it is found that, the demand of the three items are normal distribution and its results are shown in Table (II). The company may be found shortage, and then it must to pay penalty at least 1 % per each month later for backorders and 3 % for lost sales. Tables (III-a) and (III-b) show the average inventory shortage quantity and its back, lost and their percents. Tables (IV-a) and (IV-b) show the average inventory shortage costs. Table (V) shows the average of total order and holding costs for the three items.

Table (II): The SPSS analysis of data for the three items.

Normal Parameters	Item 1	Item 2	Item 3
Mean	10703.75	11181.875	7375.000
Std. Deviation	2299.5869	2245.706	2048.611

Asymp. Sig. (2-tailed)	0.063	0.078	0.452

Table (I-a) : The actual inventory quantity and demand rate, from May 2004 to April 2008.

Year	N. of cycle	Month	Item 1		Item 2		Item 3	
			Q	D	Q	D	Q	D
2004	1	May	5800	6000	10500	10500	8000	9000
		June	9000	8000	9000	10000	5500	5000
		July	11800	12000	12000	12000	8000	9000
		Aug	11800	12000	12000	12500	6000	5000
		Sept.	8000	8500	10000	9000	4000	4000
		Oct.	7200	7000	7500	7000	3000	4000
	2	Nov.	10000	10000	10000	10500	5500	5000
		Dec.	11000	12000	9000	9000	5500	5000
2005		Jan.	12800	12800	11000	11000	5000	5500
		Feb.	11000	10000	7500	7500	4000	5000
		March	6000	6500	12500	12500	5000	5000
		April	9500	8500	13000	12500	7000	6000
	3	May	12000	12000	11000	12000	9500	10000
		June	12000	12500	10000	9000	6500	6000
		July	8500	9000	12500	12800	9000	10000
		Aug.	7000	7500	17000	16000	7000	6000
		Sept.	11000	12000	9000	10000	5000	5000
		Oct.	13400	11000	7800	8000	4000	5000
	4	Nov.	12850	13500	12500	12000	6500	6000
		Dec.	12830	13000	11000	12000	6500	6000
2006		Jan.	12850	12500	11850	10500	7000	7500
		Feb.	12830	11850	6830	8000	6000	7000
		March	12820	12000	11820	12500	7000	7000
		April	10730	11030	12730	12230	9000	8000

Table (I-b): The actual inventory quantity and demand rate, from May 2004 to April 2008.

Year	N. of cycle	Month	Item 1		Item 2		Item 3	
			Q	D	Q	D	Q	D
2006	5	May	6500	7000	11500	12000	10000	11000
		June	9800	8500	10000	9500	7500	7000
		July	12500	13000	12800	12950	10000	11000
		Aug.	12200	13000	17000	16000	8500	7000
		Sept.	9000	8600	9000	9500	6000	6000
		Oct.	7000	7300	8500	8750	5000	6000
	6	Nov.	10000	12000	13000	12000	7500	7000
		Dec.	12000	10500	11500	12500	7500	7000
2007		Jan.	13000	14000	12000	11000	8000	8500
		Feb.	13000	13000	7000	8000	7000	8000
		March	13000	12000	12000	13000	8000	8000
		April	11000	10000	13000	13000	10000	9000
		May	7000	7000	12000	13000	11500	12000
		June	10000	11000	10000	9000	8500	8000
		July	13000	13000	13000	14000	11000	12000
		Aug.	12000	13000	17000	16000	9000	8000
		Sept.	9000	9000	11000	9000	7000	7000
		Oct.	10000	8000	8000	9000	7000	7000
	8	Nov.	10000	12000	13000	12000	8500	8000
		Dec.	12000	10000	11500	12000	8500	8000
2008		Jan.	14000	14500	12500	12000	9000	9500
		Feb.	13000	13200	8000	7500	8000	9000
		March	13000	13000	13000	13000	9000	9000
		April	11000	10000	14000	14000	11000	10000

Table (III-a) The average of inventory shortage quantity from May 2004 to Nov 2008.

Data / Month	Item 1				Item 2				Item 3			
	Shortage	Back	Lost	Fine	Shortage	Back	Lost	Fine	Shortage	Back	Lost	Fine
May (1)	200	-	-	-	-	-	-	-	1000	-	-	-
June	-	200	-	1%	1000	-	-	-	-	-	-	-
July	-	-	-	-	-	-	-	-	500	-	-	-
Aug	-	-	-	-	500	-	-	-	-	500	1000	1% 3%
Seb	100	-	-	-	-	500	1000	1% 3%	-	-	-	-
Oct	-	100	-	1%	-	-	-	-	500	-	500	3%
Nov (2)	-	-	-	-	500	-	-	-	-	-	-	-
Dec	1000	-	-	-	-	-	-	-	-	-	-	-
Jan	-	-	-	-	-	-	-	-	-	-	-	-
Feb	-	1000	-	2%	-	-	500	3%	500	-	-	-
March	500	-	-	-	-	-	-	-	-	-	-	-
April	-	500	-	1%	-	-	-	-	-	500	-	2%
May(3)	-	-	-	-	1000	-	-	-	500	-	-	-
June	500	-	-	-	-	1000	-	1%	-	500	-	1%
July	500	-	-	-	300	-	-	-	1000	-	-	-
Aug	500	-	-	-	-	300	-	1%	-	1000	-	1%
Seb	1000	-	500	3%	300	-	-	-	-	-	-	-
Oct	-	1000 500	500	1% 2% 3%	200	-	300 200	3% 3%	500	-	500	3%
Nov (4)	650	-	-	-	-	-	-	-	-	-	-	-
Dec	170	-	-	-	500	-	-	-	-	-	-	-
Jan	-	170	-	1%	-	500	-	1%	-	-	-	-
Feb	-	-	650	3%	320	-	-	-	500	-	-	-
March	-	-	-	-	680	-	-	-	-	-	-	-
April	-	-	-	-	-	320	680	2% 3%	-	500	-	2%

Table (III-b) The average of inventory shortage quantity from May 2004 to Nov 2008.

Data / Month	Item 1				Item 2				Item 3			
	Shortage	Back	Lost	Fine	Shortage	Back	Lost	Fine	Shortage	Back	Lost	Fine
May (5)	500	-	-	-	500	-	-	-	1000	-	-	-
June	-	500	-	1%	-	500	-	1%	-	-	-	-
July	-	-	-	-	150	-	-	-	500	-	-	-
Aug	500	-	-	-	-	150	-	1%	-	500	1000	1% 3%
Seb	-	-	-	-	-	-	-	-	-	-	-	-
Oct	-	-	500	3%	-	-	-	-	-	-	-	-
Nov (6)	2000	-	-	-	-	-	-	-	-	-	-	-
Dec	-	-	-	-	-	-	-	-	-	-	-	-
Jan	-	-	-	-	-	-	-	-	-	-	-	-
Feb	-	-	2000	3%	-	-	-	-	500	-	-	-
March	-	-	-	-	1000	-	-	-	-	-	-	-
April	-	-	-	-	-	1000	-	1%	-	500	-	2%
May (7)	-	-	-	-	1000	-	-	-	500	-	-	-
June	1000	-	-	-	-	1000	-	1%	-	500	-	1%
July	-	-	-	-	1000	-	-	-	1000	-	-	-
Aug	1000	-	-	-	-	1000	-	1%	-	1000	-	1%
Seb	-	-	1000	3%	-	-	-	-	-	-	-	-
Oct	-	1000	-	2%	-	-	-	-	-	-	-	-
Nov (8)	2000	-	-	-	-	-	-	-	-	-	-	-
Dec	-	2000	-	1%	-	-	-	-	-	-	-	-
Jan	500	-	-	-	-	-	-	-	-	-	-	-
Feb	200	-	-	-	-	-	-	-	500	-	-	-
March	-	-	-	-	-	-	-	-	-	-	-	-
April	-	200	500	2% 3%	-	-	-	-	-	500	-	2%

Table (IV-a) The average of inventory shortage cost from May 2004 to Nov 2008.

Data / Month	Item 1		Item 2		Item 3	
	Back	Lost	Back	Lost	Back	Lost
May (1)	-	-	-	-	-	-
June	0.0273	-	-	-	-	-
July	-	-	-	-	-	-
Aug	-	-	-	-	0.2752	1.6513
Seb	-	-	0.0732	0.4394	-	-
Oct	0.0149	-	-	-	-	1.2405
Nov (2)	-	-	-	-	-	-
Dec	-	-	-	-	-	-
Jan	-	-	-	-	-	-
Feb	0.2273	-	-	0.2896	-	-
March	-	-	-	-	-	-
April	0.0676	-	-	-	0.4630	-
May (3)	-	-	-	-	-	-
June	-	-	0.1429	-	0.2987	-
July	-	-	-	-	-	-
Aug	-	-	0.0266	-	0.5504	-
Seb	-	0.2316	-	-	-	-
Oct	0.2976	0.2232	-	0.264	-	1.2405
Nov (4)	-	-	-	-	-	-
Dec	-	-	-	-	-	-
Jan	0.01833	-	0.06	-	-	-
Feb	-	0.22162	-	-	-	-
March	-	-	-	-	-	-
April	-	-	0.0693	0.221	0.463	-

Table (IV-b) The average of inventory shortage cost from May 2004 to Nov 2008.

Data / Month	Item 1		Item 2		Item 3	
	Back	Lost	Back	Lost	Back	Lost
May (5)	-	-	-	-	-	-
June	0.0683	-	0.715	-	-	-
July	-	-	-	-	-	-
Aug	-	-	0.01332	-	0.2752	1.6513
Seb	-	-	-	-	-	-
Oct	-	0.2232	-	-	-	-
Nov (6)	-	-	-	-	-	-
Dec	-	-	-	-	-	-
Jan	-	-	-	-	-	-
Feb	-	0.682	-	-	-	-
March	-	-	-	-	-	-
April	-	-	0.1083	-	0.4630	-
May (7)	-	-	-	-	-	-
June	-	-	0.143	-	0.299	-
July	-	-	-	-	-	-
Aug	-	-	0.089	-	0.550	-
Seb	-	0.463	-	-	-	-
Oct	0.298	-	-	-	-	-
Nov (8)	-	-	-	-	-	-
Dec	0.2382	-	-	-	-	-
Jan	-	-	-	-	-	-
Feb	-	-	-	-	-	-
March	-	-	-	-	-	-
April	0.0541	0.203	-	-	0.463	-

Table (V) The monthly average order cost and holding cost for each item.

Month	Order Cost		Holding Cost	
	Item 1, 2	Item 3	Item 1, 2	Item 3
May	23..52	70..56	84.960	254.880
June	22.80	68.40	84.408	253.440
July	22.80	68.40	84.984	254.952
Aug	22.80	68.40	84.816	254.448
Seb	25.20	75.60	84.648	253.944
Oct	22.80	68.40	84.816	254.448
Nov	24.00	72.00	84.696	254.088
Dec	25.68	77.04	83.856	251.568
Jan	24.00	72.00	85.152	255.456
Feb	24.00	72.00	84..912	254.736
March	26.40	79.20	85.272	255.816
April	24.00	72.00	85.824	257.472

APPENDIX B: MATHEMATICA PROCEDURES

In this appendix the iteration steps of computer mathematical program to compute the optimal values of Q^*, r^* and $E(TC(Q, r))$.

1-Constraint iteration procedure:

Step 1: Input all the inventory model data for example, expected demand value, holding unit cost, lead time value, order unit cost, mean, etc. at β=0 and λ=0 and put, $r_0 = \mu$ as an initial value so, $s_0 = 0$, then calculate the first order quantity Q_1.
Step 2: Use the calculated order quantity in step 1 to calculate r_1 and s_1.
Step 3: Use the calculated r_1 and s_1 in step 2 to calculate a new order quantity Q_2.
Step 4: Repeat steps 1 and 2. If two values of respectively calculated order quantity are equaled, then it is the optimal Q^*.
Step 5: Using the calculated optimal order quantity Q^* and optimal reorder point r^* to calculate the condition shortage quantity.

2-Normal iteration procedure:

Step 1: Input all the inventory model data for example, expected demand value, holding unit cost, lead time value, order unit cost, mean, etc. at one β value and assumption value of λ and put, $r_0 = \mu$ as an initial value so, $s_0 = 0$, then calculate the first order quantity Q_1.
Step 2: Use the calculated order quantity in step 1 to calculate r_1 and s_1.
Step 3: Use the calculated r_1 and s_1 in step 2 to calculate a new order quantity Q_2.
Step 4: Repeat steps 1 and 2. If two values of respectively calculated order quantity are equaled, then it is the optimal Q^*.
Step 5: Using the calculated optimal order quantity Q^* and optimal reorder level r^* to calculate the expected total cost.
Step 6: Repeat all steps at changes values of λ to be the condition is active. If the condition is active, then it is the minimum expected total cost at this value of β.
Step 7: Repeat all steps at other values of β.