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Research article

Moments of Dual Generalized Order Statistics and Characterization for Transmuted Exponential Model

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Abstract: The relationships for moments of dual generalized order statistics (DGOS) for a transmuted exponential (TEx) model are given in this study. These relations are useful when determining the single and product moments of DGOS for a TEx model recursively. Some expressions for recursive computation of moments for special cases of DGOS from a TEx model are given. Some characterizations of the distribution are also given at the end.

Keywords: Exponential distribution; Dual generalized order statistics; Transmuted distributions; Moments.

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1. Introduction

The DGOS is a classical method to study the properties of random variables (RVs) that are arranged from highest to lowest. Ref. [1] have given the joint distribution of n DGOS as

$$f_{1,2,\ldots,n:n,m,k}\left(z_1,\ldots,z_n\right)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[F\left(z_n\right)\right]^{k-1} f\left(z_n\right)\left[\prod_{i=1}^{n-1}\left\{F\left(z_i\right)\right\}^m f\left(z_i\right)\right], \tag{1}$$

where $\gamma_j=k+(n-j)(m+1)$ and $F\left(z_i\right)$ is cumulative distribution function (cdf) of *i*th RV. Ref. [6] have further shown that the marginal distribution of a single *DGOS* and joint distribution of two *DGOS* are provided as

$$f_{r:n,m,k}(z)=\frac{C_{r-1}}{(r-1)!} f(z)\left[F(z)\right]^{\gamma_r-1} g_m^{r-1}\left[F(z)\right], \tag{2}$$

and

$$\begin{aligned}f_{r,s:n,m,k}\left(z_1,z_2\right)=&\frac{C_{s-1}}{(r-1)!(s-r-1)!} f\left(z_1\right) f\left(z_2\right)\left[F\left(z_1\right)\right]^m g_m^{r-1}\left[F\left(z_1\right)\right]\\&\times\left[F\left(z_2\right)\right]^{\gamma_s-1}\left[h_m\left\{F\left(z_1\right)\right\}-h_m\left\{F\left(z_2\right)\right\}\right]^{s-r-1},\end{aligned} \tag{3}$$

where $C_{r-1}=\prod_{j=1}^{r}\gamma_j$

and $h_m(z)=\begin{cases}\dfrac{z^{m+1}}{m+1}; & m\neq -1\\ \ln z; & m=-1.\end{cases}$; $g_m(z)=\begin{cases}\dfrac{1}{m+1}\left(1-z^{m+1}\right); & m\neq -1\\ -\ln z; & m=-1.\end{cases}$

Numerous models of RVs that are arranged in decreasing order statistics (OS) appear as a sub-model of *DGOS*. For example, the decreasing OS looks to be a particular example of *DGOS* when $m=0$ and $k=1$, [2] The lower record values (REVs), proposed by [3] and [4], emerge as a sub-model for $m=-1$.

The studies on DGOS mostly are focused on obtaining some methodologies for recursive computation of moments for specific choices of distributions in (2) and (3). Studies are also conducted to obtain some characterizations of distributions based upon the moments of DGOS. Ref. [5] provided some identities for recursive computation of moments of DGOS. The relations for moments of DGOS for parent Inverse Weibull model have been discussed in [6]. The recurrence relations (RRs) for moments of DGOS for exponentiated Weibull model have been investigated by [7] whereas the relations for moments of power function model was investigated by [8]. Topp-Leone Weibull generated family of distributions with applications was found in [19]. The RRs for moments of DGOS for a inverted Kumaraswamy model was found in [9]. The RRs for moments of ordered variables in transmuted (T) models have received little attention. The RRs for moments of OS in a TEx model were recently discovered by [10]. In [11], the RRs for moments of generalized OS for a TEx model were also found. Detailed information on RRs and distribution characterization employing DGOS and lower REVsmay be explored in [12-13].

The RRs for moments of DGOS for a T model have yet to be investigated, and we got the RRs for moments of DGOS for a TEx model in this study. For single, inverse, product, and ratio moments, the relationships have been derived. These relations can be utilized to determine exceptional situations' comparable relations. The study also discusses several characterizations of the TEx model utilizing DGOS single and product moments. Below is a basic overview of the TEx model.

2. Transmuted Exponential Model

In reliability and life testing, the exponential (Ex) model is a prominent probability model. The exponential distribution's probability density function (pdf) and cdf are provided via

$$f(z;\alpha)=\alpha \mathrm{e}^{-\alpha z} \text{ and } F(z)=1-\mathrm{e}^{-\alpha z} \ ; \alpha, z>0 .$$

The Ex model has been widely researched, and numerous modifications have been presented in the literature. Ref. [14] discussed the TEx model through using quadratic transmutation approach established by [15]. The pdf and cdf of the TEx model are provided via

$$f(z)=\alpha \mathrm{e}^{-\alpha z}\left[1+\lambda-2\lambda\left(1-\mathrm{e}^{-\alpha z}\right)\right]; \ z,\alpha>0, -1<\lambda<1 \tag{4}$$

and

$$F(z)=(1+\lambda)\left(1-\mathrm{e}^{-\alpha z}\right)-\lambda\left(1-\mathrm{e}^{-\alpha z}\right)^2 \ ; \ z,\alpha>0, -1<\lambda<1. \tag{5}$$

Ref. [16] provide more information on T distributions. Ref. [10] demonstrated a relationship between the pdf and cdf of the TEx model as

$$\left[1-F(z)\right]=\frac{1}{\alpha}f(z)-\lambda\sum_{j=0}^{\infty}(-2)^{j}\frac{\alpha^{j}z^{j}}{j!}. \tag{6}$$

Ref. [11] utilized the formula (6) to find the RRs for generalized OS moments for a TEx model. We reported the RRs for moments of DGOS for the TEx model in this study. To do this, we slightly alter formula (6) to have the equivalent description:

$$F(z)=1-\frac{1}{\alpha}f(z)+\lambda\sum_{j=0}^{\infty}\frac{(-2)^{j}\alpha^{j}}{j!}z^{j}. \tag{7}$$

The RRs for moments of DGOS for TEx model will be obtained utilizing the formula (7). The sections that follow describe these connections and characterizations.

3. Relation for Single Moments

We shall establish the RRs for single moments of DGOS for a TEx model in this section. The theory Primarily proves these RRs.

Theorem 1: The single moments of *DGOS* for TEx model are provided via

$$\begin{aligned}\mu_{r:n,m,k}^{p}&=\mu_{r-1:n,m,k}^{p}+\frac{\gamma_{r(k-1)}C_{r-1}}{\gamma_{r}C_{r-1(k-1)}}\left(\mu_{r:n,m,k-1}^{p}-\mu_{r-1:n,m,k-1}^{p}\right)+\frac{p}{\alpha\gamma_{r}}\mu_{r:n,m,k}^{p-1}\\&+\lambda\sum_{j=0}^{\infty}\frac{(-2)^{j}\alpha^{j}}{j!}\frac{p\gamma_{r(k-1)}C_{r-1}}{(p+j)\gamma_{r}C_{r-1(k-1)}}\left(\mu_{r:n,m,k-1}^{p+j}-\mu_{r-1:n,m,k-1}^{p+j}\right),\end{aligned} \tag{8}$$

where $\gamma_{j(k-1)}=(k-1)+(n-j)(m+1)$ and $C_{r-1(k-1)}=\prod_{j=1}^{r}\gamma_{j(k-1)}$.

Proof: The single DGOS moments are connected as follows; see, for illustration, [17] and [13].

$$\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}=-\frac{pC_{r-1}}{\gamma_{r}(r-1)!}\int_{-\infty}^{\infty}z^{p-1}\left[F(z)\right]^{\gamma_{r}}g_{m}^{r-1}\left[F(z)\right]dz, \tag{9}$$

where $\mu_{r:n,m,k}^{p}=E\left(Z_{r:n,m,k}^{p}\right)$ and $Z_{r:n,m,k}^{p}$ is rth *DGOS*. Using (7) in (9) we have

$$\begin{aligned}\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}&=-\frac{pC_{r-1}}{\gamma_{r}(r-1)!}\int_{0}^{\infty}z^{p-1}\left[F(z)\right]^{\gamma_{r}-1}g_{m}^{r-1}\left[F(z)\right]dz\\&+\frac{pC_{r-1}}{\alpha\gamma_{r}(r-1)!}\int_{0}^{\infty}z^{p-1}f(z)\left[F(z)\right]^{\gamma_{r}-1}g_{m}^{r-1}\left[F(z)\right]dz\\&-\lambda\sum_{j=0}^{\infty}\frac{(-2)^{j}\alpha^{j}}{j!}\frac{pC_{r-1}}{\gamma_{r}(r-1)!}\int_{0}^{\infty}z^{p+j-1}\left[F(z)\right]^{\gamma_{r}-1}g_{m}^{r-1}\left[F(z)\right]dz\end{aligned}$$

or

$$\mu^p_{r:n,m,k}-\mu^p_{r-1:n,m,k}=-\frac{\gamma_{r(k-1)}C_{r-1}}{\gamma_r C_{r-1(k-1)}}\frac{pC_{r-1(k-1)}}{\gamma_{r(k-1)}(r-1)!}\int_0^\infty z^{p-1}\left[F(z)\right]^{\gamma_{r(k-1)}}g_m^{r-1}\left[F(x)z\right]dz$$
$$+\frac{pC_{r-1}}{\alpha\gamma_r(r-1)!}\int_0^\infty z^{p-1}f(z)\left[F(z)\right]^{\gamma_r-1}g_m^{r-1}\left[F(z)\right]dz$$
$$-\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}\frac{p\gamma_{r(k-1)}C_{r-1}}{\gamma_r(p+j)C_{r-1(k-1)}}\frac{(p+j)C_{r-1(k-1)}}{\gamma_{r(k-1)}(r-1)!}$$
$$\times\int_0^\infty z^{p+j-1}\left[F(z)\right]^{\gamma_{r(k-1)}}g_m^{r-1}\left[F(z)\right]dz,$$

where $\gamma_{j(k-1)}=(k-1)+(n-j)(m+1)$ and $C_{r-1(k-1)}=\prod_{j=1}^{r}\gamma_{j(k-1)}$.

Again utilizing (9), and simplifying, we get

$$\mu^p_{r:n,m,k}=\mu^p_{r-1:n,m,k}+\frac{\gamma_{r(k-1)}C_{r-1}}{\gamma_r C_{r-1(k-1)}}\left(\mu^p_{r:n,m,k-1}-\mu^p_{r-1:n,m,k-1}\right)+\frac{p}{\alpha\gamma_r}\mu^{p-1}_{r:n,m,k}$$
$$+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}\frac{p\gamma_{r(k-1)}C_{r-1}}{(p+j)\gamma_r C_{r-1(k-1)}}\left(\mu^{p+j}_{r:n,m,k-1}-\mu^{p+j}_{r-1:n,m,k-1}\right),$$

which is (8) and hence the theorem.

The relations for single moments of DGOS for Ex model can indeed be computed from (8) by utilizing $\lambda=0$. Some corollaries which immediately follows from Theorem 1 are given below.

Corollary 1: The relation for single moments of lower REVs for TEx model are computed by utilizing $m=-1$ in (8) and is

$$\mu^p_{K(r)}=\mu^p_{K(r-1)}+\left(\frac{k}{k-1}\right)^{r-1}\left(\mu^p_{K-1(r)}-\mu^p_{K-1(r-1)}\right)+\frac{p}{\alpha k}\mu^{p-1}_{K(r)}$$
$$+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}\frac{p}{(p+j)}\left(\frac{k}{k-1}\right)^{r-1}\left(\mu^{p+j}_{K-1(r)}-\mu^{p+j}_{K-1(r-1)}\right).\quad(10)$$

The relations for single moments of lower REVs for Ex model can indeed be readily computed from (10) by utilizing $\lambda=0$.

Corollary 2: Utilizing m = 0 and k = 1 in (8), the relation for single moments of reversed OS for TEx model is

$$\mu^p_{r:n}=\mu^p_{r-1:n}+\frac{n+1}{n-r+1}\left(\mu^p_{r:n}-\mu^p_{r-1:n}\right)+\frac{p}{\alpha(n-r+1)}\mu^{p-1}_{r:n}$$
$$+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}\frac{p(n+1)}{(p+j)(n-r+1)}\left(\mu^{p+j}_{r:n}-\mu^{p+j}_{r-1:n}\right).\quad(11)$$

The relations for single moments of reversed OS for Ex model are readily computed from (11) by utilizing $\lambda=0$.

4. Relation for Product Moments

The relation for product moments of DGOS for a TEx model is found in this section. The theory Primarily proves the RRs.

Theorem 2: The product moments of DGOS for TEx model are provided via

$$\mu_{r,s:n,m,k}^{p,q} = \mu_{r,s-1:n,m,k}^{p,q} + \frac{\gamma_{s(k-1)} C_{s-1}}{\gamma_s C_{s-1(k-1)}} \left(\mu_{r,s:n,m,k-1}^{p,q} - \mu_{r,s-1:n,m,k-1}^{p,q} \right) + \frac{q}{\alpha \gamma_s} \mu_{r,s:n,m,k}^{p,q-1}$$
$$+\lambda \sum_{j=0}^{\infty} \frac{(-2)^j \alpha^j}{j!} \frac{q \gamma_{s(k-1)} C_{s-1}}{(q+j) \gamma_s C_{s-1(k-1)}} \left(\mu_{r,s:n,m,k-1}^{p,q+j} - \mu_{r,s-1:n,m,k-1}^{p,q+j} \right), \tag{12}$$

where $C_{s-1(k-1)} = \prod_{j=1}^{s} \gamma_{j(k-1)}$.

Proof: The product moments of DGOS are connected as follows; see, for illustration, [17] and [13].

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{q C_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{z_1} z_1^p z_2^{q-1} f(z_1) \left[F(z_1) \right]^m$$
$$\times g_m^{r-1} \left[F(z_1) \right] \left[h_m(z_1) - h_m(z_2) \right]^{s-r-1} \left[F(z_2) \right]^{\gamma_s} dz_2 dz_1, \tag{13}$$

where $\mu_{r,s:n,m,k}^{p,q} = E\left(Z_{r:n,m,k}^{p} Z_{s:n,m,k}^{q} \right)$. Using (7) in (13) we have

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{q C_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{z_1} z_1^p z_2^{q-1} f(z_1) \left[F(z_1) \right]^m g_m^{r-1} \left[F(z_1) \right]$$
$$\times \left[h_m(z_1) - h_m(z_2) \right]^{s-r-1} \left[1 - \frac{1}{\alpha} f(z_2) + \lambda \sum_{j=0}^{\infty} \frac{(-2)^j \alpha^j}{j!} z_2^j \right]$$
$$\times \left[F(z_2) \right]^{\gamma_s - 1} dz_2 dz_1$$

or

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{q C_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{z_1} z_1^p z_2^{q-1} f(z_1) \left[F(z_1) \right]^m$$
$$\times g_m^{r-1} \left[F(z_1) \right] \left[h_m(z_1) - h_m(z_2) \right]^{s-r-1} \left[F(z_2) \right]^{\gamma_s - 1} dz_2 dz_1$$
$$+ \frac{q C_{s-1}}{\alpha \gamma_s (r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} z_1^p z_2^{q-1} f(z_1) f(z_2) \left[F(z_1) \right]^m$$
$$\times g_m^{r-1} \left[F(z_1) \right] \left[h_m(z_1) - h_m(z_2) \right]^{s-r-1} \left[F(z_2) \right]^{\gamma_s - 1} dz_2 dz_1$$
$$-\lambda \sum_{j=0}^{\infty} \frac{(-2)^j \alpha^j}{j!} \frac{q C_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{z_1} z_1^p z_2^{q+j-1} f(z_1)$$
$$\times \left[F(z_1) \right]^m g_m^{r-1} \left[F(z_1) \right] \left[h_m(z_1) - h_m(z_2) \right]^{s-r-1}$$
$$\times \left[F(z_2) \right]^{\gamma_s - 1} dz_2 dz_1$$

Again utilizing (13), and re–arranging, we get

$$\mu_{r,s:n,m,k}^{p,q}=\mu_{r,s-1:n,m,k}^{p,q}+\frac{\gamma_{s(k-1)}C_{s-1}}{\gamma_s C_{s-1(k-1)}}\left(\mu_{r,s:n,m,k-1}^{p,q}-\mu_{r,s-1:n,m,k-1}^{p,q}\right)+\frac{q}{\alpha\gamma_s}\mu_{r,s:n,m,k}^{p,q-1}$$
$$+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}\frac{q\gamma_{s(k-1)}C_{s-1}}{(q+j)\gamma_s C_{s-1(k-1)}}\left(\mu_{r,s:n,m,k-1}^{p,q+j}-\mu_{r,s-1:n,m,k-1}^{p,q+j}\right),$$

which is (12) and hence the theorem.

The RRs for product moments of DGOS for Ex model can indeed be readily investigated from (12) by utilizing $\lambda = 0$.

Some corollaries which immediately follow from Theorem 2 are given below.

Corollary 3: Utilizing m = –1 in (12), the relation for product moments of lower REVs for TEx model is computed as

$$\mu_{K(r,s)}^{p,q}=\mu_{K(r,s-1)}^{p,q}+\left(\frac{k}{k-1}\right)^{s-1}\left(\mu_{K-1(r,s)}^{p,q}-\mu_{K-1(r,s-1)}^{p,q}\right)+\frac{q}{\alpha(n-s+1)}\mu_{K(r,s)}^{p,q-1}$$
$$+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}\frac{q}{(q+j)}\left(\frac{k}{k-1}\right)^{s-1}\left(\mu_{K-1(r,s)}^{p,q+j}-\mu_{K-1(r,s-1)}^{p,q+j}\right). \qquad (13)$$

The relations for product moments of lower REVs for Ex model can indeed be readily computed from (13) by utilizing $\lambda = 0$.

Corollary 4: Utilizing m = 0 and k = 1 in the formula (12), the relation for product moments of reversed OS for TEx model is computed as

$$\mu_{r,s:n}^{p,q}=\mu_{r,s-1:n}^{p,q}+\frac{n+1}{n-s+1}\left(\mu_{r,s:n}^{p,q}-\mu_{r,s-1:n}^{p,q}\right)+\frac{q}{\alpha(n-s+1)}\mu_{r,s:n}^{p,q-1}$$
$$+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}\frac{q(n+1)}{(q+j)(n+s+1)}\left(\mu_{r,s:n}^{p,q+j}-\mu_{r,s-1:n}^{p,q+j}\right). \qquad (14)$$

The relations for product moments of reversed OS for Ex model can indeed be easily computed from (14) by utilizing $\lambda = 0$.

5. Characterizations

In this part, we shall characterize the TEx model through utilizing single and product moments of DGOS. The accompanying theorems provide these characterizations.

Theorem 3: The moments of a RV Z are connected as follows, which is both a necessary and sufficient requirement for it to have pdf and cdf (4) and (5), respectively.

$$\mu_{r:n,m,k}^{p}=\mu_{r-1:n,m,k}^{p}+\frac{\gamma_{r(k-1)}C_{r-1}}{\gamma_r C_{r-1(k-1)}}\left(\mu_{r:n,m,k-1}^{p}-\mu_{r-1:n,m,k-1}^{p}\right)+\frac{p}{\alpha\gamma_r}\mu_{r:n,m,k}^{p-1}$$
$$+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}\frac{p\gamma_{r(k-1)}C_{r-1}}{(p+j)\gamma_r C_{r-1(k-1)}}\left(\mu_{r:n,m,k-1}^{p+j}-\mu_{r-1:n,m,k-1}^{p+j}\right).$$

Proof: The required condition is easily derived from Theorem 1. Assume (7) and (9) for the adequate condition, and therefore

$$-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} z^{p-1}\left[F(z)\right]^{\gamma_r} g_m^{r-1}\left[F(z)\right]dz=-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} z^{p-1}\left[F(z)\right]^{\gamma_r-1}$$
$$\times\left\{1-\frac{1}{\alpha}f(z)+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}z^j\right\}g_m^{r-1}\left[F(z)\right]dz$$

or
$$-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} z^{p-1}\left[F(z)\right]^{\gamma_r-1} g_m^{r-1}\left[F(z)\right]\left[F(z)-\left\{1-\frac{1}{\alpha}f(z)+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}z^j\right\}\right]dz=0$$

Utilizing Müntz–Száz theorem; see [18]; to previous formula we get

$$F(z)=1-\frac{1}{\alpha}f(z)+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}z^j,$$

where is (7) and this relationship exists among pdf and cdf of a TEx model, therefore the theorem.

Theorem 4: A necessary and sufficient requirement for a RV Z to have pdf and cdf (4) and (5), accordingly, is that the product moments of its DGOS are connected as

$$\mu_{r,s:n,m,k}^{p,q}=\mu_{r,s-1:n,m,k}^{p,q}+\frac{\gamma_{s(k-1)}C_{s-1}}{\gamma_s C_{s-1(k-1)}}\left(\mu_{r,s:n,m,k-1}^{p,q}-\mu_{r,s-1:n,m,k-1}^{p,q}\right)+\frac{q}{\alpha\gamma_s}\mu_{r,s:n,m,k}^{p,q-1}$$
$$+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}\frac{q\gamma_{s(k-1)}C_{s-1}}{(q+j)\gamma_s C_{s-1(k-1)}}\left(\mu_{r,s:n,m,k-1}^{p,q+j}-\mu_{r,s-1:n,m,k-1}^{p,q+j}\right).$$

Proof: The required component follows shortly after Theorem 2. We think (13) to be adequate.

$$\mu_{r,s:n,m,k}^{p,q}-\mu_{r,s-1:n,m,k}^{p,q}=-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{z_1} z_1^p z_2^{q-1} f(z_1)\left[F(z_1)\right]^m$$
$$\times g_m^{r-1}\left[F(z_1)\right]\left[h_m(z_1)-h_m(z_2)\right]^{s-r-1}\left[F(z_2)\right]^{\gamma_s}dz_2dz_1.$$

Utilizing previous formula with (7) we get

$$=-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^c\int_0^{z_1} z_1^p z_2^{q-1} f(z_1)\left[F(z_1)\right]^m g_m^{r-1}\left[F(z_1)\right]\left[h_m(z_1)-h_m(z_2)\right]^{s-r-1}$$
$$\times\left[F(z_2)\right]^{\gamma_s}dz_2dz_1$$
$$=-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^c\int_0^{z_1} z_1^p z_2^{q-1} f(z_1)\left[F(z_1)\right]^m g_m^{r-1}\left[F(z_1)\right]$$
$$\times\left[h_m(z_1)-h_m(z_2)\right]^{s-r-1}\left[F(z_2)\right]^{\gamma_s-1}\left\{1-\frac{1}{\alpha}f(z_2)+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}z_2^j\right\}dz_2dz_1$$

or

$$-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^c\int_0^{z_1} z_1^p z_2^{q-1} f(z_1)\left[F(z_1)\right]^m g_m^{r-1}\left[F(z_1)\right]\left[F(z_2)\right]^{\gamma_s-1}$$

$$\times\left[h_m(z_1)-h_m(z_2)\right]^{s-r-1}\left[F(z_2)-\left\{1-\frac{1}{\alpha}f(z_2)+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}z_2^j\right\}\right]dz_2dz_1=0$$

Utilizing Müntz–Száz theorem; see [18]; to previous formula we get

$$F(z_2)=1-\frac{1}{\alpha}f(z_2)+\lambda\sum_{j=0}^{\infty}\frac{(-2)^j\alpha^j}{j!}z_2^j,$$

where is (7) and this relationship exists among pdf and cdf of a TEx model, therefore the theorem.

6. Concluding Remarks

In this study, we found the RRs for single and product moments of DGOS for a TEx model, and the relations for specific situations. These relations are useful to recursively compute the higher order moments from the lower order moments. Some characterizations of the transmuted are also given for a TEx model on the basis of single and product moments of DGOS.

References

1. Ahsanullah, M. and Nevzorov, V. B. (2001) Ordered Random Variables, Nova Science Publishers, USA.
2. Al–Sobhi, M., Hanif Shahbaz, S. and Sherwani, R. A. K. (2021) Relations for Moments of Generalized Order Statistics for Transmuted Exponential Distribution and Characterization, Pakistan Journal of Statistics and Operation Research, 17(1), 43–50.
3. Al–Sobhi, M., Hanif Shahbaz, S. and Shahbaz, M. Q. (2020). Recursive Computation for Moments of Order Statistics for Transmuted Exponential distribution, Advances and Applications in Statistics, 63(1), 175–189.
4. Arnold, B. C., Balakrishnan, N., & Nagaraja, H. N. (2008). A first course in order statistics. Society for Industrial and Applied Mathematics.
5. Athar, H. and Faizan, M. (2011). Moments of lower generalized order statistics for Power Function distribution and a characterization, International Journal of Statistics and Data Science, 11 (1), 125-134.
6. Burkschat, M., Cramer, E. and Kamps, U. (2003). Dual generalized order statistics. Metron, 61(1), 13–26.
7. Chandler, K. (1952). The distribution and frequency of record values. Journal of the Royal Statistical Society: Series B (Methodological), 14(2), 220-228.
8. Cramer, E., & Kamps, U. (2000). Relations for expectations of functions of generalized order statistics. Journal of Statistical Planning and Inference, 89(1-2), 79-89.

9. Daghistani, A. M., Al–Zahrani, B. and Shahbaz, M. Q. (2019) Relations for Moments of Dual Generalized Order Statistics for a new Inverse Kumaraswamy Distribution, Pakistan Journal of Statistics and Operation Research, 15(4), 989–997.
10. Dziubdziela, W., & Kopociński, B. (1976). Limiting properties of the k-th record values. Applicationes Mathematicae, 2(15), 187-190.
11. Hwang, J. S., & Lin, G. D. (1984). Extensions of Müntz-Szász theorem and applications. Analysis, 4(1-2), 143-160.
12. Khan, R. U., Anwar, Z. and Athar, H. (2008). Recurrence relations for single and product moments of dual generalized order statistics from exponentiated Weibull distribution. Aligarh Journal of Statistics, 28(1), 37–45.
13. Kotb, M. S., El-Din, M. M. M. and Newer, H. A. (2013). Recurrence relations for single and product moments of lower generalized order statistics from a general class of distributions and its characterizations, International Journal of Pure and Applied Mathematics, 87(2), 229–247.
14. Owoloko, E. A., Oguntunde, P. E., & Adejumo, A. O. (2015). Performance rating of the transmuted exponential distribution: an analytical approach. SpringerPlus, 4(1), 1-15.
15. Pawlas, P., and Szynal, D. (2001). Recurrence relations for single and product moment of lower generalized order statistics from Inverse Weibull distribution, Demonstratio Mathematica, 34(2), 353–358.
16. Rahman, M. M., Al–Zahrani, B., Hanif Shahbaz, S. and Shahbaz, M. Q. (2020) Transmuted Distributions: A Review, Pakistan Journal of Statistics and Operation Research, 16(1), 83–94.
17. Shahbaz, M. Q., Ahsanullah, M., Shahbaz, S. H., & Al-Zahrani, B. M. (2016). Ordered Random Variables: Theory and Applications.
18. Shaw, W. T. and Buckley, I. R. C. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. Research report.
19. Elbatal, I. A., Helal, T. S., Elsehetry, A. M., & Elshaarawy, R. S. (2022). Topp-Leone Weibull Generated Family of Distributions with Applications. Journal of Business and Environmental Sciences, 1(1), 183-195.

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