Computational Journal of Mathematical and Statistical Sciences
1(1), 51–62
DOI:10.21608/cjmss.2022.272724
https://cjmss.journals.ekb.eg/

Research article

Order Statistics of Inverse Pareto Distribution

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Abstract: In this paper we study the distribution of order statistics of the inverse Pareto distribution. We consider the single and product moment of order statistics from inverse Pareto distribution. Also, we establish some recurrence relation for single moments of order statistics. The exact analytical expressions of entropy, residual entropy and past residual entropy for order statistics of inverse Pareto distribution is derived.

Keywords: Inverse Pareto distribution; Order statistics; Moments; Entropy; Past residual entropy.
Mathematics Subject Classification: 60E05; 62E10; 62E15.

Received: 4 October 2022; **Revised:** 7 November 2022; **Accepted:** 30 November 2022; **Published:** 5 December 2022.

1. Introduction

Order statistics have been used in wide range of problems, including robust statistical estimation and detection of outliers, characterization of probability distribution, goodness of fit-tests, quality control, and analysis of censored sample. The use of recurrence relations for the moments of order statistics is quite well known in statistical literature (see for example Arnold et al. [1], Malik et al. [2]. For improved form of these results, Samuel and Thomas [3], Arnold et al. [1], and Ali and Khan [4] have reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, Cauchy, logistic, gamma and exponential. Very recently, Dar and Abdullah [5] have studied the sampling distribution of order statistics of the two parametric Lomax distribution and derived the exact analytical expressions of entropy, residual entropy and past residual entropy for order statistics of Lomax distribution.

2. Inverse Pareto distribution

(See Lei Guo and Wenhao Gui [6]) A random variable X with range of values $(0, \infty)$ is said to have the inverse Pareto distribution, if its probability density function (pdf) is given by

$$f(x) = \frac{\alpha x^{\alpha-1}}{(1+x)^{\alpha+1}}, x, \alpha \geq 0. \tag{2.1}$$

Here α is the shape parameter. Inverse Pareto distribution holds both decreasing and upside-down bathtub shape hazard rate. The above distribution is monotonically decreasing and highly skewed to the right. The cumulative distribution function (cdf) and survival function (sf) associated with Equation (2.1) is given, respectively, by

$$F(x) = \left(\frac{x}{x+1}\right)^{\alpha}, \tag{2.2}$$

$$\overline{F}(x) = 1 - \left(\frac{x}{x+1}\right)^{\alpha}. \tag{2.3}$$

Noting that all inverse distributions possess the upside-down bathtub shape for their hazard rates, we consider an inverted version of the Pareto distribution that can be effectively used to model the upside-down bathtub shape hazard rate data. Inverse distributions are special cases of the class of ratio distributions, in which the numerator random variable has a degenerate distribution. If the random variable (r.v.) T has Pareto distribution, then the r.v. X = 1/T has an inverse Pareto distribution (IPD) inverse Pareto distribution was used to describe demand for the pay-as-bid auctions model in Holmberg [7]. Naghettini et al. (1996) [8] estimated the upper tail of flood peak frequency distribution by the inverse Pareto distribution. For other applications, see Wildani et al. [9].

The following functional relationship exists between the pdf and cdf of the inverse Pareto distribution (IPD)

$$f(x) = \frac{\alpha}{x(x+1)} F(x) \tag{2.4}$$

3. Distribution of Order Statistics

Let $X_1, X_2, X_3, \ldots\ldots\ldots, X_n$, be a random sample of size from the inverse Pareto distribution and let $X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq \ldots\ldots\ldots \leq X_{n:n}$ denotes the corresponding order statistics. Then the pdf of $X_{r:n} 1 \leq r \leq n$ is given by [see David and Nagaraja [10] and Arnold et al. [1].

$$f_{r:n}(x) = C_{r:n} \times [F(x)]^{r-1} \times [1 - F(x)]^{n-r} \tag{3.1}$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$ The probability density functions of smallest (r = 1) and largest (r = n) order statistics can be easily obtained from Equation (2.1) and are given, respectively, by

$$f_{1:n}(x) = n[1 - F(x)]^{n-1} f(x)$$

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x)$$

Using Equations (2.1) and (2.2), and taking r = 1 in Equation (3.1) yields the pdf of the minimum order statistics for the inverse pareto distribution

$$f_{1:n}(x) = n\alpha \sum_{i=0}^{n-1} \sum_{j=0}^{\infty} \binom{n-1}{i} (-1)^{i+j-1} j(x)^{\alpha(i+1)+j-2}$$

Similarly using Equations (2.1) and (2.2), and taking r = n in Equation (3.1) yields the pdf of the largest order statistics for the inverse Pareto distribution

$$f_{n:n}(x) = n\alpha \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{i-1} i(x)^{n\alpha+j-2} \quad (3.2)$$

The joint pdf of $X_{r:s}$ and $X_{r:s}$ for $1 \leq r \leq s \leq n$ is given by (see Arnold et al. [1])

$$f_{r:s:n}(x) = C_{r:s:n} \times [F(x)]^{r-1} \times [F(y) - F(x)]^{s-r-1} \times [1 - F(y)]^{n-s} \quad (3.3)$$

for $-\infty \leq x \leq y \leq +\infty$ where $C_{r:s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$

Following two theorems gives the distribution of the order statistics from the distribution.

3.1 Probability density function of rth order statistics of inverse Pareto distribution

Let $f(x)$ and $F(x)$ be the cdf and pdf of the inverse Pareto distribution. Then the density function of the r^{th} order statistics say $f_{r:n}(x)$ is given by

$$f_{r:n}(x) = n\alpha \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} \binom{n-r}{i} (-1)^{i+j-1} j(x)^{\alpha(r+i)+j-2} \quad (3.4)$$

Proof: First it should be noted that equation (3.1) can be written as

$$f_{r:n}(x) = C_{r:n} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^{i} [F(x)]^{r+i-1} f(x) \quad (3.5)$$

The proof follows by substituting equations (2.1) and (2.2) into equation (3.4).

3.2 Joint Probability density function of order statistics of inverse Pareto distribution

Let $X_{r:n}$ and $X_{s:n}$ for $1 \leq r \leq s \leq n$ be the r^{th} and s^{th} order statistics from the inverse Pareto distribution. Then the joint pdf of $X_{r:n}$ and $X_{s:n}$ is given by

$$f_{r:s:n}(x) = C_{r:s:n} \sum_{i=0}^{s-r-1} \sum_{j=0}^{n-s} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{i+j+k+l-2} (kl) x^{\alpha(r+i)+k-2} y^{\alpha(s-r-i+j)+l-2} \quad (3.6)$$

Proof: Another form of representing equation (3.3) is as follows:

$$f_{r:s:n}(x)=C_{r:s:n}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}\binom{s-r-1}{i}\binom{n-s}{j}(-1)^{i+j}[F(y)]^{s-r-1-i+j}[F(x)]^{r+i-1}f(x)f(y)$$

The proof immediately follows by substituting equations (2.1) and (2.2) into equation (3.6).

4. Single and Product Moments

In this section, we derive explicit expressions for both of the single and product moments of order statistics from the inverse Pareto distribution.

4.1 Single moment

Let $X_1, X_2, X_3, \ldots\ldots. X_n$ be a random sample of size n from the distribution and let $X_{1:n}, X_{2:n}, X_{3:n} \ldots\ldots. X_{n:n}$ denote the corresponding order statistics. Then k^{th} the moments of the r^{th} order statistics for $k = 1,2 \ldots..$ denoted by $\mu_{r:n}^{(k)}$ is given by

$$\mu_{r:n}^{(k)}=\alpha\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^{i}\frac{\beta(\alpha r+\alpha i+k;k-1)}{\beta(r;n-r+1)} \tag{4.1}$$

where $\beta(..;..)$ is the beta function.
Proof: We know that

$$\mu_{r:n}^{(k)}=\int_0^{\infty}x^k f_{r:n}(x)dx$$

$$\mu_{r:n}^{(k)}=C_{r,n}\int_0^{\infty}x^k\,[F(x)]^{r-1}\times[1-F(x)]^{n-r}\times f(x)dx \tag{4.2}$$

Now substituting Equations (2.1) and (2.2) into Equation (4.2), yields Equation (4.1). the Equation (4.1) can be exploited to drive the mean and the variance of the rth order statistics. For example, when k = 1 we can obtain the mean of the rth order statistics as follows:

$$\mu_{r:n}^{(1)}=\alpha C_{r,n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^{i}$$

For k = 2, one can get the second order moment of the rth order statistics as

$$\mu_{r:n}^{(2)}=\alpha C_{r,n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^{i}\frac{1}{\alpha r+\alpha i+2}$$

Therefore, the variance of the rth order statistics can be obtained easily by using the relation

$$Var(X_{r:n})=\mu_{r:n}^{(2)}-\left[\mu_{r:n}^{(1)}\right]^2$$

$$= \alpha^2 \left[C_{r,n} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \frac{1}{\alpha r + \alpha i + 2} - C_{r,n} \left(\sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \right)^2 \right]$$

Similarly the third and fourth order moments of the rth order statistic, $\mu_{r:n}^{(3)}$ and $\mu_{r:n}^{(4)}$, can be obtained in similar ways. The mean, variance and other statistical measures of the extreme order statistics are always of great interest. Taking r = 1, one can obtain the mean of smallest order statistics:

$$\mu_{1:n}^{(1)} = \frac{\alpha}{n} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i$$

Also, second order moment of the smallest order statistics can be obtained as follows:

$$\mu_{1:n}^{(2)} = \frac{\alpha}{n} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \frac{1}{(\alpha + i) - 2}$$

Therefore, the variance of the smallest order statistics is

$$Var(X_{1:n}) = \mu_{1:n}^{(2)} - \left[\mu_{1:n}^{(1)} \right]^2$$

$$= \left(\frac{\alpha}{n}\right) \left[\sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \frac{1}{\alpha(i-1) - 2} - \frac{\alpha}{n} \left(\sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \right)^2 \right]$$

Similarly, the mean, the second order moment and hence the variance of the largest order statistics (r = n) is given by

$$\mu_{n:n}^{(1)} = \frac{\alpha}{n}$$

and

$$\mu_{n:n}^{(1)} = \frac{\alpha}{n(n\alpha - 2)}$$

Therefore, the variance of the largest order statistics is

$$Var(X_{n:n}) = \mu_{n:n}^{(2)} - \left[\mu_{n:n}^{(1)} \right]^2$$

$$Var(X_{n:n}) = \frac{\alpha}{n} \left[\frac{1}{n\alpha - 2} - \frac{\alpha}{n} \right]$$

4.2 Recurrence relation for single moments:

Let $X_1, X_2, X_3, \ldots\ldots. X_n$ be a random sample of size n from inverse Pareto distribution and let $X_{1:n}, X_{2:n}, X_{3:n} \ldots\ldots. X_{n:n}$ denote the corresponding order statistics. Then $1 \le r \le n$ we have the following moment relation:

$$\mu_{r:n}^{(k)} = \sum_{i=0}^{\infty} (-1)^{i-1} \frac{r}{k-i-1} \left[\mu_{r+1:n}^{(k-i-1)} - \mu_{r:n}^{(k-i-1)}\right] \tag{4.3}$$

Proof: Using equations (2.4) and (4.2) gives

$$\mu_{r:n}^{(k)} = C_{r:n} \int_{i=0}^{\infty} x^k \, [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x) dx$$

$$\mu_{r:n}^{(k)} = C_{r:n} \sum_{i=0}^{\infty} (-1)^{i-1} \int_{i=0}^{\infty} x^{k-i-2} \, [F(x)]^{r} [1 - F(x)]^{n-r} dx$$

By using integration by parts, we easily obtain the desired result.

4.3 Recurrence relation for Product moments:

for $1 \le r \le s \le n \le$, and $n \in N$, we have

$$\mu_{r:s:n}^{(k_1,k_2)} = \alpha^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j-2} \frac{r}{(k_1 - i - 1, k_2 - j - 1)} \tag{4.4}$$
$$\left[n\left\{\mu_{r:s-1:n-1}^{(k_1+i,k_2+j)} - \mu_{r:s:n-1}^{(k_1+i,k_2+j)}\right\} - (n-s-1)\left\{\mu_{r:s-1:n}^{(k_1+i,k_2+j)} - \mu_{r:s:n}^{(k_1+i,k_2+j)}\right\}\right]$$

Proof: We start by noting that

$$\mu_{r:s:n}^{(k_1,k_2)} = C_{r:s:n} \int_0^{\infty} \int_x^{\infty} x^{k_1} y^{k_2} \times [F(x)]^{r-1} \times [F(y) - F(x)]^{s-r-1} \times [1 - F(y)]^{n-s} f(x) f(y) dy dx$$

or

$$\mu_{r:s:n}^{(k_1,k_2)} = C_{r:s:n} \int_0^{\infty} x^{k_1} \, [F(x)]^{r-1} f(x) I_X dx \tag{4.5}$$

where

$$I_X = \int_x^{\infty} y^{k_2} \, [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(y) dy$$

Applying equation (2.4) gives

$$I_X = \alpha \sum_{j=0}^{\infty} (-1)^{j-1}$$

$$\left[\int_x^{\infty} y^{k_2+j-2} [F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} - \int_x^{\infty} y^{k_2+j-2} [F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s+1}\right]$$

Now, integrating by parts and then substituting I_X into equation (4.5) gives directly the desired result.

5. Entropy

(see Shannon [11]) An entropy of a continuous random variable X with density function $f_X(x)$ is defined as

$$H(X) = -\int_0^{\infty} f_X(x) log f_X(x) \tag{5.1}$$

Analytical expression for univariate distribution is discussed in references such as Laz and Rathie [12], Nadarajah and Zagrafos [13]. Also, the information properties of order statistics have been studied by Wong and Chen [14], Park and Ebrahimi et al. [15]. The measure given in (5.1) is not suitable for measuring the uncertainty of a component with information only about its current age. A more realistic approach which make the use of the age into account is described by Ebrahimi et al. [15], which is defined as follows:

$$H(X) = -\int_t^{\infty} \frac{f(x)}{\overline{F}(t)} log \frac{f(x)}{\overline{F}(t)} \tag{5.2}$$

It is obvious that for $t = \infty$, Equation (5.2) is reduced to Equation (5.1). In many realistic situation uncertainty is not necessarily related to future but can also refer to past. Based on this idea, Crescenzo and Longobardi [16] develop the concept of past entropy over (0, t)

5.1 Entropy based order statistics

Let $X_1, X_2, X_3, \ldots\ldots . X_n$ be a random sample of size n from a distribution $F_X(x)$ with density function $f(x)$ and let $Y_1 \leq Y_2 \leq \ldots\ldots . \leq Y_n$ denote the corresponding order statistics. Then the pdf of Y_r, $1 \leq r \leq n$, is given by

$$f_{Y_r}(y) = C_{r:n} \times [F_X(y)]^{r-1} \times [1 - F_X(y)]^{n-r}, 0 \leq x \leq \infty$$

for $C_{r:n} = \frac{1}{\beta(r,n-r+1)} = \frac{n!}{(r-1)!(n-r)!}$ and $\beta(.\,.,\ldots)$ is the beta function as before.

Further, let U is the uniform distribution defined over the unit interval. The order statistics of a sample taken randomly from uniform distribution $U_1, U_2, U_3, \ldots\ldots . U_n$ are denoted by $W_1 \leq W_2 \leq \ldots\ldots . \leq W_n$. The random variable W_r, $r = 1,2,\cdots,n$ has a Beta distribution with density function

$$g_r(w) = C_{r:n}[w]^{r-1}(1 - w)^{n-r}, 0 \leq w \leq 1$$

In the following subsections, we derive the exact form of entropy, residual entropy and past residual entropy for the inverse Pareto distribution based on order statistics.

5.2 Entropy

Using the transformation $W_r = F_X(Y_r)$, the entropies of order statistics can be computed by

$$H(Y_r) = H_n(Y_r) - E_{g_r}[\log f_X(F_X^{-1}(W_r)] \tag{5.3}$$

where f_X probability density is function of the random variable X and $H_n(W_r)$ denote the entropy of the beta distribution and is given by

$$\begin{aligned} H_n(W_r) = \log\beta(r, n-r+1) - (r-1)[\psi(r) - \psi(n+1)] \\ - (n-r)[\psi(n-r+1) - \psi(n+1)] \end{aligned} \tag{5.4}$$

where ψ is the digamma function and is defined by $\psi(\theta) = \left(\frac{d}{d\theta}\right)\log\Gamma(\theta)$.

Remark 4.1: For r = 1, i.e smallest order statistics and for r = n, i.e largest order statistics, it can be easily shown that

$$H_n(W_1) = H_n(W_n) = 1 - \log(n) - \frac{1}{n} \tag{5.5}$$

Remark 4.2: It should be noted that $\psi(n+1) - \psi(n) = \frac{1}{n}$.

Theorem 4.1

Let $X_1, X_2, X_3 \ldots\ldots\ldots X_n$, be a random sample of size n from inverse Pareto distribution with a distribution function given in equation (2.2) and let $Y_1 \le Y_2 \le \ldots\ldots \le Y_n$ denote the corresponding order statistics. Then the entropy of r^{th} order statistics of inverse Pareto distribution is given by

$$\begin{aligned} H_n(W_r) = \log\beta(r, n-r+1) - (r-1)[\psi(r) - \psi(n+1)] \\ - (n-r)[\psi(n-r+1) - \psi(n+1)] \\ \left[\log\alpha + \left(1 - \frac{1}{\alpha}\right)[\psi(r) - \psi(n+1)]\right] \\ + \frac{n!}{(r-1)!}\sum_{i=0}^{n-r}\frac{(-1)^i}{(r+i)(n-r-i)!\,(i-1)!}[\psi(1) - \psi(\alpha r + \alpha i + 1)] \\ - \frac{n!}{(r-1)!}\sum_{i=0}^{n-r}\frac{(-1)^i}{(r+i)(n-r-i)!\,(i-1)!\,2^{\alpha(r+i)}}[\psi(1) - \psi(\alpha r + \alpha i + 1)] \end{aligned} \tag{5.6}$$

Proof: Using equation (2.2) and the probability integral transformation $Yr = F^{-1}(W_r)$, one can easily arrive at

$$F^{-1}(W_r) = \frac{\alpha W_r^{1-\frac{1}{\alpha}}\left(W_r^{\frac{1}{\alpha}} - 1\right)}{\left(2W_r^{\frac{1}{\alpha}} - 1\right)}$$

Therefore, after applying equation (5.3) we get the following:

$$E_{g_r}[\log f_X(F_X^{-1}(W_r)] = \left[\log\alpha + \left(1-\frac{1}{\alpha}\right)[\psi(r)-\psi(n+1)]\right] + \frac{n!}{(r-1)!}\sum_{i=0}^{n-r}\frac{(-1)^i}{(r+i)(n-r-i)!\,(i-1)!}[\psi(1)-\psi(\alpha r+\alpha i+1)] - \frac{n!}{(r-1)!}\sum_{i=0}^{n-r}\frac{(-1)^i}{(r+i)(n-r-i)!\,(i-1)!\,2^{\alpha(r+i)}}[\psi(1)-\psi(\alpha r+\alpha i+1)] \quad (5.7)$$

Corollary 4.1: For r = 1, i.e smallest order statistics, we have

$$H(Y_1) = 1-\frac{1}{n}-\log n-\log\alpha+\left(1-\frac{1}{\alpha}\right)[\psi(n+1)+\gamma] - \sum_{i=0}^{n-1}\frac{(-1)^i}{(i+1)}[\psi(\alpha+\alpha i+1)+\gamma]\left[1-\frac{1}{2^{\alpha(r+i)}}\right]$$

where $-\psi(1)=\gamma=0.5772$ is the Eulers constant.
Corollary 4.2: For r = n, i.e largest order statistics, we have

$$H(Y_n) = 1-\frac{1}{\alpha n}-\log n\alpha-\frac{1}{(n+i)}[\psi(\alpha n+\alpha i+1)+\gamma]\left[1-\frac{1}{2^{\alpha(n+i)}}\right]$$

5.3 *Past Entropy*

If X denote the lifetime of a component, then the past entropy of X is defined by

$$H^0(X) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)}\, log\frac{f(x)}{\bar{F}(t)} \quad (5.8)$$

It is obvious that for t = 0, Equation (5.8) is reduced to Equation (5.11).

Clearly the residual entropy of first order statistics is obtained by substituting $r=1$ and using the probability integral transformation $U=F_X(x)$ in Equation (5.7). Then, we have

$$H(X_{1,n};t) = \frac{n-1}{n}-\log n + \log[\bar{F}(t)] - \frac{n}{\bar{F}^n(t)}\int_{\bar{F}(t)}^{1}(1-u)^{n-1}\log[f\{F^{-1}(u)\}]\,du \quad (5.9)$$

The residual entropy of the first order statistics for inverse Pareto distribution can be easily obtained by using Equations (1.1), (1.2), and (1.3), and then put $f(F^{-1}(u)) = \frac{\alpha W_r^{1-\frac{1}{\alpha}}\left(W_r^{\frac{1}{\alpha}}-1\right)}{\left(2W_r^{\frac{1}{\alpha}}-1\right)}$ into Equation (5.8).

$$H(X_{1,n;t}) = \frac{n}{n-1} - \log n\alpha + \log[\bar{F}(t)] + \left(1-\frac{1}{\alpha}\right)\left[\log F(t) - \sum_{i=0}^{n}(-1)^i\{F(t)\}^{i-n}\right]$$
$$+\log\left[F^{\frac{1}{\alpha}}(t)-1\right] - \alpha\log\left[2F^{\frac{1}{\alpha}}(t)-1\right] - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{n}{i}(-1)^{i+j}\frac{i\,[F(t)]^{i+\frac{j}{\alpha}-n}}{\alpha i + j}[1+2^i] \quad (5.10)$$

where $F(t)$ and $\bar{F}(t)$ are the cumulative distribution function and survival function for inverse Pareto distribution given by Equations (1.2) and (1.3) respectively. The case for r = n follows on similar lines.

5.4 Past Residual Entropy

Analogous to relation (5.2), the past residual entropy of the rth $f(F^{-1}(u))$ order statistics is defined as

$$H^0(X_{r,n;t}) = -\int_0^t \frac{f_{r,n}(x)}{F_{r,n}(t)}\log\frac{f_{r,n}(x)}{F_{r,n}(t)} \quad (5.11)$$

The past residual entropy of nth order statistics is obtained by substituting $r = n$ and using the probability integral transformation $U = F_X(x)$ in Equation (2.7), we have

$$H^0(X_{n,n;t}) = \frac{n-1}{n} - \log n + \log[F(t)]$$
$$-\frac{n}{F^n(t)}\int_t^{\infty} u^{n-1}\log[f(F^{-1}(u))]du \quad (5.12)$$

The past residual entropy of the nth order statistics for inverse Pareto distribution can be easily obtained by using (1.1), (1.2), (1.3) and $f(F^{-1}(u)) = \frac{\alpha W_r^{1-\frac{1}{\alpha}}\left(W_r^{\frac{1}{\alpha}}-1\right)}{\left(2W_r^{\frac{1}{\alpha}}-1\right)}$ in Equation (5.12).

$$H^0(X_{n,n;t}) = \frac{n-1}{n} - \log n\alpha + \log[F(t)] - \left(1-\frac{1}{\alpha}\right)\left[\log F(t) - \frac{1}{n}\right]$$
$$+\log\left[F^{\frac{1}{\alpha}}(t)-1\right] - \alpha\log\left[2F^{\frac{1}{\alpha}}(t)-1\right] - \sum_{i=0}^{\infty}(-1)^i\frac{[F(t)]^{\frac{i}{\alpha}}}{\alpha n + i}[1+2^i]$$

The case for r = 1 follows on similar lines.

6. Conclusion

In this paper we study the sampling distribution from the order statistics of inverse Pareto distribution. Also, we consider the single and product moment of order statistics from inverse Pareto distribution. We establish recurrence relation for single moments of order statistics. Also, we have derived the entropy, residual and past residual entropies for order statistics of the inverse Pareto distribution.

Conflict of Interests

The authors declare that there is no conflict of interests.

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