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Exact and Approximate to Exact Methods for Solution Linear Boundary Value Problems Using Laplace Transform

Hussein Abd Allah Soliman[1,a]

[1] Department of Basic Science, Engineering Division,
International Academy for Engineering and Media Science, 6th October, Egypt

E-mail: [a]Hussein.AbdAllah.Soliman@iaems.edu.eg

Abstract

It is a new method, a mixture of numerical and exact methods, each of which has a role to obtain the solution. It is known that the Laplace transforms method gives a closed-form for initial value problems but in the present study, we were able to use it with the aid of high-accurate numerical methods to solve linear boundary value problems. The novelty of the present method that it is converted Linear Boundary Value Problems to initial Value Problems using accurate numerical methods and then uses Laplace transforms method to find approximate to the exact solution. Approximate to Exact Method (AEM) is an algorithm, with a very strong accuracy that approaches the exact solution because it is a mixture of numerical methods of very high accuracy with a closed-form method. The uniqueness, convergence, and stability of the new technique are verified and tested by comparisons with a fourth-order accurate finite difference (FOFDM) solution.

Keywords: Exact method, AEM, BVP, Laplace Transform

1. Introduction

Two-point boundary value problems have received considerable attention due to their importance in many areas of sciences and engineering. These types of differential equations arise very frequently in fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process, and geophysics [30].

The most popular technique for solving differential equations numerically is the Runge-Kutta method. Three new Runge–Kutta methods are presented for numerical integration of systems of linear inhomogeneous ordinary differential equations (ODEs) with constant coefficients. Such ODEs arise in the numerical solution of partial differential equations governing linear wave phenomena [31]. A novel second-order prediction differential model is designed, and numerical solutions of this novel model are presented using the integrated strength of the Adams and explicit Runge–Kutta schemes [32]. A new special two-derivative Runge-Kutta type (STDRKT) method involves the fourth derivative of the solution for solving third-order ordinary differential equations [33].

The finite difference method proposed for the solution of two-point boundary value problems has been widely applied [24-26]. They used the finite difference method (FDM) of second-order accuracy to solve the nonlinear system of differential equations. they observed that the velocity reached the steady-state faster than temperature and nanoparticles concentration. Attia et al. [34] studied the effects of the-Drcian Forchheimer and Hall current resistances on the unsteady flow and heat transfer between two porous plates. they solved the governing partial differential equations, numerically, by the finite difference method FDM. Joule and viscous dissipations are considered in the energy equation. Ewis [35] used a second-order accurate finite difference method to solve the governing equations of natural convection of non- Newtonian (RivlinEricksen) fluid flow and heat transfer under the influences of non-Darcy resistance force, constant pressure gradient, dissipation, and radiation.

Various analytical and numerical techniques proposed for the solution of differential equations are available in the literature; some of these are Differential Transform Method [1-6], Rung-Kutta 4th Order Method [7], Bernoulli Polynomials [8], Cubic Spline Method [9], Sinc Collocation Method [10], Modified Picard Technique [11], Block Method [12-14], Adomian Decomposition Method [15-20], Homotopy Perturbation Method [21-23].

In this work, presented a new method to solve linear boundary value problems using Laplace Transform. The new method presents a solution for linear boundary value problems very approximate to the exact solution, so named Approximate to Exact Method (AEM). Approximate to Exact Method (AEM) is an algorithm, with a very strong accuracy that approaches the exact solution because it is a mixture of numerical methods of very high accuracy with a closed-form method. The novelty of the present method that it is converted linear boundary value problems to initial value problems using accurate numerical methods and then uses Laplace transforms method to find approximate to the exact solution. The uniqueness, convergence, and stability of the new technique are verified and tested by comparisons with a fourth-order accurate finite difference (FOFDM) solution. The provided comparisons highlight the effectiveness of the new approach, which is convergent, stable, and highly accurate.

Exact and approximate to exact methods

Problem 1: Consider the following problem of a two-point boundary value
$\theta''(\eta) - \theta(\eta) = \eta,$

The appropriate boundary conditions are described as follows:
$\theta(0) = 0$ and $\theta(1) = 2.$

Exact Solution using Laplace transform

Take the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms of derivatives:

$$\mathcal{L}\{\theta''(\eta)\} - \mathcal{L}\{\theta(\eta)\} = \mathcal{L}\{\eta\}$$
$$s^2\mathcal{L}\{\theta\} - s\,\theta(0) - \theta'(0) - \mathcal{L}\{\theta\} = \frac{1}{s^2} \tag{1}$$

Let $\theta'(0) = \delta$ and from the given boundary conditions $\theta(0) = 0$, then by substitution in Eq. (1) and rearranging gives:
$$\mathcal{L}\{\theta\} = \frac{\delta\, s^2+1}{s^2(s^2-1)} \tag{2}$$

$$\theta = \mathcal{L}^{-1}\left(\frac{\delta\, s^2+1}{s^2(s^2-1)}\right) \tag{3}$$

When we convert $\frac{\delta\, s^2+1}{s^2(s^2-1)}$ to a partial fraction, we obtain,
$$\frac{\delta\, s^2+1}{s^2(s^2-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-1} \tag{4}$$

By resolving Eq. (4), we get
$$A = 0,\ B = -1,\ C = -\frac{\delta+1}{2},\text{ and } D = \frac{\delta+1}{2}. \tag{5}$$

When we substitution by (4) and (5) in Eq. (3) gives,
$$\theta = \mathcal{L}^{-1}\left(\frac{-1}{s^2} - \frac{\delta+1}{2(s+1)} + \frac{\delta+1}{2(s-1)}\right) \tag{6}$$

The inverse Laplace equation gets,
$$\theta = -\,\eta - \frac{\delta+1}{2}e^{-\eta} + \frac{\delta+1}{2}e^{\eta} \tag{7}$$

And by using $\theta(1) = 2$ in Eq. (7), we get
$\delta = \frac{1+6e-e^2}{e^2-1}$, then by substitution in Eq. (7)

$$\theta = \frac{1}{2}\left(\left(\frac{1+6e-e^2}{e^2-1}+1\right)e^{\eta} - \left(\frac{1+6e-e^2}{e^2-1}+1\right)e^{-\eta} - 2\eta\right) \tag{8}$$

Then, Eq. (8) in the simplest form

$$\theta = \frac{e^2\eta-\eta+3e^{1-\eta}-3e^{\eta+1}}{1-e^2} \tag{9}$$

Approximate to Exact method

- **First stage of method**

Using Fourth-Order Runge-Kutta Method to find θ_i, $1 \le i \le 4$. (10)

- **Second stage of method**

Using Eq. (10) and applying the finite difference method first derivate formulae from fourth-order to find θ_0' as the following [27-28]:

$\theta_i' = \frac{-25\theta_i+48\theta_{i+1}-36\,\theta_{i+2}+16\theta_{i+3}-3\theta_{i+4}}{12h} + O\,(h^4)$, then

Let, $\theta_0' = \frac{-25\theta_0+48\theta_1-36\,\theta_2+16\theta_3-3\theta_4}{12h} = \beta$. (11)

- **Third stage of the method**

From Eq. (11), then problem (1) converts from BVP to IVP as follow:

$$\theta''(\eta) - \theta(\eta) = \eta,\ \theta\,(0) = 0 \text{ and } \theta'(0) = \beta. \tag{12}$$

Take the Laplace transform of both sides of Eq. (12):

$$\mathcal{L}\{\theta''(\eta)\} - \mathcal{L}\{\theta(\eta)\} = \mathcal{L}\{\eta\}$$

$$s^2\mathcal{L}\{\theta\} - s\,\theta(0) - \theta'(0) - \mathcal{L}\{\theta\} = \frac{1}{s^2} \tag{13}$$

After substitution of $\theta\,(0) = 0$ and $\theta'(0) = \beta$ in Eq. (13) , then

$s^2\mathcal{L}\{\theta\} - \beta - \mathcal{L}\{\theta\} = \frac{1}{s^2}$, and rearranging gives:

$$\mathcal{L}\{\theta\} = \frac{\beta\,s^2+1}{s^2(s^2-1)} \tag{14}$$

Using partial fraction and inverse Laplace transform we obtain,

$$\theta = -\eta - \frac{\beta+1}{2}e^{-\eta} + \frac{\beta+1}{2}e^{\eta} \tag{15}$$

By substitution of compute value of β from Eq. (11), in Eq. (15), then the approximate form to exact for solution of problem (1) is:

$$\theta = -\eta - \frac{1.276377153441123}{e^{\eta}} + 1.276377153441123e^{\eta} \tag{16}$$

The fourth-order finite difference method (FOFDM)

The finite domain $(0 < \theta < 1)$ of the solution is divided into $(n-1)$ subintervals such that the mesh size is$\left(h = \frac{1}{n-1}\right)$ with counter $i = 1, 2, 3, \ldots \ldots, n$. The linearized linear ordinary differential equation of (Problem 1) is transformed into a system of algebraic equations using the fourth-order difference schemes. The following (fourth-order) schemes are obtained by Taylor's expansions about $\theta_i = (i-1)h$. At $n = 1001 \to h = \frac{1}{1000} = 0.001$, fourth-order difference schemes (17-24) should be applied to equation (Problem 1) to minimize round-off errors in computations [29].

- ✓ At $\theta = \theta_1$
$$\theta' = \frac{1}{12h}(-25\theta_1 + 48\theta_2 - 36\,\theta_3 + 16\theta_4 - 3\theta_5) \tag{17}$$
- ✓ At $\theta = \theta_2$
$$\theta' = \frac{1}{12h}(-3\theta_1 - 10\theta_2 + 18\,\theta_3 - 6\theta_4 + \theta_5) \tag{18}$$
- ✓ At $\theta = \theta_i$
$$\theta' = \frac{1}{12h}(\theta_{i-2} - 8\theta_{i-1} + 8\,\theta_{i+1} - \theta_{i+2}) \tag{19}$$
- ✓ At $\theta = \theta_{n-1}$
$$\theta' = \frac{1}{12h}(-\theta_{n-4} + 6\theta_{n-3} - 18\,\theta_{n-2} + 10\theta_{n-1} + 3\theta_n) \tag{20}$$
- ✓ At $\theta = \theta_n$
$$\theta' = \frac{1}{12h}(25\theta_n - 48\theta_{n-1} + 36\,\theta_{n-2} - 16\theta_{n-3} + 3\theta_{n-4}) \tag{21}$$
- ✓ At $\theta = \theta_2$
$$\theta'' = \frac{1}{12h^2}(10\theta_1 - 15\theta_2 - 4\,\theta_3 + 14\theta_4 - 6\theta_5 + \theta_6) \tag{22}$$
- ✓ At $\theta = \theta_i$
$$\theta'' = \frac{1}{12h^2}(-\theta_{i-2} + 16\theta_{i-1} - 30\theta_i + 16\,\theta_{i+1} - \theta_{i+2}) \tag{21}$$
- ✓ At $\theta = \theta_{n-1}$
$$\theta'' = \frac{1}{12h^2}(\theta_{n-5} - 6\theta_{n-4} + 14\theta_{n-3} - 4\theta_{i-2} + 15\theta_{i-1} + 10\theta_i) \tag{24}$$

Problem 2: Consider the following problem of a two-point boundary value

$\varphi''(\eta) + 5\varphi'(\eta) + 4\varphi(\eta) = 1,$

The appropriate boundary conditions are described as follows:
$\varphi(0) = 0$ and $\varphi(1) = 0$.

Exact Solution using Laplace transform

Take the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms of derivatives:

$$\mathcal{L}\{\varphi''(\eta)\} + 5\mathcal{L}\{\varphi'(\eta)\} + 4\mathcal{L}\{\varphi(\eta)\} = \mathcal{L}\{1\}$$
$$s^2\mathcal{L}\{\varphi\} - s\,\varphi(0) - \varphi'(0) + 5[s\,\mathcal{L}\{\varphi\} - \varphi(0)] + 4\mathcal{L}\{\varphi\} = \frac{1}{s} \tag{1}$$

Let $\varphi'(0) = \alpha$ and from the given boundary conditions $\varphi(0) = 0$, then by substitution in Eq. $(1')$ and rearranging gives:
$$\mathcal{L}\{\varphi\} = \frac{1+\alpha\, s}{s(s^2+5s+4)} \tag{2}$$

$$\varphi = \mathcal{L}^{-1}\left(\frac{1+\alpha\, s}{s(s^2+5s+4)}\right) \tag{3}$$

When we convert $\frac{1+\alpha\, s}{s(s^2+5s+4)}$ to a partial fraction, we obtain,
$$\frac{1+\alpha\, s}{s(s^2+5s+4)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+1} \tag{4'}$$

By resolving Eq. $(4')$, we get
$$A = \frac{1}{4},\ B = \frac{-4\,\alpha+1}{12},\ \text{and } C = \frac{\alpha-1}{3}. \tag{5}$$

When we substitution by $(4')$ and $(5')$ in Eq. $(3')$ gives,
$$\varphi = \mathcal{L}^{-1}\left(\frac{1}{4\,s} + \frac{-4\,\alpha+1}{12(s+4)} + \frac{\alpha-1}{3(s+1)}\right) \tag{6}$$

The inverse Laplace equation gets,
$$\varphi = \frac{1}{4} + \frac{-4\,\alpha+1}{12} e^{-4\eta} + \frac{\alpha-1}{3} e^{-\eta} \tag{7}$$

And by using $\varphi(1) = 0$ in Eq. $(7')$, we get
$\alpha = -\frac{(-1+e)(1+2e+3e^2)}{4(1+e+e^2)}$, then by substitution in Eq. $(7')$,

$$\varphi(\eta) = \frac{1}{12}\left(3 + 4e^{-\eta}\left(-1 - \frac{(-1+e)(1+2e+3e^2)}{4(1+e+e^2)}\right) + e^{-4\eta}\left(1 + \frac{(-1+e)(1+2e+3e^2)}{1+e+e^2}\right)\right) \tag{8}$$

Then, Eq. $(8')$ in the simplest form:
$$\varphi(\eta) = \frac{e^{-4\eta}(e^{\eta}-1)(e^{3\eta}-e^{\eta+3}-e^{2\eta+3}+e^{3\eta+1}+e^{3\eta+2}-e^3)}{4(1+e+e^2)} \tag{9}$$

Approximate to Exact method

- **First stage of the method**

  Using Fourth-Order Runge-Kutta Method to find φ_i, $1 \leq i \leq 4$. (10)

- **Second stage of the method**

Using Eq. (10) and applying the finite difference method first derivate formulae from fourth-order to find φ_0' as the following [27-28]:

$\varphi_i' = \frac{-25\varphi_i+48\varphi_{i+1} -36\ \varphi_{i+2}+16\varphi_{i+3} -3\varphi_{i+4}}{12h} + O\ (h^4)$, then

Let, $\varphi_0' = \frac{-25\varphi_0+48\varphi_1 -36\ \varphi_2+16\varphi_3 -3\varphi_4}{12h} = \omega$. (11)

- **Third stage of the method**

From Eq. (11′), then problem (2) converts from BVP to IVP as follows:

$\varphi''(\eta) + 5\varphi'(\eta) + 4\varphi(\eta) = 1$, $\varphi\ (0) = 0$, and $\varphi'(0) = \ \omega$. (12)

Take the Laplace transform of both sides of Eq. (12′):

$\mathcal{L}\{\varphi''(\eta)\} + 5\mathcal{L}\{\varphi'(\eta)\} + 4\mathcal{L}\{\varphi(\eta)\} = \mathcal{L}\{1\}$

$s^2\mathcal{L}\{\varphi\} - s\ \varphi(0) - \varphi'(0) + 5[s\ \mathcal{L}\{\varphi\} - \varphi(0)] + 4\mathcal{L}\{\varphi\} = \frac{1}{s}$ (13)

After substitution of $\varphi\ (0) = 0$, and $\varphi'(0) = \ \omega$ in Eq. (13) , then

$s^2\mathcal{L}\{\varphi\} - \omega + 5[s\ \mathcal{L}\{\varphi\}] + 4\mathcal{L}\{\varphi\} = \frac{1}{s}$, and rearranging gives:

$\mathcal{L}\{\varphi\} = \frac{1+\omega\ s}{s(s^2+5s+4)}$ (14)

Using partial fraction and inverse Laplace transform we obtain,

$\varphi = \frac{1}{4} + \frac{-4\ \omega+1}{12} e^{-4\eta} + \frac{\omega-1}{3} e^{-\eta}$ (15)

By substitution of computing the value of ω from Eq. (11), in Eq. (15), then the approximate form to exact for the solution of problem (2) is

$$\varphi = \frac{1}{4} + \frac{0.452078099641477}{e^{4\eta}} - \frac{0.702078099641477}{e^{\eta}} \quad (16)$$

The fourth-order finite difference method (FOFDM)

The finite domain $(0 < \varphi < 1)$ of the solution is divided into $(n - 1)$ subintervals such that the mesh size is$\left(h = \frac{1}{n-1}\right)$ with counter $i\ =\ 1, 2, 3, \dots\dots, n$. The linearized linear ordinary differential equation of (Problem 1) is transformed into a system of algebraic equations using the fourth-order difference schemes. The following (fourth-order) schemes are obtained by Taylor's expansions about $\varphi_i = (i - 1)h$. At $n = 1001 \rightarrow h = \frac{1}{1000} =$

0.001, fourth-order difference schemes (17-24) should be applied to equation (Problem 1) to minimize round-off errors in computations [29].

- ✓ At $\varphi = \varphi_1$

$$\varphi' = \frac{1}{12h}(-25\varphi_1 + 48\varphi_2 - 36\,\varphi_3 + 16\varphi_4 - 3\varphi_5) \tag{17}$$

- ✓ At $\theta = \theta_2$

$$\varphi' = \frac{1}{12h}(-3\varphi_1 - 10\varphi_2 + 18\,\varphi_3 - 6\varphi_4 + \varphi_5) \tag{18}$$

- ✓ At $\varphi = \varphi_i$

$$\varphi' = \frac{1}{12h}(\varphi_{i-2} - 8\varphi_{i-1} + 8\,\varphi_{i+1} - \varphi_{i+2}) \tag{19}$$

- ✓ At $\varphi = \varphi_{n-1}$

$$\varphi' = \frac{1}{12h}(-\varphi_{n-4} + 6\varphi_{n-3} - 18\,\varphi_{n-2} + 10\varphi_{n-1} + 3\varphi_n) \tag{20}$$

- ✓ At $\varphi = \varphi_n$

$$\varphi' = \frac{1}{12h}(25\varphi_n - 48\varphi_{n-1} + 36\,\varphi_{n-2} - 16\varphi_{n-3} + 3\varphi_{n-4}) \tag{21}$$

- ✓ At $\varphi = \varphi_2$

$$\varphi'' = \frac{1}{12h^2}(10\varphi_1 - 15\varphi_2 - 4\,\varphi_3 + 14\varphi_4 - 6\varphi_5 + \varphi_6) \tag{22}$$

- ✓ At$\varphi = \varphi_i$

$$\varphi'' = \frac{1}{12h^2}(-\varphi_{i-2} + 16\varphi_{i-1} - 30\varphi_i + 16\,\varphi_{i+1} - \varphi_{i+2}) \tag{21}$$

- ✓ At $\varphi = \varphi_{n-1}$

$$\varphi'' = \frac{1}{12h^2}(\varphi_{n-5} - 6\varphi_{n-4} + 14\varphi_{n-3} - 4\varphi_{i-2} + 15\varphi_{i-1} + 10\varphi_i) \tag{24}$$

2. Results and Discussion

To demonstrate the accuracy of the new method (AEM), the current results are presented as tables. The convergence and stability of the new method are tested by comparison with an accurate numerical solution, called fourth-order finite difference method (FOFDM). Although these results in tables (1-2) show that the (FOFDM) is a very effective and powerful method, the same results in the indicated tables confirm that the new method (AEM) is more accurate and powerful. Tables (1-2) show absolute error 1 (A.E.$_1$) between (AEM) and the available exact solution. Also, tables (1-2) show absolute error 2 (A.E.$_2$) between (FOFDM) and the available exact solution. From the results shown in the tables (1-2), it became clear to us the following:

I. These results show the uniqueness, convergence, and stability of the new technique by comparisons with the fourth-order accurate finite-difference solution (FOFDM).

II. The results of a new method (AEM) are very close to the results of the exact method more than the results of (FOFDM).

III. Absolute error 1 (A.E.$_1$) is observed as less than absolute error 2 (A.E.$_2$), which confirms what was mentioned in (I).

IV. Approximate to Exact Method (AEM) is really an excellent agreement with the exact method.

Table 1: Comparison (AEM) and (FOFDM) with the exact solution for problem (1).

η	Exact	AEM	FOFDM	A.E.$_1$	A.E.$_2$
0	0	0	0.000000000272727	0	$2.7*10^{-10}$
0.1	0.155701110316105	0.155701102519554	0.155659825944795	$7.7*10^{-9}$	$4.1*10^{-5}$
0.2	0.313961363288364	0.313961347617232	0.313888389421445	$1.5*10^{-8}$	$7.2*10^{-5}$
0.3	0.477365514332794	0.477365490630238	0.477270128959705	$2.3*10^{-8}$	$9.5*10^{-5}$
0.4	0.648549800726239	0.648549768755037	0.648441057534417	$3.1*10^{-8}$	$1.0*10^{-4}$
0.5	0.830228325955111	0.830228285395283	0.830115144944074	$4.0*10^{-8}$	$1.1*10^{-4}$
0.6	1.025220223375321	1.025220173820932	1.025111480128820	$4.9*10^{-8}$	$1.0*10^{-4}$
0.7	1.236477871046454	1.236477812001547	1.236382485562844	$5.9*10^{-8}$	$9.5*10^{-5}$
0.8	1.467116440123709	1.467116370997342	1.467043466091897	$6.9*10^{-8}$	$7.2*10^{-5}$
0.9	1.720445072537871	1.720444992638206	1.720403787946480	$7.9*10^{-8}$	$4.1*10^{-5}$
1	2	2	2.000000000000000	0	0

Table 2: Comparison (AEM) and (FOFDM) with the exact solution for problem (2).

η	Exact	AEM	FOFDM	A.E.$_1$	A.E.$_2$
0	0	0	0.000000273503978	0	$2.7*10^{-7}$
0.1	-0.082229522555420	-0.082229522375807	-0.082229340425243	$1.7*10^{-10}$	$1.8*10^{-7}$
0.2	-0.121681147309955	-0.121681147027037	-0.121681026315111	$2.8*10^{-10}$	$1.2*10^{-6}$
0.3	-0.133948941948419	-0.133948941611720	-0.133948861830254	$3.3*10^{-10}$	$8.0*10^{-8}$
0.4	-0.129344030251789	-0.129344029893033	-0.129343977440309	$3.5*10^{-10}$	$5.2*10^{-8}$
0.5	-0.114649775646167	-0.114649775285288	-0.114649741053991	$3.6*10^{-10}$	$3.4*10^{-8}$
0.6	-0.094297030954426	-0.094297030603582	-0.094297008498160	$3.5*10^{-10}$	$2.2*10^{-8}$
0.7	-0.071150769178821	-0.071150768845070	-0.071150754787926	$3.3*10^{-10}$	$1.4*10^{-8}$
0.8	-0.047036325842467	-0.047036325529554	-0.047036316794971	$3.1*10^{-10}$	$9.0*10^{-9}$
0.9	-0.023091197853915	-0.023091197563458	-0.023091192331235	$2.9*10^{-10}$	$5.5*10^{-9}$
1	0	0	0	0	0

3. Conclusion

This paper presented a new method to solve linear boundary value problems using Laplace Transform. The new method presents a solution for linear boundary value problems very approximate to the exact solution, so named Approximate to Exact Method (AEM). Approximate to Exact Method (AEM) is an algorithm, with a very strong accuracy that approaches the exact solution because it is a mixture of numerical methods of very high accuracy with a closed-form method. The novelty of the present method that it is converted linear boundary value problems to initial value problems using accurate numerical methods and then uses Laplace transforms method to find approximate to the exact solution. The uniqueness, convergence, and stability of the new technique are verified and tested by comparisons with a fourth-order accurate finite difference (FOFDM) solution. The provided comparisons highlight the effectiveness of the new approach, which is convergent, stable, and highly accurate.

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Statement for Conflict of Interest

There are no conflicts to declare.

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