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On the Oscillation of Impulsive Neutral First-order Differential Equations with Variable Arguments

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Abstract

Throughout the article, we study the oscillation of a general class of first-order neutral differential equations in presence of variable delays under the effect of impulses.

Due to its importance in applications, there are many papers concerning with the property of oscillation and non-oscillation of neutral delay differential equations. Although, a lot of works are concerning with the oscillation of neutral delay differential equations without impulse or impulsive neutral with constant delays, however few papers dealt with the impulsive neutral and those with variable delays. In this paper, we establish sufficient conditions of certain neutral equations with variable delay arguments. New oscillation criteria are deduced. Our results are based on using equivalence transformation and two useful lemmas to prove the obtained criteria.

The results of this paper improve those of [20] by adding several non-linear delay functions to the equations instead of having one delay term. Where it is assumed that the two variable delays satisfying a *Lipschitz* condition.

Moreover we discuss more general non-linear delay functions comparing with those used in [14]. Our results improve and extend some recent results in the literature. An illustrative example is given.

Keywords: Oscillation; impulsive; neutral first order differential equation; variable delay

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1. Introduction

Consider the following neutral impulsive equations:

$$\begin{cases} [r(\eta) - P(\eta)r(\lambda(\eta))] + Q(\eta)\Omega(r(\rho(\eta))) - V(\eta)\Psi(r(\theta(\eta))) = 0, \eta \neq \eta_j, j = 1, 2, \dots; \\ r(\eta_j^+) = I_j(r(\eta_j)), r(\lambda(\eta_j^+)) = I_j(r(\lambda(\eta_j))). \end{cases} \quad (1)$$

In which, $\{\eta_j\}_{j \in \mathbb{N}}$ represents the moments of impulse, λ, ρ and θ are the delay functions, $P(\eta), Q(\eta)$ and $V(\eta)$ are continuous non-negative functions. Also, $P(\eta) \in PC([\eta_0, \infty), R^+), Q(\eta), V(\eta) \in C([\eta_0, \infty), R^+), \lambda(\eta), \rho(\eta)$ and $\theta(\eta) \in C([\eta_0, \infty), R)$ are increasing functions such that

$$\lim_{\eta \rightarrow \infty} \lambda(\eta) = \lim_{\eta \rightarrow \infty} \rho(\eta) = \lim_{\eta \rightarrow \infty} \theta(\eta) = \infty.$$

Throughout the paper, we assume that:

(A₁) The functions $\Omega(\eta)$ and $\Psi(\eta)$ are non-decreasing functions and satisfying:

$$r\Omega(z) > 0, r\Psi(z) > 0.$$

(A₂) There exist some positive constants β_1 and β_2 such that

$$|\Omega(r)| \geq \beta_1|r|, \text{ and } |\Psi(r)| \geq \beta_2|r|$$

(A₃) For $I_j \in C(R, R)$, there exists a sequence of positive real numbers b_j satisfying

$$b_j \leq \frac{I_j(r)}{r} \leq 1.$$

In the last decades, there has been a great interest in studying differential equations, see for example ([12,16, 24, 27]). Recently many authors were concerning with the oscillation of neutral impulsive differential equations (see [1, 2, 6, 8-11,13,17-19,21-23, 25]) due to its importance in some models of real phenomena which include derivatives on the part of the used variable. Although, many papers discussing the oscillation of linear and nonlinear impulsive equations with constant delays ([3, 4, 6]), however few of them dealt with variable delays ([5, 7, 8, 15, 26]).

In 2019, Santra et al. [20] discussed the oscillatory behavior of the non-linear neutral equations with constant delay:

$$\begin{cases} [r(\eta) - P(\eta)r(\eta - \lambda)]' + Q(\eta)\Omega(r(\eta - \rho)) = 0, \eta \neq \eta_j, j = 1, 2, \dots; \\ r(\eta_j^+) = I_j(r(\eta_j)), r(\eta_j^+ - \lambda) = I_j(r(\eta_j - \lambda)). \end{cases} \quad (2)$$

They also investigated several criteria of Eq. (2) with several ranges for the neutral coefficient $P(\eta)$.

In [14], the authors established sufficient conditions of the oscillation of the neutral equations with variable coefficients

$$\begin{cases} [r(\eta) - P(\eta)r(\lambda(\eta))] + Q(\eta)r(\rho(\eta)) - V(\eta)r(\theta(\eta)) = 0, \eta \neq \eta_j, j = 1, 2, \dots; \\ r(\eta_j^+) + b_j r(\eta_j) = a_j r(\eta_j), \eta = \eta_j. \end{cases} \quad (3)$$

In this article, we discuss the oscillatory behavior of (1) in presence of positive and negative variable delay coefficients.

2. Preliminaries.

Now, we give two useful lemmas to prove our results.

Lemma 2.1.

Suppose that $r(\eta)$ be an eventually positive solution of (1) and for a continuous function $\gamma(\eta)$,

$$\begin{aligned} \varpi(\eta) = & r(\eta) - P(\eta)r(\lambda(\eta)) - \\ & \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(v)\Omega(r(\rho(v))) dv - \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(v)\Psi(r(\theta(v))) dv, \end{aligned} \quad (4)$$

where $\gamma(\eta) < t, \theta^{-1}(\gamma(\eta)) < t$ and $\rho^{-1}(\gamma(\eta)) > \eta$ for $\eta \in (\eta_j, \eta_{j+1}]$, $0 < \eta_0 < \eta_1 < \dots < \eta_j \rightarrow \infty$ at $j \rightarrow \infty$.

Let $\varpi(\eta)$ satisfies the following conditions:

$$(C_1) \beta_1 Q(\rho^{-1}(\gamma(\eta))) [\rho^{-1}(\gamma(\eta))] - \beta_2 V(\theta^{-1}(\gamma(\eta))) [\theta^{-1}(\gamma(\eta))] \geq 0.$$

$$(C_2) \begin{cases} P(\eta_j^+) \geq p(\eta_j), & \lambda(\eta_j) \neq \eta_i. \\ P(\eta_j^+) \geq 1/b_j p(\eta_j), & \lambda(\eta_j) = \eta_i. \end{cases}$$

$$(C_3) \text{ If } \limsup_{\eta \rightarrow \infty} \{P(\eta) + \beta_1 \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(v)dv + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(v)dv\} \leq 1,$$

$$\text{then } \begin{cases} \varpi(\eta) > 0, & \text{for } \eta \geq \eta_0, \\ \varpi(\eta_j^+) \geq 0, & \text{for } \eta \in (\eta_j, \eta_{j+1}]. \end{cases}$$

Proof.

Suppose that $r(\eta)$ be an eventually positive solution of Eq. (1) such that $r(\eta), r(\lambda(\eta)) > 0$, $r(\rho(\eta))$ and $r(\theta(\eta))$ also be positive. From (1) and (2), we get

$$\begin{aligned} \varpi'(\eta) &= [r(\eta) - P(\eta)r(\lambda(\eta))] - [Q(\eta)\Omega(r(\rho(\eta)))_{\eta}^{\rho^{-1}(\gamma(\eta))} - [V(\eta)\Psi(r(\theta(\eta)))]_{\theta^{-1}(\gamma(\eta))}^{\eta}] \\ &= -Q(\rho^{-1}(\gamma(\eta))\Omega(r(\rho(\rho^{-1}(\gamma(\eta)))) - V(\theta^{-1}(\gamma(\eta)))\Psi(r(\theta(\theta^{-1}(\gamma(\eta)))). \end{aligned} \quad (5)$$

By (A_1) , we obtain

$$\begin{aligned} \varpi'(\eta) &\leq -\beta_1 Q(\rho^{-1}(\gamma(\eta))r(\rho(\rho^{-1}(\gamma(\eta)))) - \beta_2 V(\theta^{-1}(\gamma(\eta)))r(\theta(\theta^{-1}(\gamma(\eta)))) \leq \\ &-\beta_1 Q(\rho^{-1}(\gamma(\eta))[\rho^{-1}(\gamma(\eta))] - \beta_2 V(\theta^{-1}(\gamma(\eta)))[\theta^{-1}(\gamma(\eta))]r(\gamma(\eta)) \end{aligned} \quad (6)$$

Hence by the condition (C_1) , we conclude that

$$\varpi'(\eta) \leq 0, \quad (7)$$

for all $\eta \in (\eta_j, \eta_{j+1}]$. First, we show that $\varpi(\eta_j^+) \leq \varpi(\eta_j)$. Since

$$\begin{aligned} \varpi(\eta_j^+) &= r(\eta_j^+) - P(\eta_j^+)r(\lambda(\eta_j^+)) \\ &\quad - \int_{\eta_j}^{\rho^{-1}(\gamma(\eta_j))} Q(v)\Omega(r(\rho(v)))dv - \int_{\theta^{-1}(\gamma(\eta_j))}^{\eta_j} V(v)\Psi(r(\theta(v)))dv \end{aligned}$$

$$\begin{aligned}
&= I_j \left(r(\eta_j) \right) - P(\eta_j^+) I_j(r(\lambda(\eta_j))) \\
&\quad - \int_{\eta_j}^{\rho^{-1}(\gamma(\eta_j))} Q(v) \Omega \left(r(\rho(v)) \right) dv - \int_{\theta^{-1}(\gamma(\eta_j))}^{\eta_j} V(v) \Psi \left(r(\theta(v)) \right) dv, \quad (8)
\end{aligned}$$

For $\lambda(\eta_j) = \eta_i$, applying the condition (C_2)

$$\begin{aligned}
\varpi(\eta_j^+) &\leq r(\eta_j) - b_j P(\eta_j^+) r(\lambda(\eta_j)) \\
&\quad - \int_{\eta_j}^{\rho^{-1}(\gamma(\eta_j))} Q(v) \Omega \left(r(\rho(v)) \right) dv - \int_{\theta^{-1}(\gamma(\eta_j))}^{\eta_j} V(v) \Psi \left(r(\theta(v)) \right) dv \\
&\leq r(\eta_j) - P(\eta_j) r(\lambda(\eta_j)) \\
&\quad - \int_{\eta_j}^{\rho^{-1}(\gamma(\eta_j))} Q(v) \Omega \left(r(\rho(v)) \right) dv - \int_{\theta^{-1}(\gamma(\eta_j))}^{\eta_j} V(v) \Psi \left(r(\theta(v)) \right) dv.
\end{aligned}$$

So,

$$\varpi(\eta_j^+) \leq \varpi(\eta_j) \quad (9)$$

For $(\eta_j) \neq \eta_i$, we get

$$\begin{aligned}
\varpi(\eta_j^+) &= I_j \left(r(\eta_j) \right) - P(\eta_j^+) r(\lambda(\eta_j)) \\
&\quad - \int_{\eta_j}^{\rho^{-1}(\gamma(\eta_j))} Q(v) \Omega \left(r(\rho(v)) \right) dv - \int_{\theta^{-1}(\gamma(\eta_j))}^{\eta_j} V(v) \Psi \left(r(\theta(v)) \right) dv \\
&\leq r(\eta_j) - P(\eta_j) r(\lambda(\eta_j)) \\
&\quad - \int_{\eta_j}^{\rho^{-1}(\gamma(\eta_j))} Q(v) \Omega \left(r(\rho(v)) \right) dv - \int_{\theta^{-1}(\gamma(\eta_j))}^{\eta_j} V(v) \Psi \left(r(\theta(v)) \right) dv \\
&\leq r \left(\lambda(\eta_j) \right),
\end{aligned}$$

which implies that $\varpi(\eta_j^+) \leq \varpi(\eta_j)$. (10)

Hence $-\infty \leq h < \infty, |h| = \sup\{\varpi(\eta_j^+), \lim_{j \rightarrow \infty} \varpi(\eta_j)\}, \eta \in [\eta_h, \infty)$ for $h \geq \eta_0$.

Now, we claim that $\varpi(\eta_j) \geq 0$ for $j=h, h+1, \dots$. If not, there exists $m \geq h$ such that:

$$\varpi(\eta_m) < 0 \text{ for } \eta \geq \eta_m.$$

Since $\varpi(\eta)$ is non-increasing on $[\eta_0, \infty)$, then there exists a positive constant ε satisfying $\varpi(\eta_m) = -\varepsilon < 0$.

Further from (1), we get

$$r(\eta) \leq -\varepsilon + P(\eta)r(\lambda(\eta)) + \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(v)\Omega(r(\rho(v))) dv - \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(v)\Psi(r(\theta(v))) dv. \quad (11)$$

There exist two cases:

Case 1: If $r(\eta)$ is unbounded then \exists a sequence of points $\{S_n\}$ such that $S_n \geq \eta_m$, then $\lim_{n \rightarrow \infty} r(S_n) = \infty$ and $r(S_n) = \max\{r(\eta), \eta_m \leq \eta \leq S_n\}$.

Then the inequality (11) becomes

$$\begin{aligned} r(S_n) &\leq -\varepsilon + P(S_n)r(\lambda(S_n)) \\ &\quad + \int_{S_n}^{\rho^{-1}(\gamma(S_n))} Q(v)\Omega(r(\rho(v))) dv + \int_{\theta^{-1}(\gamma(S_n))}^{S_n} V(v)\Psi(r(\theta(v))) dv \\ &\leq -\varepsilon + \{P(S_n) + \beta_1 \int_{S_n}^{\rho^{-1}(\gamma(S_n))} Q(v)dv + \beta_2 \int_{\theta^{-1}(\gamma(S_n))}^{S_n} V(v)dv\}r(S_n) \end{aligned}$$

$$\text{Then} \quad r(S_n) \leq -\mu + r(S_n), \quad (12)$$

which implies that $0 \leq -\varepsilon$. This is a contradiction with the assumption that $\varepsilon > 0$.

Case 2: If $r(\eta)$ is bounded such that $\limsup_{\eta \rightarrow \infty} r(\eta) = M < \infty$.

Here, we can take the sequence of points $\{S_n\}$ as $\lim_{n \rightarrow \infty} r(S_n) = M$ and $r(\xi(S_n)) = \max\{r(t): \delta_1(S_n) \leq \eta \leq \delta_2(S_n)\}$,

where $\delta_1(S_n) = \min\{\lambda(S_n), \rho(S_n)\}$ and $\delta_2(S_n) = \max\{\lambda(S_n), \theta(S_n)\}$. It is clear that

$\lim_{n \rightarrow \infty} \xi(S_n) = \infty$ and $\limsup_{n \rightarrow \infty} r(\xi(S_n)) \leq M$. Then we get,

$$\begin{aligned}
r(S_n) &\leq -\varepsilon + P(S_n)r(\lambda(S_n)) \\
&\quad + \int_{S_n}^{\rho^{-1}(\gamma(S_n))} Q(v)\Omega(r(\rho(v)))dv + \int_{\theta^{-1}(\gamma(S_n))}^{S_n} V(v)\Psi(r(\theta(v)))dv \\
&\leq -\varepsilon + \{P(S_n) + \beta_1 \int_{S_n}^{\rho^{-1}(\gamma(S_n))} Q(v)dv + \beta_2 \int_{\theta^{-1}(\gamma(S_n))}^{S_n} V(v)dv\}r(\xi(S_n)) \\
&\leq -\varepsilon + r(\xi(S_n)). \tag{13}
\end{aligned}$$

Taking limit sup. at $n \rightarrow \infty$, we obtain $0 \leq -\varepsilon$ which is again a contradiction. From the two cases, we can see that $\varpi(\eta_j) \geq 0$ for $\eta \in (\eta_j, \eta_{j+1}]$, $j=h, h+1, \dots$

Finally, to prove that $\varpi(\eta) > 0$ for $j=1, 2, \dots$. If it not satisfied, suppose that there exists $m \geq 0$ at which $\varpi(\eta_m) = 0$.

Integrating (7) from η_m to η_{m+1} , we get

$$\begin{aligned}
\varpi(\eta_{m+1}) &= \\
\varpi(\eta_m^+) - \int_{\eta_m}^{\eta_{m+1}} \{\beta_1 Q(\rho^{-1}(\gamma(\eta)))[\rho^{-1}(\gamma(\eta))] - \beta_2 V(\theta^{-1}(\gamma(\eta)))[\theta^{-1}(\gamma(\eta))]\}'r(\gamma(\eta))d\eta \\
&< \varpi(\eta_m^+) \leq \varpi(\eta_m) = 0, \tag{14}
\end{aligned}$$

which is a contradiction. Ultimately, $\varpi(\eta) > 0$ for $\eta \geq \eta_0$.

Lemma 2.2.

Suppose that the functions $\phi(t)$ and $B(t) \in C(R^+, R^+)$ such that $B(t) < t$ for $t \geq t_0$ and $\lim_{t \rightarrow \infty} B(t) = \infty$. If the function $A(t)$ satisfies

$$\liminf_{t \rightarrow \infty} \int_{B(t)}^t \phi(s)ds > \frac{1}{e}, \tag{15}$$

then the inequality $r'(t) + \phi(t)r(B(t)) \leq 0$ has no eventually positive solution.

3. Main Results.

Now, we give sufficient conditions for the oscillation of Eq. (1).

Theorem 3.1.

Let $(C_1) - (C_3)$ hold. If

$$\liminf_{\eta \rightarrow \infty} \int_{\gamma(\eta)}^{\eta} \{\beta_1 Q(\rho^{-1}(\gamma(\eta))) [\rho^{-1}(\gamma(\eta))] - \beta_2 V(\theta^{-1}(\gamma(\eta))) [\theta^{-1}(\gamma(\eta))]\}' \cdot \left\{ 1 + P(\gamma(s)) + \beta_1 \int_{\gamma(\eta)}^{\rho^{-1}(\gamma(\eta))} Q(v) dv + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\gamma(\eta)} V(v) dv \right\} ds > \frac{1}{e}, \quad (16)$$

then Eq. (1) is oscillatory, where $\theta^{-1}(\gamma(\eta)) < \eta$, and $\rho^{-1}(\gamma(\eta)) > \eta$.

Proof. Suppose that $r(\eta)$ be an eventually positive solution of Eq. (1).

But since $\varpi(\eta) \leq r(\eta)$, then by Lemma 2.1, we have

$$\begin{aligned} r(\eta) &= \varpi(\eta) + P(\eta)r(\lambda(\eta)) + \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(v)\Omega(r(\rho(v))) dv + \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(v)\Psi(r(\theta(v))) dv \\ r(\eta) &\geq \varpi(\eta) + P(\eta)r(\lambda(\eta)) + \beta_1 \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(v)\varpi(\rho(v)) dv + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(v)\varpi(\theta(v)) dv \\ &\geq \varpi(\eta) + P(\eta)\varpi(\lambda(\eta)) + \beta_1 \varpi(\gamma(\eta)) \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(v) dv + \beta_2 \varpi(\theta(\eta)) \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(v) dv \\ &\geq \varpi(\eta) + P(\eta)\varpi(\lambda(\eta)) + \beta_1 \varpi(\eta) \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(v) dv + \beta_2 \varpi(\eta) \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(v) dv \\ &\geq \varpi(\eta) \{1 + P(\eta) + \beta_1 \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(v) dv + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(v) dv\}. \end{aligned}$$

Then

$$r(\gamma(\eta)) \geq \varpi(\gamma(\eta)) \{1 + P(\gamma(\eta)) + \beta_1 \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(v) dv + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(v) dv\}. \quad (17)$$

Substituting from (17) into (7), we obtain

$$\varpi'(\eta) \leq -\{\beta_1 Q(\rho^{-1}(\gamma(\eta))) [\rho^{-1}(\gamma(\eta))] - \beta_2 V(\theta^{-1}(\gamma(\eta))) [\theta^{-1}(\gamma(\eta))]\}'.$$

$$\left\{ 1 + P(\gamma(s)) + \beta_1 \int_{\gamma(\eta)}^{\rho^{-1}(\gamma(\eta))} Q(v)dv + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\gamma(\eta)} V(v)dv \right\} ds \leq 0. \quad (18)$$

But from (15) and (16), we obtain that (18) has no positive solution. This is a contradiction, then the proof is complete.

Corollary 3.2. Let the conditions $(C_1) - (C_3)$ be hold. If

$$\begin{aligned} & \{\beta_1 Q(\rho^{-1}(\gamma(\eta))[\rho^{-1}(\gamma(\eta))]' - \beta_2 V(\theta^{-1}(\gamma(\eta)))[\theta^{-1}(\gamma(\eta))]' \} \\ & \geq [e \times \min_{\eta \geq \eta_0} \{\eta - \gamma(\eta)\}]^{-1} \geq e^{-1}, \end{aligned} \quad (19)$$

then Eq. (1) oscillates.

Proof.

Going through as in Corollary 3.2. of [14], the proof is complete.

4. Example.

Consider the impulsive neutral equation:

$$\begin{cases} \left[r(\eta) - \frac{1+e^{-\eta}}{9} r(\eta - 2\pi) \right]' + \frac{8-e^{-\eta}}{9} r\left(\eta - \frac{5\pi}{2}\right) (1 + |r(\eta - 2)|^\mu) \\ - \frac{e^{-\eta}}{9} r(\eta - 2\pi)(1 + |r(\eta - 2)|^\mu) = 0, \eta \neq \eta_j, j = 1, 2, \dots; \\ r(\eta_j^+) = \frac{j}{j+1} r(\eta_j), \quad r(\eta_j^+ - \lambda) = I_j(r(\eta_j) - \lambda), \end{cases} \quad (20)$$

where $I_j = j/j + 1$ and $\mu > 0$. Taking $\gamma(\eta) = t - 9\pi/4$, $P(\eta_j) = 1/20j$ and $b_j = j + 1/j$.

Since $\Omega(r(\rho(\eta))) \geq r(\eta - 5\pi/4)(1 + |r(\eta - 2)|^\mu)$ and $\Psi(r(\theta(\eta))) \geq r(\eta - 2\pi)(1 + |r(\eta - 2)|^\mu)$, we can take $\beta_1 = \beta_2 = 1$. Now since

$$\beta_1 Q(\rho^{-1}(\gamma(\eta))[\rho^{-1}(\gamma(\eta))]' - \beta_2 V(\theta^{-1}(\gamma(\eta)))[\theta^{-1}(\gamma(\eta))]' = \frac{8}{9} - \frac{1}{9}(e^{-\pi/4} + e^{\pi/4}) > 0.$$

For $\eta_j = j$, then $P(\eta_j^+) = \lim_{\eta \rightarrow j^+} \frac{1+e^{-\eta}}{9} \geq \frac{1}{9}$ and $\frac{1}{b_j} P(\eta) = \frac{j+1}{j} \times \frac{1}{20j} = \frac{j+1}{20j^2}$ which satisfies the condition (C_2) . Finally, we check the condition

$$\lim_{\eta \rightarrow \infty} \inf \int_{\gamma(\eta)}^{\eta} \{ \beta_1 Q(\rho^{-1}(\gamma(\eta))) [\rho^{-1}(\gamma(\eta))]' - \beta_2 V(\theta^{-1}(\gamma(\eta))) [\theta^{-1}(\gamma(\eta))] \} .$$

$$\left\{ 1 + P(\gamma(s)) + \beta_1 \int_{\gamma(\eta)}^{\rho^{-1}(\gamma(\eta))} Q(v) dv + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\gamma(\eta)} V(v) dv \right\} ds$$

$$\lim_{\eta \rightarrow \infty} \inf \int_{\eta-9\pi/4}^{\eta} \left[\frac{8}{9} - \frac{1}{9} (e^{-s-\frac{\pi}{4}} + e^{-s+\frac{\pi}{4}}) \right] \times \left[1 + \frac{1}{9} + \frac{1}{9} e^{-s+\frac{\pi}{4}} + \frac{2\pi}{9} - \frac{2e^{-s+\frac{9\pi}{4}}}{9} + \frac{e^{-s+\frac{10\pi}{4}}}{9} + \frac{e^{-s+2\pi}}{9} \right] ds \geq \frac{1}{e}.$$

Using Theorem 3.1, then Eq. (20) oscillates.

Conclusion. The aim of this paper is to discuss the oscillation of a general class of first-order impulsive neutral equations with positive and negative delays (1). New results are given to improve and extend some recent papers like [14] and [20]. An example is given to illustrate our results.

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الملخص العربي

حول السلوك التذبذبي لمجموعة من المعادلات التفاضلية المحايدة من الرتبة الأولى تحتوى على معاملات تأخير متغيرة وتحت تأثير نبضات.

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الملخص العربي

في هذه المقالة، قمنا بدراسة خاصية التذبذب لفئة عامة من المعادلات التفاضلية المحايدة من الدرجة الأولى في وجود معاملات تأخير متغيرة تحت تأثير النبضات. نظرًا لأهميتها في التطبيقات، هناك العديد من الأبحاث المتعلقة بخاصية التذبذب وعدم التذبذب في المعادلات التفاضلية للتأخير المحايد. على الرغم من أن الكثير من المقالات تتعلق بدراسة تذبذب المعادلات التفاضلية ذات معاملات للتأخير المحايد دون وجود اندفاع أو في حالة وجود تأخيرات ثابتة، إلا أن القليل من الأبحاث تناولت المعادلات التفاضلية المحايدة ذات التأخيرات المتغيرة.

في هذه المقالة، قدمنا شروطًا كافية للسلوك التذبذبي لبعض المعادلات المحايدة ذات معاملات تأخير متغيرة. تعمل هذه النتائج على تحسين نتائج سابقة بإضافة عدة دوال تأخير غير خطية إلى المعادلات بدلاً من وجود معامل تأخير واحد. علاوة على ذلك استخدمنا دوال تأخير غير خطية أكثر عمومية مقارنة بتلك المستخدمة في [14]. تعمل نتائجنا على تحسين وتوسيع بعض النتائج الحديثة، أيضا تم اعطاء مثال توضيحي.