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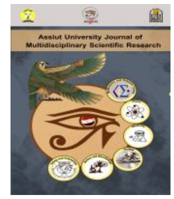

Endo-Noetherian Skew Generalized Power Series Rings

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ABSTRACT

Endo-Noetherian modules were introduced by A. Kaidi and E. Sanchez as a generalization of Noetherian modules. A left $\mathbb{R}$-module $\mathcal{M}$ which satisfies the ascending chain condition for endomorphic kenels is said to be endo-Noetherian. A ring $\mathbb{R}$ is said to be left endo-Noetherian if ${}_{\mathbb{R}}\mathbb{R}$ is endo-Noetherian as a left module. The authors studied the property of endo-Noetherian for the polynomial ring $\mathbb{R}[x]$ and the formal power series ring $\mathbb{R}[[x]]$. In this article, we show under what conditions on a ring $\mathbb{R}$, a strictly ordered monoid $(\aleph, \preccurlyeq)$, and a monoid homomorphism $\sigma: \aleph \longrightarrow \mathrm{End}(\mathbb{R})$, the skew generalized power series ring $\mathbb{R}[[\aleph, \sigma]]$ is left endo-Noetherian if and only if $\mathbb{R}$ is left endo-Noetherian.

1. Introduction

In this article, $\mathbb{R}$ denotes a ring with identity (not necessarily to be commutative). A. Kaidi and E. Sanchez [1] introduced a new class of rings called endo-Noetherian rings as a generalization of Noetherian rings which was identified by Emmy Noether in 1921 [2] and the name Noetherian is in her honor. Also, the endo-Noetherian property is a generalization of the iso-Noetherian property (see Definition 3). A left $\mathbb{R}$ -module $\mathcal{M}$ is called endo-Noetherian if any ascending chain of endomorphic kernels $Ker\ (f_1) \subseteq Ker\ (f_2) \subseteq ...$, stabilizes, where $f_i \in End\ (\mathcal{M})$ for all $i \in \mathbb{N}^*$, i.e., there exists $n \in \mathbb{N}^*$ such that for each $k \geq n$, $Ker\ (f_n) = Ker\ (f_k)$. If ${}_{\mathbb{R}}\mathbb{R}$ is endo-Noetherian, then the ring $\mathbb{R}$ is called left endo-Noetherian. Given a ring $\mathbb{R}$ and an element $a \in \mathbb{R}$, let $f_a : \mathbb{R} \to \mathbb{R}$ defined by $f_a(x) := x\,a$, and putting $\ell.\mathrm{ann}_{\mathbb{R}}(a) = ker f_a = \{\, x \in \mathbb{R} : x\,a = 0\}$. From the fact that $End\ ({}_{\mathbb{R}}\mathbb{R}) =$

$\{f_a : a \in R\}$ we have the equivalent definition, R is left endo-Noetherian if and only if for every ascending chain $\ell.\mathrm{ann}_R(a_1) \subseteq \ell.\mathrm{ann}_R(a_2) \subseteq \ldots$ where $a_i \in R$, there exists $n \in N^*$, where $\ell.\mathrm{ann}_R(a_n) = \ell.\mathrm{ann}_R(a_k)$ for each $k \geq n$ [1, Proposition 1.16].

In [3] Gouaid et al. studied when the formal power series rings and polynomial rings over an Armendariz ring be endo-Noetherian.

According to [4], a skew generalized power series ring $R[[\aleph,\sigma]]$ (see section 2 for definition) embraces a wide range of many ring extensions, including, generalized power series rings, (skew) monoid rings, (skew) Laurent series rings, (skew) power series rings, (skew) Laurent polynomial rings and (skew) polynomial rings. All of the aforementioned subclasses have analogues for any investigation of the interplay between the skew generalized power series rings and the left endo-Noetherian property, and this analogous result follows directly from a single argument.

The purpose of this article is to study the left endo-Noetherian property of the skew generalized power series rings and the direct product of left endo-Noetherian rings. In Section 2, we introduce the relations among the class of endo-Noetherian rings and some other related classes. Also, we expose briefly the skew generalized power series ring $R[[\aleph, \sigma]]$ construction and some of its particular cases.

2. Preliminaries

Definition 1. [5] An R-module $\mathcal{M}$ is said to be Hopfian if each surjective endomorphism of $\mathcal{M}$ is an automorphism. A ring R is said to be left Hopfian if $_RR$ is Hopfian.

Definition 2. [5] An R -module $\mathcal{M}$ is said to be strongly Hopfian if the ascending chain Ker $f \subseteq$ Ker $f^2 \subseteq \ldots$ stabilizes for each $f \in \mathrm{End}(\mathcal{M})$. A ring R is said to be left strongly Hopfian if $_RR$ is strongly Hopfian.

Equivalently [5, Proposition 2.9], R is left strongly Hopfian if there exists $n \in N^*$ such that $\ell.\mathrm{ann}_R(a^n) = \ell.\mathrm{ann}_R(a^{n+1})$ for each $a \in R$.

Definition 3. [6] An R-module $\mathcal{M}$ is called iso-Noetherian if for each ascending chain of submodules of $\mathcal{M}$, $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \ldots$, there exists an index $n \geq 1$ such that $\mathcal{M}_n \simeq \mathcal{M}_i$ for each $i \geq n$. If R is iso-Noetherian as an R-module, then R is said to be an iso-Noetherian ring.

The following diagram shows the relation between endo-Noetherian rings and the previous rings.

Noetherian $\Longrightarrow$ iso-Noetherian $\Longrightarrow$ endo-Noetherian $\Longrightarrow$ strongly Hopfian $\Longrightarrow$ Hopfian.

As the accompanying examples show, the converse of the preceding implication is not true:

Examples:

1. An example of a module that is endo-Noetherian, but is not Noetherian. $\mathbb{Q}_{\mathbb{Z}}$ is an endo-Noetherian module. But $\mathbb{Q}_{\mathbb{Z}}$ is not a Noetherian module because we have an infinite increasing sequence.

$$\mathbb{Z} \subset (1/2)\,\mathbb{Z} \subset (1/4)\,\mathbb{Z} \subset \ldots$$

2. An example of a ring that is endo-Noetherian, but is not Noetherian [3, Example 2.1]. Let $\mathbb{R}$ be a commutative ring with identity and $\mathcal{M}$ an $\mathbb{R}$-module. The trivial extension of $\mathbb{R}$ by $\mathcal{M}$ is a commutative ring $\mathbb{R}\,(+)\mathcal{M} = \{(x, y) \mid x \in \mathbb{R}$ and $m \in \mathcal{M}\}$ under the usual addition and the multiplication defined as $(x_1, y_1)(x_2, y_2) = (x_1x_2, x_1y_2 + x_2y_1)$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}\,(+)\,\mathcal{M}$. Now, let $\mathcal{K}$ be a vector space over a field $\mathcal{F}$. Since for every $(a,b) \in \mathcal{F} \times \mathcal{K} \setminus \{(0,0)\}$,

$$ann_{\mathcal{F}(+)\mathcal{K}}\big((a,b)\big) = \begin{cases} 0(+)\mathcal{K}, & if \;\; a = 0 \\ (0,0), & if \;\; a \neq 0 \end{cases},$$

one concludes that $\mathcal{F}$ (+) $\mathcal{K}$ is endo-Noetherian. Also, if $\mathcal{K}$ is an infinite dimensional vector space, then by Theorem 4.8 in [7], $\mathcal{F}$ (+) $\mathcal{K}$ is not a Noetherian ring.

3. An example of a ring that is strongly Hopfian, but is not endo-Noetherian [3, Example 2.7]. Let $(\mathcal{F}_i)_{i\in I}$ be an infinite family of fields and $\mathbb{R} = \prod_{i\in I}\mathcal{F}_i$. One concludes that $\mathbb{R}$ is a strongly Hopfian ring since $\mathbb{R}$ is a zero-dimensional ring. However, $\mathbb{R}$ is not endo-Noetherian (see Theorem 2 below).
4. An example of a Hopfian module that is not Noetherian [5, Remark 2.2(2)]. The additive group $\mathbb{Q}$ of rational numbers is a non-Noetherian $\mathbb{Z}$-module, which is Hopfian.
5. [5, Proposition 2.7(4)] A semi-simple Z-module that is Hopfian but not strongly Hopfian exists.

However, if $\mathbb{R}$ is a semisimple ring, then by [1, Remark 1.9] and [5, Corollary 3.11] we have

Noetherian ⇔ iso-Noetherian ⇔ endo-Noetherian ⇔ strongly Hopfian ⇔ Hopfian.

Now, we recall some definitions and facts that we need to recall in the construction of the skew generalized power series ring (which was defined in [8]).

A monoid $\aleph$ with an order $\preccurlyeq$ is called an ordered monoid $(\aleph, \preccurlyeq)$ if for every $u, v, w \in \aleph$ such that $u \preccurlyeq v$, then $wu \preccurlyeq wv$ and $uw \preccurlyeq vw$. An ordered monoid $(\aleph, \preccurlyeq)$ is called strictly ordered if for every $u, v, w \in \aleph$ such that $u \prec v$, then $wu \prec wv$ and $uw \prec vw$. A partially ordered set $(\aleph, \preccurlyeq)$ is said to be Artinian if every strictly decreasing sequence of elements of $\aleph$ is finite, and $(\aleph, \preccurlyeq)$ is said to be narrow if there is no infinite subset of pairwise order-incomparable elements of $\aleph$. Thus, $(\aleph, \preccurlyeq)$ is Artinian and narrow if and only if every nonempty subset of $\aleph$ has a finite number of minimal elements. Easily, one can see that if $(A_i)_{i\in I}$ is a finite family of Artinian and narrow subsets of an ordered set, then the union $\cup_i A_i$ is also Artinian and narrow. Also, any subset of an Artinian and narrow set is Artinian and narrow. Let R be a ring and $(\aleph, \preccurlyeq)$ a strictly ordered monoid, if g is a map from $\aleph$ into R, then the support of g is denoted by supp $(g) = \{x \in \aleph \mid g(x) \neq 0\}$.

Let $(\aleph, \preccurlyeq)$ be a strictly ordered monoid and R a ring. The ring of generalized power series $[[R^{\aleph, \preccurlyeq}]]$ is the set of all maps $g: \aleph \longrightarrow R$, where supp$(g)$ is Artinian and narrow [see,8]. The addition is pointwise and the multiplication is defined by

$$(gh)(z) = \sum_{(x,y)\in X_z(g,h)} g(x)\,h(y),$$

where

$$X_z(g,h) = \{(x,y) \in \aleph \times \aleph : z = xy; x \in supp(g), y \in supp(h)\}.$$

and $(gh)(z) = 0$ if $X_z(g,h) = \emptyset$ for each $g, h \in [[R^{\aleph, \preccurlyeq}]]$. One can embed the ring R into $R[[\aleph, \sigma]]$ by the map $r \longmapsto c_r$, where

$$c_r(u) = \begin{cases} r, & if \;\; u = 1 \\ 0, & \text{otherwise} \end{cases}$$

and embedding the monoid $\aleph$ into the multiplicative monoid of the ring $R[[\aleph, \sigma]]$ by the map $s \longmapsto e_s$, where

$$e_s(u) = \begin{cases} 1, & if\ u = s \\ 0, & \text{otherwise} \end{cases}.$$

In [9] Mazurek and Ziembowski extended the constructure of the generalized power series ring to a skew version as follows.

Let R be a ring, $(\aleph, \preccurlyeq)$ a strictly ordered monoid, and σ: $\aleph \longrightarrow End$ (R) a monoid homomorphism assigns to each $u \in \aleph$ an endomorphism σ_u *of* R such that $\sigma_u = \sigma(u)$. The skew generalized power series ring R [[$\aleph$, σ]] is the set of all maps $g: \aleph \longrightarrow$ R whose support is Artinian and narrow, the addition is pointwise and the convolution multiplication is defined by

$$(gh)(w) = \sum_{(u,v) \in X_w(g,h)} g(u)\sigma_u\, h(v),$$

and $(gh)(w) = 0$ if $X_w(g,h) = \emptyset$ for each $g, h \in$ R[[$\aleph$, σ]].

The following examples are special constructures of R[[$\aleph$, σ]].

Let R be a ring and θ an endomorphism of R. Then for the additive monoid, $\aleph$ = N of nonnegative integers, the map σ: $\aleph \longrightarrow End$ (R) given by $\sigma(n) = \theta^n$ for any $n \in \aleph$, is a monoid homomorphism. If furthermore, θ is an automorphism of R, then it defines also a monoid homomorphism σ: $\aleph \longrightarrow Aut$ (R) for $\aleph = \mathbb{Z}$, the additive monoid of integers.

a) Let $(\aleph, \preccurlyeq)$ be the additive monoid of non-negative integers with the trivial order (N, =). Then R[[$\aleph$, σ]] is isomorphic to the skew polynomial ring R[x, θ].
b) Let $(\aleph, \preccurlyeq)$ be the additive monoid of non-negative integers with the usual order (N,≤). Then R[[$\aleph$, σ]] is isomorphic to the skew power series ring R[[x, θ]].
c) Let $(\aleph, \preccurlyeq)$ be the group of integers with the trivial order ($\mathbb{Z}$, =), and σ is an automorphism of R. Then R[[$\aleph$, σ]] is isomorphic to the skew Laurent polynomial ring R[x, x^{-1}, θ].
d) Let $(\aleph, \preccurlyeq)$ be the group of integers with the usual order ($\mathbb{Z}$, ≤), and σ is an automorphism of R. Then R[[$\aleph$, σ]] is isomorphic to the skew Laurent series ring R[[x, x^{-1}, θ]].

If θ is the identity map of R, then the above points (a)-(d) show that the ring of polynomials R[x], the ring of power series R[[x]], the ring of Laurent polynomials R[x, x^{-1}], the ring of Laurent series R[[x, x^{-1}]] are special cases of the skew generalized power series ring construction.

Definition 4. [10] An endomorphism φ of a ring R called compatible if ab = 0 ⇔ a φ(b) = 0, for every a, b ∈ ℝ.

Let ℝ be a ring, (ℵ, ≼) a strictly ordered monoid, and σ: ℵ ⟶ *End* (ℝ) a monoid homomorphism. A ring ℝ is said to be ℵ-compatible if σ_x is compatible for all $x \in ℵ$; we say that ℝ is (ℵ, σ)-compatible to indicate the homomorphism σ.

In the following, we show that the ℵ-Armendariz ring, power serieswise Armendariz ring, and Armendariz ring are special cases of (ℵ, σ)- Armendariz ring [See 11, Example 2.2]. Thus, any result on (ℵ, σ)- Armendariz rings has its counterpart in each of these constructures.

Definition 5. [11] Let ℝ be a ring, (ℵ, ≼) a strictly ordered monoid, and σ: ℵ ⟶ *End* (ℝ) a monoid homomorphism. A ring ℝ is (ℵ, σ)-Armendariz if whenever $f, g \in$ ℝ[[ℵ, σ]] satisfy fg = 0, we have $f(x)\ \sigma_x(g(y)) = 0$ for every $x, y \in ℵ$.

a) [12] Let σ be trivial. The ring ℝ is ℵ - Armendariz if whenever fg = 0 for generalized power series $f, g \in [[ℝ^{ℵ,≼}]]$, then $f(u)(g(v)) = 0$ for every u, $v \in ℵ$.
b) [13] Let σ be trivial, and (ℵ, ≼) the additive monoid of non-negative integers with the usual order (N,≤). The ring ℝ is power serieswise Armendariz if whenever $g(x)\ h(x) = 0$ for power series $g(x) = \sum_{i=0}^{\infty} a_i x^i,\ h(x) = \sum_{j=0}^{\infty} b_j x^j \in ℝ[[x]]$, then $a_i b_j = 0$ for every i and j.
c) [14] Let σ be trivial, and (ℵ, ≼) the additive monoid of non-negative integers with the trivial order (N, =). The ring ℝ is Armendariz if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i,\ g(x) = \sum_{j=0}^{n} b_j x^j \in ℝ[x]$ satisfy $f(x)g(x) = 0$, we have $a_i b_j = 0$ for every i and j.

3. Endo-Noetherian skew generalized power series rings

The conditions for the power series rings and polynomial rings over an Armendariz ring to be endo-Noetherian were investigated in [3]. We extend these results to the skew generalized power series rings.

Theorem 1. Let ℝ be a ring, (ℵ, ≼) a strictly ordered monoid, and σ: ℵ ⟶ *End* (ℝ) a monoid homomorphism. Let ℝ is ℵ-compatible, and (ℵ,σ)- Armendariz. Then, the following assumptions are equivalent:

a) ℝ[[ℵ, σ]] is left endo-Noetherian.

b) ℝ satisfies the ascending chain condition on left annihilators of each subset B of ℝ which defines an element f in ℝ[[ℵ, σ]].

Proof. (a)$\Longrightarrow$(b) Let $(B_i)_{i\in I}$ be a sequence of subsets of R, where $\ell.\mathrm{ann}_R(B_1) \subseteq \ell.\mathrm{ann}_R(B_2) \subseteq \ldots$, where B_i is the content of $f_i \in R[[\aleph, \sigma]]$, i.e., $B_i = C_{f_i} = \{f_i(t) \mid t \in \aleph\}$ for each i.

We will show that $\ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_1) \subseteq \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_2) \subseteq \ldots$. Let k $\in N^*$ and $g \in \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_k)$. Since R is an $(\aleph, \sigma)$- Armendariz ring, $g(u)\,\sigma_s(f_k(v)) = 0$ for all u, $v \in \aleph$, R is also $\aleph$-compatible, it follows that $g(u)f_k(v) = 0$. Hence $g(u) \in \ell.\mathrm{ann}_R(B_k) \subseteq \ell.\mathrm{ann}_R(B_{k+1})$ for each u $\in \aleph$ which implies that for each u, $v \in \aleph$, $g(u)\,f_{k+1}(v) = 0$ and $g(u)\sigma_u(f_{k+1}(v)) = 0$. Thus, we have $gf_{k+1} = 0$ and hence

$\ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_1) \subseteq \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_2) \subseteq \ldots$. As $R[[\aleph, \sigma]]$ is a left endo-Noetherian ring, there exists $n \in N^*$ such that $\ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_n) = \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_k)$ for each $k \geq n$. We will show that $\ell.\mathrm{ann}_R(B_n) = \ell.\mathrm{ann}_R(B_k)$ for each $k \geq n$. If r $\in \ell.\mathrm{ann}_R(B_k)$, then 0 = $rf_k(s) = C_r(1)\,f_k(s) = C_r(1)\sigma_1 f_k(s) = (C_r f_k)(s)$ for each s $\in \aleph$. Therefore $C_r \in \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_k) = \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_n)$. Hence r $\in \ell.\mathrm{ann}_R(B_n)$ and R satisfies the ascending chain condition on left annihilators of each subset B of R which defines an element f in $R[[\aleph, \sigma]]$.

(b) $\Longrightarrow$ (a) Let $(f_i)_{i\in I}$ be a sequence of elements of $R[[\aleph, \sigma]]$, where

$$\ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_1) \subseteq \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_2) \subseteq \ldots .$$

We will prove that $\ell.\mathrm{ann}_R(B_1) \subseteq \ell.\mathrm{ann}_R(B_2) \subseteq \ldots$. Let r $\in$ R such that $r(B_k) = 0$. Since R is $\aleph$ -compatible and an $(\aleph, \sigma)$- Armendariz ring we have

$0 = rf_k(u) = C_r(1)\,f_k(u) = C_r(1)\,\sigma_1 f_k(u) = (C_r f_k)(u)$ for each u $\in \aleph$. This implies that $C_r \in \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_k) \subseteq \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_{k+1})$. Similarly, $r \in \ell.\mathrm{ann}_R(B_{k+1})$. Now, let $g \in \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_k)$, where $k \geq n$. Since R is $\aleph$ -compatible and an $(\aleph, \sigma)$- Armendariz ring $g(u)\,\sigma_u\,f_k(v) = g(u)\,f_k(v) = 0$ for each u, $v \in \aleph$. Then for each u $\in \aleph$, $g(u) \in \ell.\mathrm{ann}_R(B_k) = \ell.\mathrm{ann}_R(B_n)$ for each $k \geq n$. We conclude that $g \in \ell.\mathrm{ann}_{R[[\aleph,\sigma]]}(f_n)$ and hence $R[[\aleph, \sigma]]$ is left endo-Noetherian.

In the above Theorem, let σ be the identity map of R. Then we obtain the following corollary.

Corollary 1. For an ℵ - Armendariz ring R the following assumptions are equivalent:

a) $[[R^{\aleph,\preccurlyeq}]]$ is left endo-Noetherian .

b) R satisfies the ascending chain condition on left annihilators of each subset B ⊆ R which defines an element f in $[[R^{\aleph,\preccurlyeq}]]$.

In particular, R is left endo-Noetherian, if $[[R^{\aleph,\preccurlyeq}]]$ is left endo-Noetherian.

In the above Theorem, let σ be the identity map of R, ℵ = N and ≼ is the usual order, we obtain the following corollary.

Corollary 2. (See also [3, Theorem 3.4]) For a power serieswise Armendariz ring R the following assumptions are equivalent:

a) R[[x]] is left endo-Noetherian.

b) R satisfies the ascending chain condition on left annihilators of a countably subset.

c) R satisfies the ascending chain condition on left annihilators of countably generated ideals of R.

d) For each sequence $\left(f_k = \sum_{j\geq 0} a_{k,j}X_j\right)_k$ of elements of R[[x]] such that 33$\ell.\mathrm{ann}_{R[[x]]}(f_1) \subseteq \ell.\mathrm{ann}_{R[[x]]}(f_2) \ldots,$ there exists $\mathrm{n} \in \mathbb{Z}^+$such that $\bigcap_{j\geq 0} \ell.\mathrm{ann}_R(a_{k,j}) = \bigcap_{j\geq 0} \ell.\mathrm{ann}_R(a_{\mathrm{n},j})$ for each $k \geq \mathrm{n}$.

In particular, R is left endo-Noetherian, if R[[x]] is left endo-Noetherian.

In the above Theorem, let σ be the identity map of R, ℵ = N, and ≼ is the trivial order, we obtain the following corollary.

Corollary 3. (See also [3, Theorem 3.1]) For an Armendariz ring R the following assumptions are equivalent:

a) R[x] is left endo-Noetherian.

b) R satisfies the ascending chain condition on left annihilators of finite subsets.

c) R satisfies the ascending chain condition on left annihilators of finitely generated ideals of R.

In particular, R is left endo-Noetherian , if R[x] is left endo-Noetherian.

The following theorem corresponds to [3, Theorem 2.5] with a similar proof.

Theorem 2. Let R be a direct product of rings $(R_i)_{i\in I}$. Then the following assumptions are equivalent:

a) R is left endo-Noetherian.

b) I is finite and R_i is left endo-Noetherian for all $i \in I$.

Proof. (a)⟹(b) Assume that $R = \prod_{i\in I} R_i$ is left endo-Noetherian. Assume that I is not finite, then the chain $\ell.\mathrm{ann}_R((0, 1, 1, ...)) \subseteq \ell.\mathrm{ann}_R((0, 0, 1, 1, ...)) \subseteq ...$ is not stabilize. Let $(a_{r\,i})_r$ be a sequence of elements of R_i for some $i \in I$ such that $\ell.\mathrm{ann}_{R_i}(a_{1\,i}) \subseteq \ell.\mathrm{ann}_{R_i}(a_{2\,i}) \subseteq ...$. Then we have $\ell.\mathrm{ann}_R((0, 0, ..., a_{1\,i}, 0, 0, ...)) \subseteq \ell.\mathrm{ann}_R((0, 0, ..., a_{2\,i}, 0, 0, ...)) \subseteq ...$.

Now, since R is left endo-Noetherian, there exists $m \in \mathbb{N}^*$ such that $\ell.\mathrm{ann}_R((0, 0, ..., a_{k\,i}, 0, 0, ...)) = \ell.\mathrm{ann}_R((0, 0, ..., a_{m\,i}, 0, 0, ...))$ for each $k \geq m$. Let $b_i \in \ell.\mathrm{ann}_{R_i}(a_{k\,i})$, then $(0, 0, ..., b_i, 0, 0, ...) \in \ell.\mathrm{ann}_R((0, 0, ..., a_{k\,i}, 0, 0, ...)) = \ell.\mathrm{ann}_R((0, 0, ..., a_{m\,i}, 0, 0, ...))$, hence $b_i \in \ell.\mathrm{ann}_{R_i}(a_{m\,i})$, $\ell.\mathrm{ann}_{R_i}(a_{k\,i}) = \ell.\mathrm{ann}_{R_i}(a_{m\,i})$, which implies that R_i is left endo-Noetherian.

(b)⟹(a) Let R_1, R_2,...and R_n are left endo-Noetherian rings. We will show that $R = \prod_{i\in I} R_i$ is left endo-Noetherian. Let $(a_{11}, a_{12},..., a_{1n})$, $(a_{21}, a_{22},..., a_{2n}),\cdots \in R$ such that $\ell.\mathrm{ann}_R((a_{11}, a_{12},..., a_{1n})) \subseteq \ell.\mathrm{ann}_R((a_{21}, a_{22},..., a_{2n})) \subseteq ...$. It follows easily that $\ell.\mathrm{ann}_{R_i}(a_{1i}) \subseteq \ell.\mathrm{ann}_{R_i}(a_{2i}) \subseteq ...$, $i = 1, 2, ..., n$. Since $R_1, R_2,..., R_n$ are left endo-Noetherian rings, there exists positive integers $m_1, m_2,...,m_n$, n such that for each $k_i \geq m_i$, $\ell.\mathrm{ann}_{R_i}(a_{k_i\,i}) = \ell.\mathrm{ann}_{R_i}(a_{m_i\,i})$, $i = 1,2,..., n$. We obtain $\ell.\mathrm{ann}_R((a_{m1}, a_{m2},..., a_{mn})) = \ell.\mathrm{ann}_R((a_{k1}, a_{k2},..., a_{kn}))$, where $m = \max_{1\leq i\leq n}\{m_i\}$, and for all $k \geq m$, we conclude that R is left endo-Noetherian.

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