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Solution of Ordinary Differential Equations Using Adomian Decomposition Method

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Abstract

In this paper, we apply the Adomian decomposition method (ADM) for solving linear and nonlinear ordinary differential equations (ODEs). The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are discussed. Some applications are solved such as relaxation-oscillation equation.

Keywords: *Adomian Method; existence; uniqueness; error analysis; relaxation-oscillation equation.*

1 Introduction

Differential equations have many applications in engineering and science, including electrical networks, fluid flow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, optical and neural network systems [1]-[11]. In this paper, Adomian decomposition method (ADM) [12]-[19] is used to solve these type of equations. This method has many advantages, it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization. The convergence of the series solution and the error analysis are discussed. Some numerical examples and applications (such as relaxation-oscillation equation) are solved.

2 Problem solving

2.1 The solution algorithm

$$
\begin{aligned}
&\mathrm{L}y(t) + f(y) = x(t), \\
&\qquad y^{(j)} = 0, \quad j = 0, 1, 2, \ldots, n-1.
\end{aligned}
\tag{1}
$$

Where

$$\mathrm{L} = L + R, \tag{2}$$

$$L = \frac{d^n}{dt^n}, \quad \text{and} \quad R = \sum_{k=0}^{n-1} a_k(t) \frac{d^k}{dt^k}. \tag{3}$$

And $f(y)$ is the nonlinear term expanded in terms of Adomian polynomials,

$$f(y) = \sum_{n=0}^{\infty} A_n, \tag{4}$$

$$A_n = \left(\frac{1}{n!}\right) \frac{d^n}{d\lambda^n} \left[f\left(\sum_{j=0}^{\infty} \lambda^j y_j \right) \right]_{\lambda=0}, \tag{5}$$

And the linear operator L as defined before in equations (2) and (3). Substitute from (3) and (2) into (1) we get,

$$(L + R)y(t) + \sum_{n=0}^{\infty} A_n = x(t), \tag{6}$$

$$Ly(t) = x(t) - Ry(t) - \sum_{n=0}^{\infty} A_n, \tag{7}$$

Applying L^{-1} to both sides of equation (7) we have,

$$y(t) = L^{-1} x(t) - L^{-1} Ry(t) - L^{-1} \left(\sum_{n=0}^{\infty} A_n \right), \tag{8}$$

Decompose $y(t) = \sum_{n=0}^{\infty} y_n(t)$ and substitute in equation (8), we get the following recursive relations,

$$y_0(t) = L^{-1} x(t), \tag{9}$$

$$y_n(t) = -L^{-1} R y_{n-1}(t) - L^{-1} A_{n-1}. \tag{10}$$

Finally, the solution of (1) is

$$y(t) = \sum_{n=0}^{\infty} y_n(t). \tag{11}$$

3 Convergence analysis

3.1. Existence and uniqueness of the solution

Define the mapping $F: E \to E$ where E is the Banach space ($C[I], \|\cdot\|$), the space of all continuous functions on I with the norm $\|y(t)\| = \max_{t\in I} |y(t)|$, $\left|\sum_{k=0}^{n-1} a_k(t)\right| \prec M$ $\forall\, 0 \le \tau \le t \le T$, M is finite constant and $f(y)$ satisfy Lipschitz condition with Lipschitz constants C such as, constants C such as,

$$|f(y) - f(z)| \le C|y - z|$$

Theorem 1:

The problem (1) has a unique solution whenever $0 < \beta < 1$ *where,* $\beta = T^n[M + C]$.

Proof:

The mapping $F: E \to E$ is defined as,

$$Fy(t) = L^{-1}x(t) - L^{-1}Ry(t) - L^{-1}(f(y))$$

Let $y(t), z(t) \in E$:

$$\begin{aligned}
\|Fy - Fz\| &= \max_{t\in I} \left|-L^{-1}Ry(t) - L^{-1}(f(y)) + L^{-1}Rz(t) + L^{-1}(f(z))d\tau\right| \\
&= \max_{t\in I} |[L^{-1}Ry(t) - L^{-1}Rz(t)] + [L^{-1}(f(y)) - L^{-1}(f(z))]d\tau| \\
&\le \max_{t\in I} \left|L^{-1}R[y(t) - z(t)]\right| + \max_{t\in I} |L^{-1}[f(y) - f(z)]|d\tau \\
&\le \max_{t\in I} |y(t) - z(t)|\left|L^{-1}R[1]\right| + C\max_{t\in I} |y(t) - z(t)||L^{-1}[1]|d\tau \\
&\le \max_{t\in I} |y(t) - z(t)|\left|\sum_{k=0}^{n-1} a_k(t)\right| T^{n-k} + CT^n \max_{t\in I} |y(t) - z(t)| \\
&\le [MT^n + CT^n]\|y - z\| \\
&\le T^n[M + C]\|y - z\| \\
&\le \beta\|y - z\|
\end{aligned}$$

Under the condition $0 < \beta < 1,$ the mapping F is contraction and hence there exists a unique solution of the problem *(1)* and this completes the proof. ■

3.2. Proof of convergence

Theorem 2:

The series solution (11) of the problem (1) using ADM converges if $|y_1| < \infty$ *and* $0 < \beta < 1,\ \beta = T^n[M + L]$.

Proof: Define the sequence $\{S_n\}$ such that, $S_n = \sum_{i=0}^{n} y_i(t)$ is the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} y_i(t)$ since,

$$f(y) = f\left(\sum_{i=0}^{\infty} y_i(t)\right) = \sum_{i=0}^{\infty} A_i(y_0, y_1, \dots, y_i)$$

So,

$$f(S_n) = \sum_{i=0}^{n} A_i(y_0, y_1, \dots, y_i)$$

Let S_n and S_m be two arbitrary partial sums with $n > m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned}
\| S_n - S_m \| &= \max_{t \in I} |S_n - S_m| = \max_{t \in I} \left| \sum_{i=m+1}^{n} y_i(t) \right| \\
&= \max_{t \in I} \left| \sum_{i=m+1}^{n} [L^{-1} R y_i(t) + L^{-1}(f(y))] \right| \\
&= \max_{t \in I} \left| \left[L^{-1} R \sum_{i=m+1}^{n} y_i(t) + L^{-1} \sum_{i=m+1}^{n} A_i d\tau \right] \right| \\
&= \max_{t \in I} \left| \left[L^{-1} R \sum_{i=m}^{n-1} y_i(t) + L^{-1} \sum_{i=m}^{n-1} A_i d\tau \right] \right| \\
&= \max_{t \in I} |[L^{-1} R[S_{n-1} - S_{m-1}] + L^{-1}[f(S_{n-1}) - f(S_{m-1})] d\tau]| \\
&\le \max_{t \in I} L^{-1} R|S_{n-1} - S_{m-1}| + L^{-1}|f(S_{n-1}) - f(S_{m-1})| d\tau \\
&\le \max_{t \in I} L^{-1} R|S_{n-1} - S_{m-1}| + L^{-1}|f(S_{n-1}) - f(S_{m-1})| d\tau \\
&\le [MT^n + CT^n] \| S_{n-1} - S_{m-1} \| \\
&\le \beta \| S_{n-1} - S_{m-1} \|
\end{aligned}$$

Let $n = m + 1$ then,

$$\| S_{m+1} - S_m \| \leq \beta \| S_m - S_{m-1} \| \leq \beta^2 \| S_{m-1} - S_{m-2} \| \leq \cdots \leq \beta^m \| S_1 - S_0 \|$$

From the triangle inequality we have,

$$\begin{aligned} \| S_n - S_m \| &\leq \| S_{m+1} - S_m \| + \| S_{m+2} - S_{m+1} \| + \cdots + \| S_n - S_{n-1} \| \\ &\leq [\beta^m + \beta^{m+1} + \cdots + \beta^{n-1}] \| S_1 - S_0 \| \\ &\leq \beta^m [1 + \beta + \cdots + \beta^{n-m-1}] \| S_1 - S_0 \| \\ &\leq \beta^m \left[\frac{1 - \beta^{n-m}}{1 - \beta} \right] \| y_1(t) \| \end{aligned}$$

Since, $0 < \beta < 1$, and $n > m$ then, $(1 - \beta^{n-m}) \leq 1$. Consequently,

$$\begin{aligned} \| S_n - S_m \| &\leq \frac{\beta^m}{1 - \beta} \| y_1(t) \| \\ &\leq \frac{\beta^m}{1 - \beta} \max_{t \in I} |y_1(t)| \end{aligned}$$

but, $|y_1(t)| < \infty$ and as $m \to \infty$ then, $\| S_n - S_m \| \to 0$ and hence, $\{S_n\}$ is a Cauchy sequence in this Banach space so, the series $\sum_{n=0}^{\infty} y_n(t)$ converges and this completes the proof. ∎

3.3. Error analysis

For ADM, we can estimate the maximum absolute truncated error of the Adomian's series solution in the following theorem.

<u>**Theorem 3**</u>: *The maximum absolute truncation error of the series solution (11) to the problem (1) is estimated to be,*

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^{m} y_i(t) \right| \leq \frac{\beta^m}{1 - \beta} \max_{t \in I} |y_1(t)|$$

<u>**Proof:**</u> From theorem 2 we have,

$$\| S_n - S_m \| \leq \frac{\beta^m}{1 - \beta} \max_{t \in I} |y_1(t)|$$

But, $S_n = \sum_{i=0}^{n} y_i(t)$ as $n \to \infty$ then, $S_n \to y(t)$ so,

$$\| y(t) - S_m \| \leq \frac{\beta^m}{1 - \beta} \max_{t \in I} |y_1(t)|$$

So, the maximum absolute truncation error in the interval J is,

$$\max_{t\in I}\left|y(t)-\sum_{i=0}^{m} y_i(t)\right| \le \frac{\beta^m}{1-\beta}\max_{t\in I}|y_1(t)|$$

And this completes the proof. ■

4. Numerical Examples

4.1. Application: relaxation-oscillation equation

The Relaxation-Oscillation equation is,

$$\begin{aligned} D^m y(t) + Ay(t) &= f(t), \qquad t>0, \\ y^{(k)}(0) &= 0, \qquad (k=0,1,\dots,m-1). \end{aligned} \tag{12}$$

We will solve it by using ADM in two cases when $m=1$ and $m=2$.

<u>Case 1 ($m=1$):</u>

In this case this problem is called the *relaxation differential equation.* If we take $A=1,$ $f(t)=H(t),$ and $y(0)=0,$ the equation (12) will be,

$$\frac{dy}{dt} + y(t) = H(t), \qquad y(0)=0, \tag{13}$$

where $H(t)$ is the unit-step function, and it has the exact solution $y(t)=1-e^{-t}$.

Using ADM we get,

$$y_0(t) = \int_0^t H(\tau)d\tau, \tag{14}$$

$$y_n(t) = -\int_0^t y_{n-1}(\tau)d\tau, \qquad n\ge 1. \tag{15}$$

from equations (14) and (15) we have,

$$y_0 = t,\ y_1 = -\frac{t^2}{2},\ y_2 = \frac{t^3}{6},\ y_3 = -\frac{t^4}{24},\ y_4 = \frac{t^5}{120},\ \dots$$

hence,

$$y(t) = \sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + y_3 + y_4 + \cdots$$

$$\begin{aligned} y(t) &= t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{24} + \frac{t^5}{120} - \cdots \\ &\approx 1 - e^{-t}. \end{aligned} \tag{16}$$

A comparison between the exact and ADM solutions is given in figures 1.a-1.c. From these figures, we see that when we increase the number of the terms n, the solution will be more accurate, moreover, it gives the exact solution.

<u>Notices:</u>

1) All computations and figures are made using *MATHEMATICA* software for all the given examples.

2) In all figures, the solid curve represents ADM solution, while the other curve for the other method.

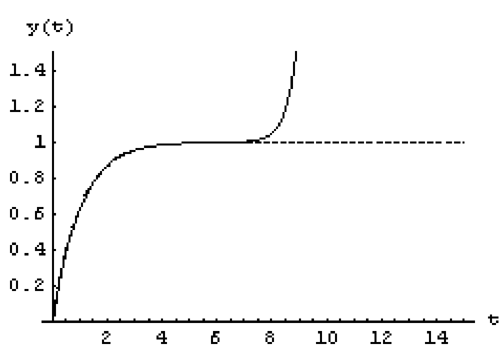

Fig2.1.a:ADMand Exact Sol. [n=20]

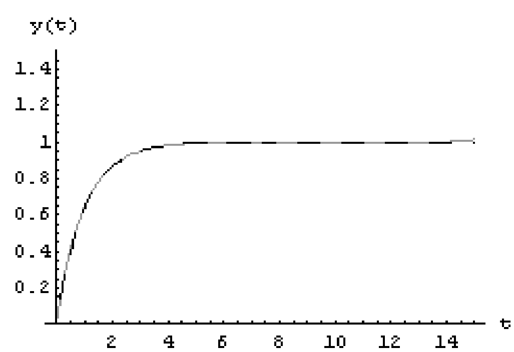

Fig2.1.b:ADMand Exact Sol. [n=40]

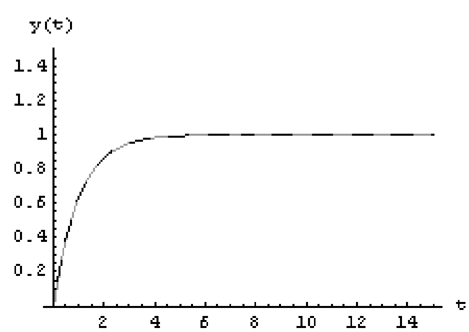

Fig2.1.c:ADMand Exact Sol. [n=60]

Case 2 ($m = 2$):

In this case this problem is called the *oscillation differential equation.* If we take $A = 1$, $f(t) = H(t), \quad y(0) = 0,$ and $y'(0) = 0,$ the equation (12) will be,

$$\frac{d^2 y}{dt^2} + y(t) = H(t), \qquad y(0) = 0,\ y'(0) = 0. \tag{17}$$

which has the exact solution $y(t) = 1 - \cos(t)$.

Using ADM we get,

$$\begin{aligned} y_0(t) &= \int_0^t \int_0^t H(\tau)\,d\tau\,d\tau, \\ y_n(t) &= -\int_0^t \int_0^t y_{n-1}(\tau)\,d\tau\,d\tau, \qquad n \geq 1. \end{aligned} \tag{18}$$

from equation (18) we have,

$$y_0 = \frac{t^2}{2},\ y_1 = -\frac{t^4}{24},\ y_2 = \frac{t^6}{720},\ y_3 = -\frac{t^8}{40320},\ y_4 = \frac{t^{10}}{3628800},\cdots$$

hence,

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + y_3 + y_4 + \cdots \\ &= \frac{t^2}{2} - \frac{t^4}{24} + \frac{t^6}{720} - \frac{t^8}{40320} + \frac{t^{10}}{3628800} - \cdots \\ &\approx 1 - \cos(t). \end{aligned} \tag{19}$$

A comparison between the exact and ADM solutions is given in figure 2 ·

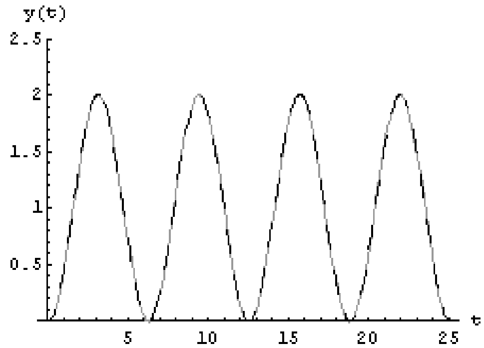

Fig2.2:ADMand Exact Sol.[n=35]

4.1.2. Numerical Example

Example *Consider the initial value problem,*

$$\begin{aligned} Dy &= y^2 + 1, \\ y(0) &= 0, \quad 0 < t < 1. \end{aligned} \tag{20}$$

Which has the exact solution $y(t) = \tan(t)$.

Applying ADM to the problem (20) we have,

$$\begin{aligned} y_0 &= \int_0^t 1 d\tau, \\ y_n &= \int_0^t A_{n-1}(\tau) d\tau, \quad n \geq 1. \end{aligned} \tag{21}$$

From equation (21) we find that,

$$y_0 = t,\ y_1 = \frac{t^3}{3},\ y_2 = \frac{2t^5}{15},\ y_3 = \frac{17t^7}{315},\ y_4 = \frac{62t^9}{2835}, \ldots$$

Hence the approximate solution of the problem (20) is given by the truncated series,

$$\begin{aligned} \Phi_5(t) &= \sum_{n=0}^{4} y_n = y_0 + y_1 + y_2 + y_3 + y_4 \\ &= t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} \approx \tan(t). \end{aligned} \tag{22}$$

A comparison between the exact and ADM solutions is given in figure 3. We see from this figure that ADM gives the exact solution.

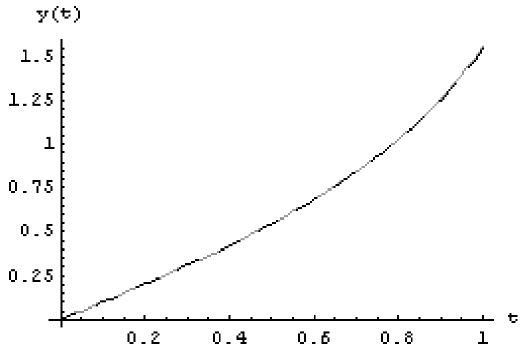

Fig2.3:ADMandExactSol.[n=5]

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