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THE EFFECT OF FRACTIONAL ORDER ON SYNCHRONIZATION OF TWO FRACTIONAL ORDER CHAOTIC AND HYPERCHAOTIC SYSTEMS

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Abstract. This paper studies the synchronization of two commensurate fractional order chaotic and hyperchaotic systems using nonlinear control technique. We discuss some stability conditions of three and four dimensional fractional order systems. We apply these stability conditions to chaos and hyperchaos synchronization. The effect of fractional order on synchronization of fractional order chaotic and hyperchaotic systems is shown; chaos synchronization of the commensurate fractional order Liu system is achieved, while it is not achieved in its integer order counterparts using the same nonlinear controllers. Furthermore, achieving chaos synchronization via nonlinear control of the novel hyperchaotic system is found just in the fractional order case when using the same nonlinear control laws. Numerical simulations are used to verify the theoretical analysis.

1. Introduction

Fractional calculus can be traced back three hundred years ago [1-2]. Recently, the concept of fractional derivative has been widely investigated [3-5] since it has tremendous potential to change the way we see, model, and control the nature around us. Denying fractional derivatives is like saying that zero, fractional, or irrational numbers do not exist. It has been found that many systems in interdisciplinary fields can be elegantly modeled with the help of the fractional derivatives such as viscoelastic systems [6], dielectric polarization [7], electrode-electrolyte polarization [8], electromagnetic waves [9], quantum evolution of complex systems [10], nonlinear oscillation of earthquakes [11], diffusion waves [12], electromagnetism [13], and mechanics [14]. Furthermore, fractional calculus has been recently found to have useful applications in many scientific fields such as general physics [1-2, 15], kinetic theories [16-17], engineering [1], statistical mechanics [18], quantum mechanics [19], finance [20], mathematical biology [21-23] and social sciences [24-25]. Meanwhile, the applications of chaos in physics and engineering have caught much attention during the past two decades [26-30]. Indeed, chaotic behaviors have recently been found in many fractional order chaotic systems [31-36]. Moreover,

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chaos synchronization is also one of the most important applications of chaos theory. Chaos synchronization in integer and fractional order systems has received more interest [26-30, 37-38] due to its potential applications to many fields of sciences especially secure communications [39]. Furthermore, a chaotic system has one positive Lyapunov exponent and a hyperchaotic system is classified as a chaotic system with more than one positive Lyapunov exponent. Thus, a hyperchaotic system shows more complex behaviors and abundant dynamics than chaotic system. Recently, some fractional order hyperchaotic systems have been appeared such as the fractional order hyperchaotic system of Rossler [34], the fractional order hyperchaotic system of Lorenz [40], the fractional order hyperchaotic system by Deng et al. [41], the fractional order hyperchaotic system of Chen [42] and the Novel fractional order hyperchaotic system [43]. Based on some stability conditions of fractional order systems, we show the effect of fractional order on the synchronization of fractional order chaotic and hyperchaotic systems. For this purpose, we give two examples one is to synchronize the commensurate fractional order chaotic Liu system and the other is to synchronize the commensurate novel fractional order hyperchaotic system, while their integer order counterparts are not synchronized using the same nonlinear controllers.

2. Preliminaries

The fractional order derivatives have several definitions. We use the Caputo definition of fractional derivative [1, 44] which is called smooth fractional derivative and is widely used in real applications:

$$ {}_0D_t^\alpha g(t) = \begin{cases} \frac{\int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{dt^m} g(\tau) d\tau}{\Gamma(m-\alpha)}, & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} g(t), & m = \alpha, \end{cases} \tag{2.1} $$

where m is the least integer which is not less than α and Γ stands for Gamma function. Here, we use the operator D^α, which is generally called "α-order Caputo differential operator", where $D^\alpha \equiv {}_0D_t^\alpha$.

Theorem 2.1 [45]. Consider the following commensurate linear autonomous fractional order system:

$$ D^\alpha X(t) = AX(t), \qquad X(0) = X_0, \tag{2.2} $$

with $X(t) = (x_1, ..., x_n)^T \in R^n$, $A \in R^{n\times n}$ and $\alpha \in (0, 1)$. The autonomous system (2.2) is:

(1) asymptotically stable if and only if $|\arg(\lambda)| > \alpha\pi/2$. Moreover, the components of the state decay towards 0 like $t^{-\alpha}$.
(2) stable if and only if $|\arg(\lambda)| \geq \alpha\pi/2$ for all eigenvalues λ. Here, λ represents an eigenvalue of the matrix A.

The stability region of the linear fractional order system is shown in figure 1 (in which $J = \sqrt{-1}$).

In the following, we will discuss the stability of the commensurate nonlinear autonomous fractional order system:

$$ D^\alpha X(t) = f(X(t)), \qquad X(0) = X_0. \tag{2.3} $$

The equilibrium solutions of system (2.3) are obtained by equating its right-hand sides to zero.

Theorem 2.2 Let $X^* = (x_1^*, \cdots, x_n^*)$ be an equilibrium point of system (2.3), and $J = \frac{\partial f}{\partial X}$ is the Jacobian matrix at the equilibrium point X^*, then X^* is locally asymptotically stable if all the eigenvalues $(\lambda_1, ..., \lambda_n)$ of the Jacobian matrix J satisfy the conditions:

$$|\arg(\lambda_i)| > \alpha\pi/2, \quad (i = 1, \cdots, n). \tag{2.4}$$

Proof. Consider the n-dimensional commensurate nonlinear autonomous fractional order system:

$$D^\alpha x_1(t) = f_1(x_1, ..., x_n), \ \ldots, \ D^\alpha x_n(t) = f_n(x_1, ..., x_n), \tag{2.5}$$

The initial values of system (2.5) are given as; $x_1(0) = x_{10}, \ \ldots, \ x_n(0) = x_{n0}$. If $x_i(t) = x_i^* + \delta_i(t)$, then

$$D^\alpha(x_i^* + \delta_i) = f_i(x_1^* + \delta_1, \cdots, x_n^* + \delta_n), \quad i = 1, \cdots, n.$$

This implies that

$$D^\alpha \delta_i(t) = f_i(x_1^* + \delta_1, ..., x_n^* + \delta_n),$$

Using the Taylor expansion and the fact that $f_i(x_1^*, \cdots, x_n^*) = 0$, then

$$D^\alpha \delta_i(t) \approx \frac{\partial f_i}{\partial x_1} |_{X=X^*} \delta_1 + \cdots + \frac{\partial f_i}{\partial x_n} |_{X=X^*} \delta_n,$$

which reduces to the following system

$$D^\alpha \delta = J\delta, \quad \delta = (\delta_1, ..., \delta_n)^T, \quad J(X^*) = (\frac{\partial f_i}{\partial x_j})_{ij} |_{X=X^*}, \tag{2.6}$$

where $J(X^*)$ satisfies the following relation:

$$B^{-1}JB = C, \qquad C = diag(\lambda_1, \ldots, \lambda_n),$$

where B is the eigenvector of J. System (2.6) has the initial values

$$\delta_1(0) = x_1(0) - x_1^*, \ \cdots, \ \delta_n(0) = x_n(0) - x_n^*.$$

Thus, system (2.6) becomes

$$D^\alpha \delta = (BCB^{-1})\delta, \qquad D^\alpha(B^{-1}\delta) = C(B^{-1}\delta),$$

hence

$$D^\alpha \xi = C\xi, \qquad \xi = B^{-1}\delta, \qquad \xi = (\xi_1, ..., \xi_n)^T,$$

therefore

$$D^\alpha \xi_1 = \lambda_1 \xi_1, \cdots, D^\alpha \xi_n = \lambda_n \xi_n.$$

The solutions of the last equations are obtained by using Mittag-Leffler functions:

$$\begin{aligned}
&\xi_1(t) = \textstyle\sum_{k=0}^{\infty} \frac{(\lambda_1)^k t^{k\alpha}}{\Gamma(k\alpha+1)} \xi_1(0) = E_\alpha(\lambda_1 t^\alpha)\xi_1(0),\\
&\vdots\\
&\xi_n(t) = \textstyle\sum_{k=0}^{\infty} \frac{(\lambda_n)^k t^{k\alpha}}{\Gamma(k\alpha+1)} \xi_n(0) = E_\alpha(\lambda_n t^\alpha)\xi_n(0).
\end{aligned}$$

Using the conditions (2.4), it follows that $\xi_1(t), \cdots, \xi_n(t)$ are decreasing and consequently $\delta_1(t), \cdots, \delta_n(t)$ are decreasing. So, the equilibrium point X^* is locally asymptotically stable if the conditions (2.4) are satisfied.

Thus, the equilibrium point X^* of the fractional order system (2.3) is as locally asymptotically stable as its integer order form.

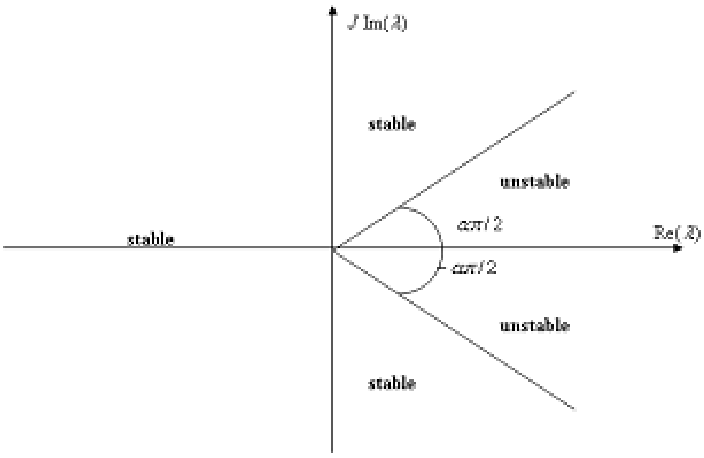


Figure 1. Stability region of linear fractional order system with order α.

2.1 Numerical simulation of fractional order differential systems

Throughout this paper, the simulations results are carried out using an efficient method for solving fractional order differential equations that is; the predictor-correctors scheme which has been investigated in [46-48], and represents a generalization of the Adams-Bashforth-Moulton algorithm. To explain the method we consider the following fractional order system:

$$D^{\alpha}y(t) = g(t, y(t)), \qquad 0 \le t \le T, \quad y^{(k)}(0) = y_0^{(k)}, \quad k = 0, ..., m-1, \tag{2.7}$$

then the initial value problem (2.7) is equivalent to Volterra integral equation of the second kind:

$$y(t) = \sum_{k=0}^{[\alpha]-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} g(\varsigma, y(\varsigma)) d\varsigma. \tag{2.8}$$

Set $h = T/N,\ t_n = nh,\ n = 0, 1, ..., N \in Z^+$. Then (2.8) can be discretized as follows:

$$y_h(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} g(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum \omega_{j,n+1} g(t_j, y_h(t_j)),$$

where

$$\omega_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \leq j \leq n, \\ 1, & j = n+1, \end{cases}$$

$$y_h^p(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \psi_{j,n+1} g(t_j, y_h(t_j)),$$

$$\psi_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j)^\alpha).$$

The error estimate is $\max_{j=0,\,1,\,\ldots,\,N} |y(t_j) - y_h(t_j)| = O(h^p)$, in which $p = \min(2,\ 1+\alpha)$.

3. Systems description

The commensurate fractional order Liu system is given as follows [35]:

$$D^\alpha x = a(y-x), \quad D^\alpha y = bx - kxz, \quad D^\alpha z = -cz + qx^2, \tag{3.1}$$

where α is the fractional order and $\alpha \in (0,\ 1]$. The parameters a, c, k, q are all positive real parameters and $b \in R$. At $\alpha = 1$, system (3.1) becomes the original integer order Liu system which exhibits chaotic behaviors using the parameter values $a = 10, b = 40, c = 2.5, q = 4$ and $k = 1$ [49]. The equilibrium points of the fractional order Liu system (3.1) are $E_0 = (0,0,0)$, $E_1 = (\bar{x}, \bar{y}, \bar{z}) = (\sqrt{bc/qk},\ \sqrt{bc/qk}, b/k)$ and $E_2 = (-\bar{x}, -\bar{y}, \bar{z})$. Using the above-mentioned parameter values and fractional order $\alpha = 0.9$, system (3.1) shows chaotic behavior (see figure 2).

Now, we consider the commensurate novel fractional order hyperchaotic system [43]:

$$D^\alpha x = \sigma(y-x) + \mu yz,\ \ D^\alpha y = \rho x - dxz + y + w,\ \ D^\alpha z = xy - \eta z,\ \ D^\alpha w = -\nu y, \tag{3.2}$$

where $\alpha \in (0,1]$ and σ, η, ρ, d, μ, ν are positive real parameters. At $\alpha = 1, \sigma = 35$, $\eta = 4$, $\rho = 25, d = 5, \mu = 35$, $\nu = 100$, system (3.2) exhibits hyperchaotic behavior [50]. System (3.2) has only the equilibrium point $E_0 = (0,\ 0,\ 0,\ 0)$. Moreover, system (3.2) shows hyperchaotic behavior using the above choice of the parameter values and fractional order $\alpha = 0.97$ (see figure 3).

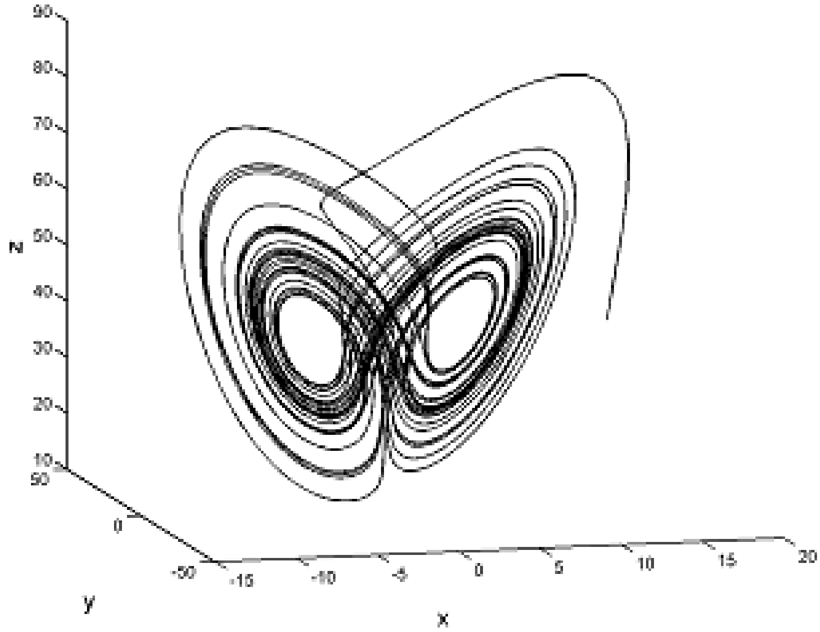


Figure 2: Chaotic attractor of the fractional order Liu system with commensurate order $\alpha = 0.9$ and parameter values $a = 10, b = 40, c = 2.5, q = 4, k = 1$.

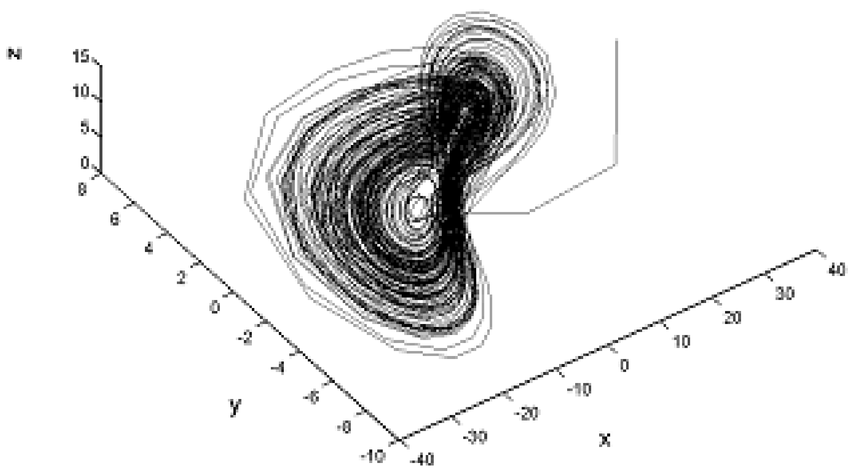


Figure 3: Hyperchaotic attractor of the novel fractional order hyperchaotic system with commensurate order $\alpha = 0.97$ and $\sigma = 35, \eta = 4, \rho = 25, d = 5, \mu = 35$ and $\nu = 100$.

4. Some stability conditions of the fractional order systems

Consider the n-dimensional commensurate fractional order system

$$D^{\alpha}x_1(t) = f_1(x_1,\ ...,x_n),\ ...\ ,D^{\alpha}x_n(t) = f_n(x_1,\ ...,x_n), \tag{4.1}$$

where $0 < \alpha \leq 1$. The point $\bar{E} = (\bar{x}_1,\ ...,\bar{x}_n)$ is defined as an equilibrium point of (4.1). The characteristic equation of the equilibrium point $\bar{E}$ is given as:

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ ...\ + a_n = 0, \tag{4.2}$$

whose discriminant is given as follows:

$$D(P) = (-1)^{n(n-1)/2}R(P, P'), \tag{4.3}$$

where P' is the derivative of P, $R(P, Q)$ is the determinant of the corresponding Sylvester $(n+l) \otimes (n+l)$ matrix and $Q(\lambda) = \lambda^n + b_1\lambda^{l-1} + \ ...\ + b_l$.

At $n = 3$, the characteristic equation of the equilibrium solution $\bar{E} = (\bar{x}_1,\ \bar{x}_2, \bar{x}_3)$ is given as

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{4.4}$$

and the discriminant $D(P)$ of $P(\lambda)$ is defined as

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2. \tag{4.5}$$

Thus, we have the following Proposition [36, 51]:

Proposition 4.1 In the case $n = 3$, if the conditions $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_1a_2 = a_3$ are satisfied, then the equilibrium point $\bar{E}$ of system (4.1) is not locally asymptotically stable when $\alpha = 1$. However, $\bar{E}$ is locally asymptotically stable when $0 < \alpha < 1$.

At $n = 4$, the characteristic equation (4.2) is reduced to:

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{4.6}$$

and its discriminant is given by

$$\begin{aligned} D(P) = &-4a_3^3a_1^3 + a_2^2a_1^2a_3^2 + 18a_3^3a_2a_1 - 6a_1^2a_3^2a_4 - 4a_2^3a_1^2a_4 - 80a_3a_2^2a_1a_4 + 144a_3^2a_2a_4 \\ &-192a_3a_4^2a_1 + 144a_2a_1^2a_4^2 + 18a_3a_2a_1^3a_4 - 27a_3^4 - 4a_2^3a_3^2 - 128a_4^2a_2^2 - 27a_1^4a_4^2 \\ &+256a_4^3 + 16a_2^4a_4. \end{aligned} \tag{4.7}$$

In this case, the equilibrium point $\bar{E} = (\bar{x}_1,\ \bar{x}_2, \bar{x}_3, \bar{x}_4)$ is locally asymptotically stable if all the roots of equation (4.6) satisfy the following conditions [45]:

$$|\arg(\lambda_i)| > \alpha\pi/2,\ \ (i = 1, 2, 3, 4). \tag{4.8}$$

Furthermore, a fractional Routh-Hurwitz condition for the stability of fractional order hyperchaotic systems is introduced in the following Proposition [43]:

Proposition 4.2 The equilibrium point $\bar{E} = (\bar{x}_1,\ \bar{x}_2, \bar{x}_3, \bar{x}_4)$ is locally asymptotically stable, for all $\alpha \in (0,\ 1)$, if the following conditions are satisfied:

$$D(P) < 0, a_1 > 0,\ a_2 > 0,\ \ a_3 > 0,\ \ a_4 > 0 \text{ and } a_2 = a_1a_4/a_3 + a_3/a_1. \tag{4.9}$$

Remark 4.1 If $\alpha = 1$, then the equilibrium point $\bar{E}$ in Proposition 4.2 is not locally asymptotically stable.

5. Application to the synchronization of the commensurate fractional order Liu system

In this section, we will show the effect of the fractional order $0 < \alpha < 1$ on the synchronization of the drive and response commensurate fractional order Liu systems using nonlinear control. The drive system is given as follows:

$$D^\alpha x_1 = a(y_1 - x_1), \quad D^\alpha y_1 = bx_1 - kx_1z_1, \quad D^\alpha z_1 = -cz_1 + qx_1^2, \tag{5.1}$$

and the response system is

$$D^\alpha x_2 = a(y_2 - x_2) + u_1, \quad D^\alpha y_2 = bx_2 - kx_2z_2 + u_2, \quad D^\alpha z_2 = -cz_2 + qx_2^2 + u_3, \tag{5.2}$$

where the controllers $u_1,\ u_2$ and u_3 are nonlinear control functions.

By subtracting (5.1) from (5.2) and setting $e_1 = x_2 - x_1,\ e_2 = y_2 - y_1,\ e_3 = z_2 - z_1$, we get:

$$D^\alpha e_1 = a(e_2 - e_1) + u_1,\ D^\alpha e_2 = be_1 + kx_1z_1 - kx_2z_2 + u_2,\ D^\alpha e_3 = -ce_3 + q(x_1 + x_2)e_1 + u_3. \tag{5.3}$$

If we choose the control laws as follow:

$$u_1 = -k_1e_1,\ u_2 = k(x_1e_3 + z_1e_1 + e_1e_3 - \bar{x}e_3 - \bar{z}e_1) - k_2e_2,\ u_3 = -q(x_1 + x_2)e_1 + 2\bar{x}qe_1 - k_3e_3, \tag{5.4}$$

then the error dynamical system (5.3) has the following characteristic equation:

$$P(\lambda) = \lambda^3 + (r_1 + r_2 + k_2)\lambda^2 + (r_1r_2 + r_1k_2 + r_2k_2)\lambda + r_1r_2k_2 + 2abc = 0, \tag{5.5}$$

where $r_1 = c + k_3 > 0$ and $r_2 = a + k_1 > 0$.

5.1 Numerical algorithms and approximate solutions

Based on the predictor-corrector scheme, systems (5.1) and (5.2) can be discretized as follow:

$$\begin{cases} x_{n+1} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\sum_{j=0}^{n} \gamma_{1,j,n+1}.a(y_j - x_j) + a(y_{n+1}^p - x_{n+1}^p)\right), \\ y_{n+1} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\sum_{j=0}^{n} \gamma_{2,j,n+1}.(bx_j - kx_jz_j) + (bx_{n+1}^p - kx_{n+1}^p z_{n+1}^p)\right), \\ z_{n+1} = z_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\sum_{j=0}^{n} \gamma_{3,j,n+1}.(-cz_j + qx_j^2) + (-cz_{n+1}^p + q(x_{n+1}^p)^2)\right), \\ \hat{x}_{n+1} = \hat{x}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\sum_{j=0}^{n} \gamma_{1,j,n+1}.(a(\hat{y}_j - \hat{x}_j) + u_1(x_j, y_j, z_j, \hat{x}_j, \hat{y}_j, \hat{z}_j))\right. \\ \qquad \left. + a(\hat{y}_{n+1}^p - \hat{x}_{n+1}^p) + u_1(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, \hat{x}_{n+1}^p, \hat{y}_{n+1}^p, \hat{z}_{n+1}^p)\right), \\ \hat{y}_{n+1} = \hat{y}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\sum_{j=0}^{n} \gamma_{2,j,n+1}.(b\hat{x}_j - k\hat{x}_j\hat{z}_j + u_2(x_j, y_j, z_j, \hat{x}_j, \hat{y}_j, \hat{z}_j))\right. \\ \qquad \left. + b\hat{x}_{n+1}^p - k\hat{x}_{n+1}^p\hat{z}_{n+1}^p + u_2(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, \hat{x}_{n+1}^p, \hat{y}_{n+1}^p, \hat{z}_{n+1}^p)\right), \\ \hat{z}_{n+1} = \hat{z}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\sum_{j=0}^{n} \gamma_{3,j,n+1}.(-c\hat{z}_j + q\hat{x}_j^2 + u_3(x_j, y_j, z_j, \hat{x}_j, \hat{y}_j, \hat{z}_j))\right. \\ \qquad \left. - c\hat{z}_{n+1}^p + q(\hat{x}_{n+1}^p)^2 + u_3(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, \hat{x}_{n+1}^p, \hat{y}_{n+1}^p, \hat{z}_{n+1}^p)\right), \end{cases} \tag{5.6}$$

in which

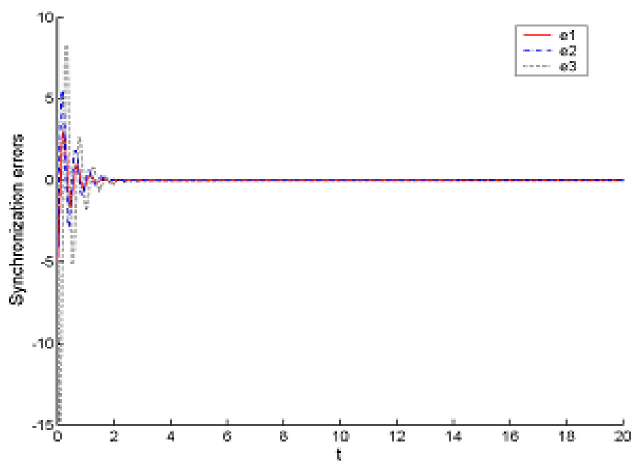
$$\begin{cases} x^p_{n+1} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta_{1,j,n+1}.a(y_j - x_j), \\ y^p_{n+1} = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta_{2,j,n+1}.(bx_j - kx_j z_j), \\ z^p_{n+1} = z_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta_{3,j,n+1}.(-cz_j + qx_j^2), \\ \hat{x}^p_{n+1} = \hat{x}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta_{1,j,n+1}.(a(\hat{y}_j - \hat{x}_j) + u_1(x_j, y_j, z_j, \hat{x}_j, \hat{y}_j, \hat{z}_j)), \\ \hat{y}^p_{n+1} = \hat{y}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta_{2,j,n+1}.(b\hat{x}_j - k\hat{x}_j \hat{z}_j + u_2(x_j, y_j, z_j, \hat{x}_j, \hat{y}_j, \hat{z}_j)), \\ \hat{z}^p_{n+1} = \hat{z}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta_{3,j,n+1}.(-c\hat{z}_j + q\hat{x}_j^2 + u_3(x_j, y_j, z_j, \hat{x}_j, \hat{y}_j, \hat{z}_j)), \end{cases}$$

and

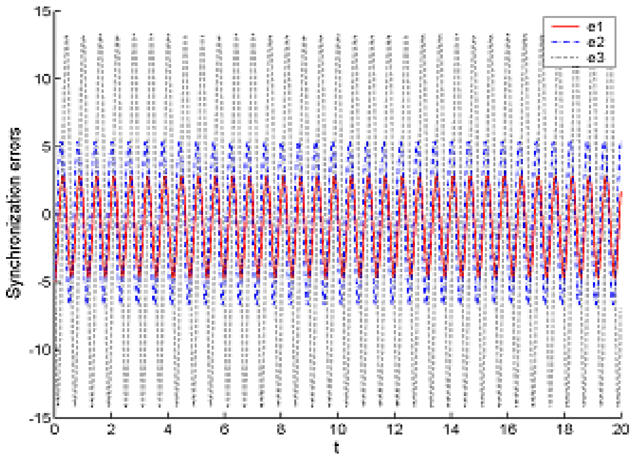
$$\gamma_{i,j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, j = 0, \\ (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}, 1 \le j \le n, \\ 1, \ j = n+1, \end{cases}$$
$$\beta_{i,j,n+1} = \frac{h^{\alpha}}{\alpha}((n-j+1)^{\alpha} - (n-j)^{\alpha}), \ 0 \le j \le n, \ i = 1, \ 2, \ 3,$$

where $(x, y, z)^T$, $(\hat{x}, \hat{y}, \hat{z})^T$ represent the states of the drive and response systems (5.1), (5.2) respectively and $h = \frac{T}{N}$, $t_n = nh$, $n = 0, 1, ..., N \in Z^+$.

Based on the above-mentioned discretization scheme, the drive and response systems (5.1), (5.2) are numerically integrated with the parameter values $a = 10, b = 40, c = 2.5, q = 4, k = 1$ and the control laws (5.4) using the feedback control gains $k_1 = 2$, $k_2 = 4.3783$, $k_3 = 1$. For this choice of the parameter values and feedback control gains, it is easy to verify that the characteristic equation (5.5) satisfies the conditions $D(P) < 0$, $a_1 > 0, a_2 > 0, a_1 a_2 = a_3$ and $a_3 > 0$ (where a_1, a_2, a_3 are the coefficients of the polynomial (5.5)). Consequently, it follows that the zero solution of the error dynamical system (5.3) is locally asymptotically stable for $\alpha \in (0, 1)$, however this zero solution is not locally asymptotically stable when $\alpha = 1$ (see Proposition 4.1). This implies that the synchronization is achieved between the fractional order drive and response systems (5.1), (5.2) using the controllers (5.4), but it is not achieved between their integer order counterparts when using the same control laws (5.4). Figures 4a and 4b show the behavior of synchronization errors between the drive and response systems (5.1), (5.2) with the controllers (5.4) and the orders $\alpha = 0.9$ and $\alpha = 1.0$ respectively.



(a)



(b)

Figure 4: Shows the synchronization errors between the drive and response systems (5.1) and (5.2) using the controllers (5.4) and feedback control gains $k_1 = 2,\ k_2 = 4.3783,\ k_3 = 1$;(a) The synchronization errors converge to zero when using fractional order $\alpha = 0.9$; (b) The synchronization errors do not tend to zero when using $\alpha = 1$.

6. Application to the synchronization of the commensurate novel fractional order hyperchaotic system

The master commensurate novel fractional order hyperchaotic system is given as follows:

$$\begin{cases} D^{\alpha}x_m = \sigma(y_m - x_m) + \mu y_m z_m, \\ D^{\alpha}y_m = \rho x_m - d x_m z_m + y_m + w_m, \\ D^{\alpha}z_m = x_m y_m - \eta z_m, \\ D^{\alpha}w_m = -\nu y_m, \end{cases} \tag{6.1}$$

and the slave system is described by:

$$\begin{cases} D^{\alpha}x_s = \sigma(y_s - x_s) + \mu y_s z_s + v_1(t), \\ D^{\alpha}y_s = \rho x_s - d x_s z_s + y_s + w_s + v_2(t), \\ D^{\alpha}z_s = x_s y_s - \eta z_s + v_3(t), \\ D^{\alpha}w_s = -\nu y_s + v_4(t), \end{cases} \tag{6.2}$$

where $v_i(t)$ $(i = 1, 2, 3, 4)$ is the feedback control function. Define the error variables as:

$$e_1 = x_s - x_m,\ e_2 = y_s - y_m,\ e_3 = z_s - z_m,\ e_4 = w_s - w_m. \tag{6.3}$$

Thus, the error dynamical system can be obtained as follows:

$$\begin{cases} D^{\alpha}e_1 = \sigma(e_2 - e_1) + \mu y_s z_s - \mu y_m z_m + v_1(t), \\ D^{\alpha}e_2 = \rho e_1 + e_2 + e_4 - d x_s z_s + d x_m z_m + v_2(t), \\ D^{\alpha}e_3 = -\eta e_3 + x_s y_s - x_m y_m + v_3(t), \\ D^{\alpha}e_4 = -\nu e_2 + v_4(t), \end{cases} \tag{6.4}$$

Let the nonlinear feedback control functions be given as:

$$\begin{aligned} &v_1(t) = -\mu(e_2e_3 + z_me_2 + y_me_3) - k_1e_1, \quad v_2(t) = d(e_1e_3 + z_me_1 + x_me_3) - k_2e_2,\\ &v_3(t) = -(e_1e_2 + y_me_1 + x_me_2 + k_3e_3), \quad v_4(t) = -k_4e_4, \end{aligned} \tag{6.5}$$

where k_1, k_2, k_3 and k_4 are positive feedback control gains. Then, the error dynamical system (6.4) is reduced to:

$$D^\alpha e_1 = \sigma(e_2 - e_1) - k_1e_1, D^\alpha e_2 = \rho e_1 + e_2 + e_4 - k_2e_2, D^\alpha e_3 = -\eta e_3 - k_3e_3,\ D^\alpha e_4 = -\nu e_2 - k_4e_4. \tag{6.6}$$

The characteristic equation of the error dynamical system (6.6) is given by:

$$\begin{aligned} P(\lambda) = {}& \lambda^4 + (-1 + k_4 + k_1 + \sigma + k_3 + \eta + k_2)\lambda^3 + (k_2k_4 - \sigma\rho + k_2k_3 + \sigma k_2 + \sigma k_4 + \sigma\eta\\ &+\sigma k_3 + k_1k_4 + \eta k_4 + k_3k_4 + k_1k_2 - \eta + k_1k_3 + \eta k_1 + \eta k_2 + \nu - \sigma - k_3 - k_4 - k_1)\lambda^2\\ &+(k_2k_3k_4 + \eta k_2k_4 + \sigma k_2k_3 + \sigma\eta k_2 + k_1k_3k_4 + \eta k_1k_4 + \sigma k_3k_4 + \sigma\eta k_4 - \sigma\eta\rho - \sigma\rho k_3\\ &+k_1k_2k_3 + \eta k_1k_2 - \eta k_4 - k_3k_4 - k_1k_3 - \eta k_1 - \sigma k_3 - \sigma\eta - k_1k_4 + \nu k_1 + k_1k_2k_4 + \sigma k_2k_4\\ &-\sigma\rho k_4 - \sigma k_4 + \sigma\nu + \nu k_3 + \nu\eta)\lambda - k_1k_3k_4 - \eta k_1k_4 - \sigma\eta\rho k_4 + \eta k_1k_2k_4 - \sigma k_3k_4\\ &+k_1k_2k_3k_4 + \nu\eta k_1 + \nu k_1k_3 - \sigma\eta k_4 - \sigma\rho k_3k_4 + \sigma\eta k_2k_4 + \sigma\nu\eta + \sigma\nu k_3 + \sigma k_2k_3k_4 = 0. \end{aligned} \tag{6.7}$$

6.1 Numerical algorithms and approximate solutions

In the following, the predictor-corrector scheme is used to integrate the fractional order master and slave systems (6.1) and (6.2) numerically under the control laws (6.5). Set $h = \frac{T}{N}, t_n = nh,\ n = 0,\ 1,\ ...,\ N \in Z^+$, let $(x, y, z,\ w)^T$ and $(\tilde{x}, \tilde{y}, \tilde{z},\ \tilde{w})^T$ refer to $(x_m, y_m, z_m, w_m)^T$ and $(x_s, y_s, z_s,\ w_s)^T$, respectively. The master and slave systems (6.1), (6.2) can be discretized as follow:

$$\begin{cases} x_{n+1} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}(\sum_{j=0}^n \gamma'_{1,j,n+1}.(\sigma(y_j - x_j) + \mu y_jz_j) + \sigma(y^p_{n+1} - x^p_{n+1}) + \mu y^p_{n+1}z^p_{n+1}),\\ y_{n+1} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}(\sum_{j=0}^n \gamma'_{2,j,n+1}.(\rho x_j - dx_jz_j + y_j + w_j) + (\rho x^p_{n+1} - dx^p_{n+1}z^p_{n+1} + y^p_{n+1} + w^p_{n+1})),\\ z_{n+1} = z_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}(\sum_{j=0}^n \gamma'_{3,j,n+1}.(x_jy_j - \eta z_j) + (x^p_{n+1}y^p_{n+1} - \eta z^p_{n+1})),\\ w_{n+1} = w_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}(\sum_{j=0}^n \gamma'_{4,j,n+1}.(-\nu y_j) + (-\nu y^p_{n+1})),\\ \tilde{x}_{n+1} = \tilde{x}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}(\sum_{j=0}^n \gamma'_{1,j,n+1}.(\sigma(\tilde{y}_j - \tilde{x}_j) + \mu\tilde{y}_j\tilde{z}_j + v_1(x_j, y_j, z_j, \tilde{x}_j, \tilde{y}_j, \tilde{z}_j))\\ \qquad +\sigma(\tilde{y}^p_{n+1} - \tilde{x}^p_{n+1}) + \mu\tilde{y}^p_{n+1}\tilde{z}^p_{n+1} + v_1(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}, \tilde{x}^p_{n+1}, \tilde{y}^p_{n+1}, \tilde{z}^p_{n+1})),\\ \tilde{y}_{n+1} = \tilde{y}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}(\sum_{j=0}^n \gamma'_{2,j,n+1}.(\rho\tilde{x}_j - d\tilde{x}_j\tilde{z}_j + \tilde{y}_j + \tilde{w}_j + v_2(x_j, y_j, z_j, \tilde{x}_j, \tilde{y}_j, \tilde{z}_j))\\ \qquad +\rho\tilde{x}^p_{n+1} - d\tilde{x}^p_{n+1}\tilde{z}^p_{n+1} + \tilde{y}^p_{n+1} + \tilde{w}^p_{n+1} + v_2(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}, \tilde{x}^p_{n+1}, \tilde{y}^p_{n+1}, \tilde{z}^p_{n+1})),\\ \tilde{z}_{n+1} = \tilde{z}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}(\sum_{j=0}^n \gamma'_{3,j,n+1}.(\tilde{x}_j\tilde{y}_j - \eta\tilde{z}_j + v_3(x_j, y_j, z_j, \tilde{x}_j, \tilde{y}_j, \tilde{z}_j))\\ \qquad +\tilde{x}^p_{n+1}\tilde{y}^p_{n+1} - \eta\tilde{z}^p_{n+1} + v_3(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}, \tilde{x}^p_{n+1}, \tilde{y}^p_{n+1}, \tilde{z}^p_{n+1})),\\ \tilde{w}_{n+1} = \tilde{w}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}(\sum_{j=0}^n \gamma'_{4,j,n+1}.(-\nu\tilde{y}_j + v_4(x_j, y_j, z_j, \tilde{x}_j, \tilde{y}_j, \tilde{z}_j))\\ \qquad -\nu\tilde{y}^p_{n+1} + v_4(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}, \tilde{x}^p_{n+1}, \tilde{y}^p_{n+1}, \tilde{z}^p_{n+1})), \end{cases}$$

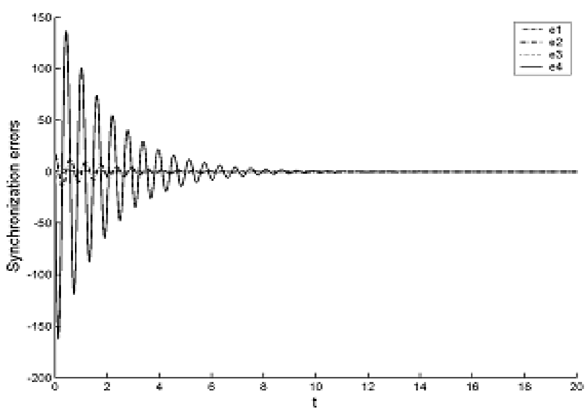
where

$$\begin{cases} x^p_{n+1} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta'_{1,j,n+1}.(\sigma(y_j - x_j) + \mu y_j z_j), \\ y^p_{n+1} = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta'_{2,j,n+1}.(\rho x_j - d x_j z_j + y_j + w_j), \\ z^p_{n+1} = z_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta'_{3,j,n+1}.(x_j y_j - \eta z_j), \\ w^p_{n+1} = w_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta'_{4,j,n+1}.(-\nu y_j), \\ \tilde{x}^p_{n+1} = \tilde{x}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta'_{1,j,n+1}.(\sigma(\tilde{y}_j - \tilde{x}_j) + \mu \tilde{y}_j \tilde{z}_j \ + v_1(x_j, y_j, z_j, \tilde{x}_j, \tilde{y}_j, \tilde{z}_j)), \\ \tilde{y}^p_{n+1} = \tilde{y}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta'_{2,j,n+1}.(\rho \tilde{x}_j - d \tilde{x}_j \tilde{z}_j + \tilde{y}_j + \tilde{w}_j + v_2(x_j, y_j, z_j, \tilde{x}_j, \tilde{y}_j, \tilde{z}_j)), \\ \tilde{z}^p_{n+1} = \tilde{z}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta'_{3,j,n+1}.(\tilde{x}_j \tilde{y}_j - \eta \tilde{z}_j + \ v_3(x_j, y_j, z_j, \tilde{x}_j, \tilde{y}_j, \tilde{z}_j)), \\ \tilde{w}^p_{n+1} = \tilde{w}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \beta'_{4,j,n+1}.(-\nu \tilde{y}_j \ + v_4(x_j, y_j, z_j, \tilde{x}_j, \tilde{y}_j, \tilde{z}_j)), \end{cases}$$

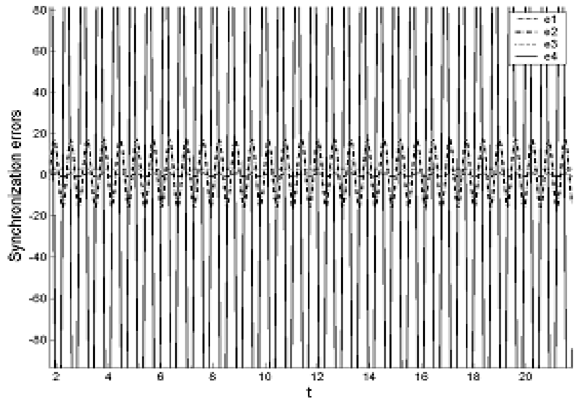
and

$$\gamma'_{i,j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0, \\ (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases}$$
$$\beta'_{i,j,n+1} = \frac{h^{\alpha}}{\alpha}((n-j+1)^{\alpha} - (n-j)^{\alpha}), \qquad 0 \le j \le n, \qquad i = 1,\, 2,\, 3,\, 4.$$

Using the control laws (6.5) with feedback control gains $k_1 = 838.886$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$ and the parameter values, $\sigma = 35$,$\eta = 4$,$\rho = 25$,$d = 5$,$\mu = 35$,$\nu = 100$, it is easy to verify that $D(P) < 0$,$a_1 > 0, a_2 > 0,\ a_3 > 0,\ a_4 > 0$ and $a_2 = a_1 a_4/a_3 + a_3/a_1$. Therefore, Proposition 4.2 implies that the zero equilibrium point of (6.6) is locally asymptotically stable, and the synchronization errors approach zero for the fractional orders $0 < \alpha < 1$. However, the zero equilibrium point of (6.6) is not locally asymptotically stable when $\alpha = 1$ (see Remark 4.1). Thus, using the above-mentioned parameter values and feedback control gains, the fractional order master and slave systems (6.1) and (6.2) are synchronized, but their integer order counterparts are not synchronized (see figure 5:a-b).



(a)



(b)

Figure 5: Synchronization errors between the master and slave systems (6.1) and (6.2) with the controllers (6.5) and feedback control gains $k_1 = 838.886$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$; (a) The synchronization errors tend to zero when using fractional order $\alpha = 0.97$; (b) The synchronization errors do not tend to zero when using $\alpha = 1$.

7. Conclusion

Some stability conditions of the fractional order systems have been used to achieve synchronization of chaotic and hyperchaotic fractional order systems. Two examples have been given to achieve chaos synchronization of the commensurate fractional order Liu and hyperchaos synchronization of the commensurate novel fractional order hyperchaotic systems using nonlinear feedback control technique. However, their integer order counterparts have not been synchronized using the same nonlinear control laws. These results show the effect of fractional order on synchronization of chaotic and hyperchaotic systems. This technique is generic and can successfully be applied to other fractional order chaotic and hyperchaotic systems. Numerical simulations have been used to show the effectiveness of the proposed synchronization techniques.

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