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IMPULSIVE PARTIAL HYPERBOLIC FRACTIONAL ORDER DIFFERENTIAL EQUATIONS IN BANACH SPACES

M. BENCHOHRA, D. SEBA

Abstract. In this paper, we prove an existence result for partial hyperbolic differential equations of fractional order with fixed time impulses. Our analysis is based on the technique of measures of noncompactness and Mönch's fixed point theorem.

1. Introduction

Fractional order models are found to be more adequate than integer order models in some real world problems. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details and examples, see the monographs of Kilbas [16], Lakshmikantham *et al.* [18], Podlubny [21], Samko [22], the papers of Abbas and Benchohra [1, 2], Agarwal *et al.* [3, 4], Ahmad and Sivasundaram [6], Benchohra *et al.* [10, 11, 12], Diethelm [14], Kilbas and Marzan [15], N'Guérékata [20], Shi and Zhang [23], Vityuk and Golushkov [25], Zhang [27], Zhou *et al.* [28] and the references therein.

Impulsive differential equations are a basic tool to study evolution processes that are subjected to abrupt changes in their state. For instance, many biological, physical, and engineering applications exhibit impulsive effects, see [9, 17, 26]. It should be noted that recent progress in the development of the qualitative theory of impulsive differential equations has been stimulated primarily by a number of interesting applied problems, see [6, 13] and references therein.

In this paper, we study a nonlinear impulsive initial value problem for differential equation of fractional order with fixed time impulses given by

$$({}^cD_0^r u)(t,y) = f(t,y,u(t,y)), \text{ if } (t,y) \in J;\ t \neq t_k,\ k = 1,\dots,m, \tag{1}$$

$$u(t_k^+, y) = u(t_k^-, y) + I_k(u(t_k^-, y)), \text{ if } y \in [0,b];\ k = 1,\dots,m, \tag{2}$$

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$$u(t,0)=\varphi(t),\ u(0,y)=\psi(y),\ t\in[0,a],\ \ y\in[0,b], \tag{3}$$

where $J=[0,a]\times[0,b]$, $a,b>0$, ${}^cD_0^r$ is the Caputo fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=a$, $f:J\times E\to E$, $I_k:E\to E$, $k=0,1,\dots,m$ are given functions, $\varphi:[0,a]\to E$ and $\psi:[0,b]\to E$ are given absolutely continuous functions with $\varphi(0)=\psi(0)$ and E is a Banach space with norm $\|.\|$.

Next we consider the following nonlocal initial value problem

$$({}^cD_0^r u)(t,y)=f(t,y,u(t,y)),\ \text{ if } (t,y)\in J;\ t\neq t_k,\ k=1,\dots,m, \tag{4}$$

$$u(t_k^+,y)=u(t_k^-,y)+I_k(u(t_k^-,y)),\ \text{ if }\ y\in[0,b];\ k=1,\dots,m, \tag{5}$$

$$u(t,0)+Q(u)=\varphi(t),\ u(0,y)+K(u)=\psi(y),\ t\in[0,a],\ \ y\in[0,b], \tag{6}$$

where $f,\ \varphi,\ \psi,\ I_k;\ k=1,...m$, are as in problem (1)-(3) and $Q,K:PC(J,E)\to E$ are continuous functions. $PC(J,E)$ is a Banach space to be specified later.

In this work we will use Mönch's fixed point theorem combined with the technique of measures of noncompactness to prove existence of solutions for the problem (1)–(3) in Banach spaces. The paper is organized as follows. In Section 2, we give some preliminaries and establish several lemmas. The main theorem is formulated and proved in Section 3. As an extension to nonlocal problems, we present a similar result for the problem (4)-(6). Then, in Section 4, an example will presented to illustrate the main results.

As far as we know, no papers exist in the literature for the problem (1)–(3) on Banach spaces. The present results extend to the Banach space setting those considered for scalar of systems of differential equations [1, 2]. We extend also the result of [12] when the impulses are absent.

2. Preliminaries

For further purpose, we give in this section some auxiliary results which will be needed in the sequel. By $C(J,E)$ we denote the Banach space of continuous functions $u:J\to E$, with the usual supremum norm

$$\|u\|_\infty=\sup\{\|u(t,y)\|,\quad (t,y)\in J\}.$$

Let also $L^1(J,E)$ be the Banach space of measurable functions $u:J\to E$ which are Bochner integrables, equipped with the norm

$$\|u(t,y)\|_{L^1}=\int_0^a\int_0^b\|u(t,y)\|dtdy.$$

$L^\infty(J,\mathbb{R})$ be the Banach space of measurable bounded functions $u:J\to\mathbb{R}$ equipped with the norm

$$\|u(t,y)\|_{L^\infty}=\inf\{C>0:|u(t,y)|\le C \text{ for a.e. } (t,y)\in J\}.$$

$$\begin{aligned}PC(J,E)=\big\{&u:J\to E:u\in C((t_k,t_{k+1}]\times[0,b],E);\ k=1,\dots,m,\ \text{ and there}\\ &\text{exist } u(t_k^-,y) \text{ and } u(t_k^+,y);k=1,\dots,m,\ \text{ such that } u(t_k^-,y)=u(t_k,y)\big\}.\end{aligned}$$

This set is a Banach space with the norm

$$\|u\|_{PC}=\sup_{(t,y)\in J}\|u(t,y)\|.$$

Set $J':=J\backslash\{(t_1,y),\dots,(t_m,y),\ y\in[0,b]\}$.

For an arbitrary set V of functions $v : J \to E$ let us denote by

$$V(t,y) = \{v(t,y), v \in V\},\ (t,y) \in J$$

and

$$V(J) = \{v(t,y) : v \in V,\ (t,y) \in J\}.$$

Let $a_1 \in [0,a]$, $z^+ = (a_1, 0) \in J$, $J_z = [a_1, a] \times [0,b]$; $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $f \in L^1(J_z, E)$, we define the left-sided mixed Riemann-Liouville integral of order r by

$$(I^r_{z^+} f)(t,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1}^t \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} f(s,x)\,ds\,dx,$$

where $\Gamma(.)$ is the Euler gamma function.

Denote by $D^2_{ty} := \frac{\partial^2}{\partial t \partial y}$ the mixed second order partial derivative.

Definition 2.1. *For a function $h \in L^1(J)$, such that $D^2_{ty} h$ is Lebesque integrable function on J the Caputo fractional-order derivative of order r, is defined by*

$$({}^c D^r_{z^+} h)(t,y) = \left(I^{1-r}_{z^+} D^2_{ty} h\right)(t,y).$$

Definition 2.2. *([7, 8]) Let E be a Banach space and let Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0,\infty)$ defined for $B \in \Omega_E$ by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } diam(B_i) \le \epsilon\}.$$

Properties: The Kuratowski measure of noncompactness satisfies the following (for more details see [7, 8]).

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
(b) $\alpha(B) = \alpha(\overline{B})$.
(c) $A \subset B \Rightarrow \alpha(A) \le \alpha(B)$.
(d) $\alpha(A+B) \le \alpha(A) + \alpha(B)$
(e) $\alpha(cB) = |c|\alpha(B)$; $c \in \mathbb{R}$.
(f) $\alpha(convB) = \alpha(B)$, where $convB$ is the convex hull of the set B.

Definition 2.3. *A map $f : J \times E \to E$ is said to be Carathéodory if*

(i) *$(t,y) \longmapsto f(t,y,u)$ is measurable for each $u \in E$;*
(ii) *$u \longmapsto f(t,y,u)$ is continuous for almost all $(t,y) \in J$.*

Now, we state two known results which are needed to prove the existence of at least one solution of (1)–(3).

Theorem 2.4. *([5, 19]) Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.5. *([24]) Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J \times J$ and f a function from $J \times E \to E$ which satisfies the Carathéodory conditions and there exists $p \in L^1(J, \mathbb{R}_+)$ such that for each $(s,t) \in J$ and each bounded set $B \subset E$ we have*

$$\lim_{h \to 0^+} \alpha(f(J_{(s,t),(h,k)} \times B)) \le p(s,t)\alpha(B); \text{ here } J_{(s,t),(h,k)} = [s-h,s] \times [t-k,t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\Big(\Big\{\int_J G((s,t),(x,y))f((s,t),u(s,t))dsdt : u \in V\Big\}\Big)$$
$$\leq \int_J \|G((s,t),(x,y))\|p(s,t)\alpha(V(s,t))dsdt.$$

3. Main Results

First of all, we define what we mean by a solution of the initial value problem (1)–(3).

Definition 3.1. *A function $u \in PC(J,E)$ is a solution of (1)–(3) if u satisfies the equation $({}^cD^r u)(t,y) = f(t,y,u(t,y))$ on J', and conditions (2)-(3) are satisfied.*

Let $h(t,y) \in C((t_k,t_{k+1}] \times [0,b], E)$, $\quad z_k = (t_k, 0)$, and

$$\mu_k(t,y) = u(t,0) + u(t_k^+, y) - u(t_k^+, 0),\ k = 0,\dots,m.$$

For the existence of solutions for the problem (1)–(3), we need the following lemmas whose proofs can be found in [2] and we give them for the sake of completeness.

Lemma 3.2. *A function $u \in C((t_k,t_{k+1}] \times [0,b], E),\ k = 0,\dots,m$ is a solution of the differential equation*

$$({}^cD^r_{z_k} u)(t,y) = h(t,y);\ (t,y) \in (t_k,t_{k+1}] \times [0,b], \tag{7}$$

if and only if $u(t,y)$ satisfies

$$u(t,y) = \mu_k(t,y) + (I^r_{z_k} h)(t,y);\ (t,y) \in (t_k,t_{k+1}] \times [0,b]. \tag{8}$$

Proof. Let $u(t,y)$ be a solution of (7). Then, from the definition of $({}^cD^r_{z_k^+} u)(x,y)$, we have

$$I^{1-r}_{z_k^+}(D^2_{ty} u)(t,y) = h(t,y).$$

It yield

$$I^r_{z_k^+}(I^{1-r}_{z_k} D^2_{ty} u)(t,y) = (I^r_{z_k^+} h)(t,y),$$

then

$$I^1_{z_k^+} D^2_{ty} u(t,y) = (I^r_{z_k^+} h)(t,y).$$

Since

$$I^1_{z_k^+}(D^2_{ty} u)(t,y) = u(t,y) - u(t,0) - u(t_k^+, y) + u(t_k^+, 0),$$

we have

$$u(t,y) = \mu_k(t,y) + (I^r_{z_k^+} h)(t,y).$$

Now let $u(t,y)$ satisfies (8). It is clear that $u(t,y)$ satisfies

$$({}^cD^r_0 u)(t,y) = h(t,y), \quad \text{on} \quad (t_k,t_{k+1}] \times [0,b].$$

In all what follows set

$$\mu(t,y) = \mu_0(t,y),\ (t,y) \in J.$$

Lemma 3.3. *Let $0 < r_1, r_2 \leq 1$ and let $h : J \to E$ be continuous. A function u is a solution of the fractional integral equation*

$$u(t,y) = \begin{cases} \mu(t,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx & \text{if } (t,y) \in [0,t_1] \times [0,b], \\ \mu(t,y) + \sum_{i=1}^k \Big(I_i(u(t_i^-,y)) - I_i(u(t_i^-,0)) \Big) & \text{if } (t,y) \in (t_k, t_{k+1}] \times [0,b], \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \int_0^y (t_i - s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx & k = 1, \ldots, m, \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^t \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx & \end{cases} \tag{9}$$

if and only if u is a solution of the fractional initial value problem

$$^cD^r u(t,y) = h(t,y), \quad (t,y) \in J', \tag{10}$$

$$u(t_k^+, y) = u(t_k^-, y) + I_k(u(t_k^-, y)), \quad k = 1, \ldots, m. \tag{11}$$

Proof. Assume that u satisfies (10)-(11). If $(t,y) \in [0,t_1] \times [0,b]$ then

$$^cD^r u(t,y) = h(t,y).$$

Thus

$$u(t,y) = \mu(t,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx.$$

If $(t,y) \in (t_1, t_2] \times [0,b]$ then Lemma 3.2 implies

$$\begin{aligned} u(t,y) &= \mu_1(t,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx \\ &= \varphi(t) + u(t_1^+, y) - u(t_1^+, 0) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx \\ &= \varphi(t) + u(t_1^-, y) - u(t_1^-, 0) + I_1(u(t_1^-, y)) - I_1(u(t_1^-, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx \\ &= \varphi(t) + u(t_1, y) - u(t_1, 0) + I_1(u(t_1^-, y)) - I_1(u(t_1^-, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx \\ &= \mu(t,y) + I_1(u(t_1^-, y)) - I_1(u(t_1^-, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^y (t_1-s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} h(s,x)dsdx. \end{aligned}$$

If $(t,y)\in(t_2,t_3]\times[0,b]$ then again from Lemma 3.2 we get

$$\begin{aligned}
u(t,y) &= \mu_2(t,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^y(t-s)^{r_1-1}(y-x)^{r_2-1}h(s,x)dsdx\\
&=\varphi(t)+u(t_2^+,y)-u(t_2^+,0)\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^y(t-s)^{r_1-1}(y-x)^{r_2-1}h(s,x)dsdx\\
&=\varphi(t)+u(t_2^-,y)-u(t_2^-,0)+I_2(u(t_2^-,y))-I_2(u(t_2^-,0))\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^y(t-s)^{r_1-1}(y-x)^{r_2-1}h(s,x)dsdx\\
&=\varphi(t)+u(t_2,y)-u(t_2,0)+I_2(u(t_2^-,y))-I_2(u(t_2^-,0))\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^y(t-s)^{r_1-1}(y-x)^{r_2-1}h(s,x)dsdx\\
&=\mu(t,y)+I_2(u(t_2^-,y))-I_2(u(t_2^-,0))+I_1(u(t_1^-,y))-I_1(u(t_1^-,0))\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^y(t_1-s)^{r_1-1}(y-x)^{r_2-1}h(s,x)dsdx\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^y(t_2-s)^{r_1-1}(y-x)^{r_2-1}h(s,x)dsdx\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^y(t-s)^{r_1-1}(y-x)^{r_2-1}h(s,x)dsdx.
\end{aligned}$$

If $(t,y)\in(t_k,t_{k+1}]\times[0,b]$ then as in the previous we get (9).

Conversely, assume that u satisfies the impulsive fractional integral equation (9). If $(t,y)\in[0,t_1]\times[0,b]$ and using the fact that ${}^cD^r$ is the left inverse of I^r we get

$${}^cD^ru(t,y)=h(t,y),\quad \text{for each } (t,y)\in[0,t_1]\times[0,b].$$

If $(t,y)\in[t_k,t_{k+1})\times[0,b]$, $k=1,\dots,m$ and using the fact that ${}^cD^rC=0$, where C is a constant, we get

$${}^cD^ru(t,y)=h(t,y),\text{for each } (t,y)\in[t_k,t_{k+1})\times[0,b].$$

Also, we can easily show that

$$u(t_k^+,y)=u(t_k^-,y)+I_k(u(t_k^-,y)),\quad y\in[0,b],k=1,\dots,m.$$

For the forthcoming analysis, we need the following assumptions:

(H1) $f:J\times E\to E$ satisfies the Carathéodory conditions.

(H2) There exists $p\in L^\infty(J,\mathbb{R}_+)$, such that,

$$\|f(t,y,u)\|\le p(t,y)\|u\|,\quad \text{for a.e. } (t,y)\in J \text{ and each } u\in E;$$

(H3) There exists $c>0$ such that

$$\|I_k(u)\|\le c\|u\| \text{ for each } u\in E.$$

(H4) For each bounded set $B\subset E$ we have

$$\alpha(I_k(B))\le c\alpha(B),\ k=1,\dots,m.$$

(H5) For each $(t,y)\in J$ and each bounded set $B\subset E$ we have

$$\lim_{(h,k)\to(0^+,0^+)}\alpha(f(J_{(t,y),(h,k)}\times B))\le p(t,y)\alpha(B).$$

Here

$$J_{(t,y),(h,k)} = [t-h,t] \times [y-k,y] \cap J.$$

Let

$$p^* = \|p\|_{L^\infty}.$$

Theorem 3.4. *Assume that assumptions* $(H1)-(H5)$ *hold. If*

$$\frac{(m+1)p^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 2mc < 1, \tag{12}$$

then the problem (1)–(3) has at least one solution.

Proof. Consider the operator $N : PC(J,E) \longrightarrow PC(J,E)$ defined by

$$\begin{aligned} N(u)(t,y) &= \mu(t,y) + \sum_{0<t_k<t} (I_k(u(t_k^-,y)) - I_k(u(t_k^-,0))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^y (t_k-s)^{r_1-1}(y-x)^{r_2-1} f(s,x,u(s,x))dsdx \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} f(s,x,u(s,x))dsdx. \end{aligned}$$

Clearly, the fixed points of the operator N are solution of the problem (1)–(3). Let

$$r_0 \geq \frac{\|\mu\|}{1 - 2mc - \frac{(m+1)p^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}} \tag{13}$$

and consider the set

$$D_{r_0} = \{u \in PC(J,E) : \|u\|_\infty \leq r_0\}.$$

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.4. The proof will be given in three steps.

Step 1: N is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J,E)$, then for each $(t,y) \in J$

$$\begin{aligned} &\|N(u_n)(t,y) - N(u)(t,y)\| \\ &\leq \sum_{k=1}^{m} (\|I_k(u_n(t_k^-,y)) - I_k(u(t_k^-,y))\| + \|I_k(u_n(t_k^-,0)) - I_k(u(t_k^-,0))\|) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^y (t_k-s)^{r_1-1}(y-x)^{r_2-1} \|f(s,x,u_n(s,x)) - f(s,x,u(s,x))\| dsdx \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1} \|f(s,x,u_n(s,x)) - f(s,x,u(s,x))\| dsdx. \end{aligned}$$

Since I_k, $k = 1, \ldots, m$ are continuous and f is of Carathéodory type, then by the Lebesgue dominated convergence theorem we have

$$\|N(u_n) - N(u)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: N maps D_{r_0} into itself.

For each $u \in D_{r_0}$, by $(H2)$ and (12) we have for each $(t,y) \in J$

$$\begin{aligned}
\|N(u)(t,y)\| &\leq \|\mu(t,y)\| + \sum_{k=1}^{m}(\|I_k(u(t_k^-,y))\| + \|I_k(u(t_k^-,0))\|) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^y (t_k-s)^{r_1-1}(y-x)^{r_2-1}\|f(s,x,u(s,x))\|dsdx \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1}\|f(s,x,u(s,x))\|dsdx \\
&\leq \|\mu\| + r_0\Big(\frac{(m+1)p^* a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 2mc\Big) \\
&\leq r_0.
\end{aligned}$$

Step 3: $N(D_{r_0})$ is bounded and equicontinuous.

By Step2, it is obvious that $N(D_{r_0}) \subset PC(J,E)$ is bounded. For the equicontinuity of $N(D_{r_0})$, let $(\tau_1,y_1),(\tau_2,y_2) \in [0,a]\times[0,b]$, $\tau_1 < \tau_2$ and $y_1 < y_2$, and let $u \in D_{r_0}$. Then

$$\begin{aligned}
&\|N(u)(\tau_2,y_2) - N(u)(\tau_1,y_1)\| \\
&\leq \|\mu(\tau_1,y_1) - \mu(\tau_2,y_2)\| + \sum_{k=1}^{m}(\|I_k(u(t_k^-,y_1)) - I_k(u(t_k^-,y_2))\|) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^{y_1}(t_k-s)^{r_1-1}[(y_2-x)^{r_2-1}-(y_1-x)^{r_2-1}]\times f(s,x,u(s,x))dsdx \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_{y_1}^{y_2}(t_k-s)^{r_1-1}(y_2-x)^{r_2-1}\|f(s,x,u(s,x))\|dsdx \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-x)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-x)^{r_2-1}]\times f(s,x,u(s,x))dsdx \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-x)^{r_2-1}\|f(s,x,u(s,x))dsdx\| \\
&\leq \|\mu(\tau_1,y_1) - \mu(\tau_2,y_2)\| + \sum_{k=1}^{m}(\|I_k(u(t_k^-,y_1)) - I_k(u(t_k^-,y_2))\|) \\
&+ \frac{p^*r_0}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^{y_1}(t_k-s)^{r_1-1}[(y_2-x)^{r_2-1}-(y_1-x)^{r_2-1}]dsdx \\
&+ \frac{p^*r_0}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_{y_1}^{y_2}(t_k-s)^{r_1-1}(y_2-x)^{r_2-1}dsdx \\
&+ \frac{p^*r_0}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-x)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-x)^{r_2-1}]dsdx \\
&+ \frac{p^*r_0}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-x)^{r_2-1}dsdx.
\end{aligned}$$

As $\tau_1 \longrightarrow \tau_2$ and $y_1 \longrightarrow y_2$, the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_{r_0} such that $V \subset \overline{conv}(N(V)\cup\{0\})$.
V is bounded and equicontinuous and therefore the function $v \to v(t,y) = \alpha(V(t,y))$ is continuous on J. Since functions ϕ and ψ are continuous on J, the set

$\overline{\{\phi(t)+\psi(y)-\phi(0),\ (t,y)\in J\}}\subset E$ is compact. Using (H4), (H5), Lemma 2.5 and the properties of the measure α we have for each $(t,y)\in J$

$$\begin{aligned}
v(t,y) &\le \alpha(N(V)(t,y)\cup\{0\})\\
&\le \alpha(N(V)(t,y))\\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^y (t_k-s)^{r_1-1}(y-x)^{r_2-1}p(s,x)\alpha(V(s,x))dsdx\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1}p(s,x)\alpha(V(s,x))dsdx\\
&+\sum_{k=1}^{m}2c\alpha(V(t_k,y))\\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^y (t_k-s)^{r_1-1}(y-x)^{r_2-1}p(s,x)v(s,x)dsdx\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^y (t-s)^{r_1-1}(y-x)^{r_2-1}p(s,x)v(s,x)dsdx\\
&+\sum_{k=1}^{m}2cv(t_k,y)\\
&\le \|v\|_\infty\Big(\frac{(m+1)p^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}+2mc\Big).
\end{aligned}$$

This means that

$$\|v\|_\infty\Big(1-\Big[\frac{(m+1)p^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}+2mc\Big]\Big)\le 0.$$

By (12) it follows that $\|v\|_\infty=0$, that is $v(t,y)=0$ for each $(t,y)\in J$, and then $V(t,y)$ is relatively compact in $PC(J,E)$. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_{r_0}. Applying now Theorem 2.4 we conclude that N has a fixed point which is a solution of the problem (1)-(3). □

Now we present (without proof) an existence result for the nonlocal problem (4)-(6).

Definition 3.5. *A function $u\in PC(J,E)$ is a solution of (4)-(6) if u satisfies $({}^cD_0^r u)(t,y)=f(t,y,u(t,y))$ on J' and conditions $(5)-(6)$ are satisfied.*

Theorem 3.6. *Further to (H1)-(H5), we assume the following conditions*

(H4) *There exists $\tilde{k}>0$ such that*

$$\|Q(u)\|\le \tilde{k}\|u\|_{PC},\quad \text{for each}\ \ u\in PC(J,E)$$

(H5) *There exists $k^*>0$ such that*

$$\|K(u)\|\le k^*\|u\|_{PC},\quad \text{for each}\ \ u\in PC(J,E)$$

(H6)

$$\alpha(Q(B))\le \tilde{k}\alpha(B),\quad \text{for any bounded set}\ \ B\subset PC(J,E)$$

(H7)

$$\alpha(K(B))\le k^*\alpha(B),\quad \text{for any bounded set}\ \ B\subset PC(J,E)$$

hold. If

$$\tilde{k} + k^* + 2mc + \frac{(m+1)p^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

then there exists a solution for problem (4)-(6) on J.

4. An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following impulsive partial hyperbolic differential equations of the form

$$({}^cD_0^r u_n)(t,y) = \frac{1}{8e^{t+y+3}} \frac{|u_n(t,y)|}{(1+|u_n(t,y)|)}, \text{ if } (t,y) \in J = [0,1]\times[0,1], t \neq \frac{1}{3}, \quad (14)$$

$$u_n(\frac{1}{3}^+, y) = u(\frac{1}{3}^-, y) + \frac{1}{6e^{t+y+4}} \frac{|u_n(\frac{1}{3}^-, y)|}{(15+|u_n(\frac{1}{3}^-, y)|)}, \text{ if } y \in [0,1], \quad (15)$$

$$u_n(t,0) = t, \ u_n(0,y) = y^2, \ t\in[0,1], \ y \in [0,1], \ n = 1,2,\dots. \quad (16)$$

Let

$$E = l^1 = \{u = (u_1, u_2, \dots, u_n, \dots) : \sum_{n=1}^{\infty} |u_n| < \infty\}$$

with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Set

$$u = (u_1, u_2, \dots, u_n, \dots) \text{ and } f = (f_1, f_2, \dots, f_n, \dots).$$

$$f_n(t,y,u_n) = \frac{1}{8e^{t+y+3}} \frac{|u_n(t,y)|}{(1+|u_n(t,y)|)}, \ (t,y)\in[0,1]\times[0,1].$$

and

$$I_k(u_n(t_k^-, y)) = \frac{1}{6e^{t+y+4}} \frac{|u_n(t_k{}^-, y)|}{(15+|u_n(t_k{}^-, y)|)}, \ y \in [0,1].$$

For each u_n and $(t,y)\in[0,1]\times[0,1]$ we have

$$|f_n(t,y,u_n)| \le \frac{1}{8e^{t+y+3}}|u_n|. \quad (17)$$

and

$$|I_k(u_n)| \le \frac{1}{6e^4}|u_n|.$$

Hence conditions (H1) and (H2) are satisfied with $p(t,y) = \frac{1}{8e^{t+y+3}}$ and $c = \frac{1}{6e^4}$. By (17), for any bounded set $B \subset l^1$, we have

$$\alpha(f(t,y,B)) \le \frac{1}{8e^{t+y+3}}\alpha(B), \text{ for each } (t,y)\in[0,1]\times[0,1].$$

Hence (H3) is satisfied. We shall show that condition (12) holds with $a = b = 1$, $m = 1$ and $p^* = \frac{1}{8e^3}$. Indeed,

$$2mc + \frac{(m+1)p^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{3e^4} + \frac{1}{4e^3\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

which is satisfied for each $(r_1, r_2) \in (0,1]\times(0,1]$. Consequently Theorem 3.4 implies that problem (14)-(16) has a solution defined on $[0,1]\times[0,1]$.

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Mouffak Benchohra
Laboratoire de Mathématiques, Université de Sidi Bel-Abbès, B.P. 89, 22000, Sidi Bel-Abbès, Algérie
E-mail address: `benchohra@univ-sba.dz, benchohra@yahoo.com`

Djamila Seba
Département de Mathématiques, Université de Boumerdès, Avenue de l'indépendance, 35000 Boumerdès, Algérie
E-mail address: `djam_seba@yahoo.fr`