

A NEW WAVELET OPERATIONAL METHOD USING BLOCK PULSE AND HAAR FUNCTIONS FOR NUMERICAL SOLUTION OF A FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

S. SAHA RAY

ABSTRACT. The fractional calculus has many applications in applied science and engineering. The solution of the differential equation containing fractional derivative is much involved. An effective and easy-to-use method for solving such equations is needed. However not only the analytical solutions exist for a limited number of cases, but also the numerical methods are very complicated and difficult. In this paper, a wavelet operational method has been applied based on the operational matrices of the orthogonal functions. By using the operational matrix of integration, a linear fractional partial differential equation has been solved numerically. In the present paper, the Haar wavelet has been used and then from matrix equation, we obtain the algebraic equations suitable for computer programming. The simplicity, clarity and powerfulness of the method has been cited through an illustration.

1. INTRODUCTION

Fractional calculus has been used to model physical and engineering processes that have been found to be best described by fractional differential equations. For that reason it is indeed required a reliable and efficient technique for the solution of fractional differential equations. Podlubny [1] used the Laplace Transform method to solve the fractional differential equations numerically with the Riemann–Liouville (RL) derivatives definition, as well as the fractional partial differential equations with constant coefficients. Podlubny [1] suggested a generalization of the definition of Green’s function to solve the problems of the fractional order systems and controllers. Recently, the analytical solution of fractional differential equation has been obtained through Adomian decomposition method by Saha Ray and his co-researchers [1-5]. Even not only its analytical solutions exist only for limited cases, but also, the numerical methods are difficult to solve. In this connection, it is worthwhile to mention that the recent papers on numerical solutions of fractional differential equations are available in open literature from the notable works of Diethelm et al. [6–9], Liu et al [10-15] and Meerschaert et al [16-18]. The discretization

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methods that they have adopted are complicated and time consuming. In recent studies, many articles have been devoted to the development of algebraic methods for the analysis, identification, and optimization of systems. The aim of these studies has been to obtain effective algorithms that are suitable for the digital computer. Their major effort has been concentrated on the methods of the orthogonal polynomial and functions. Typical examples are the applications of Walsh functions [19], block pulse functions [20], Laguerre polynomials [21], Legendre polynomials [22], Chebyshev polynomials [23], Fourier series [24] and Haar wavelets functions [25]. Kronecker operational matrices have been developed by Kilicman for some applications of fractional calculus [26]. A new analytic method has been proposed based on a piecewise orthogonal functions, namely, Block pulse, Walsh and Haar wavelets by Bouafoura et al. [27]. Recently, fractional Integral equations have been solved by Haar Wavelet Method by the learned author Lepik [28].

In this paper a new numerical method based on the operational matrices of the orthogonal functions has been applied for the solution of the fractional differential equation. In this method, the Haar wavelet and the Block-Pulse operational matrices of general order has been used to obtain the algebraic equation suitable for computer programming for the solution of fractional differential equation. In this paper a fractional partial differential equation has been solved to demonstrate the simplicity, clarity and effectiveness of the present method. The main characteristic of the operational method is to convert a differential equation into an algebraic one. It not only simplifies the problem but also speeds up the computation. To start with the integral property of the basic orthonormal matrix, $\Phi(t)$, the approximation is as follows The main characteristic of the operational method is to convert a differential equation into an algebraic one. It not only simplifies the problem but also speeds up the computation. To start with the integral property of the basic orthonormal matrix, $\Phi(t)$, the approximation is as follows

$$\int_0^t \int_0^t \dots \int_0^t \Phi(\tau) (d\tau)^k \cong Q_{\Phi}^k \Phi(t), k \in \mathbb{N} \quad (1)$$

where $\Phi(t) = [\Phi_0(t), \Phi_1(t), \dots, \Phi_{m-1}(t)]^T$ in which the elements $\Phi_0(t), \Phi_1(t), \dots, \Phi_{m-1}(t)$ are the discrete representation of the basis functions which are orthogonal on the interval $[0,1)$ and Q_{Φ} is the operational matrix for integration of $\Phi(t)$. In view of the simple structure of the operational matrix of integration Q_{Φ} , the computation of the powers of Q_{Φ} is very easy. This elegant operational property is useful for the simplification of problems.

Using the operational matrix of an orthogonal function to perform integration for solving, identifying and optimizing a linear dynamic system has several advantages: (1) the method is computer oriented, thus solving higher order differential equation becomes a matter of dimension increasing; (2) the solution is a multi-resolution type; (3) the solution is convergent, even the size of increment is very large.

2. OPERATIONAL MATRICES OF HAAR WAVELETS

The operational matrix of an orthogonal matrix $\Phi(t)$, Q_{Φ} can be expressed by

$$Q_{\Phi} = \Phi \cdot Q_B \cdot \Phi^{-1} \quad (2)$$

where Q_B is the operational matrix of the block pulse function.

$$Q_{B_m} = \frac{1}{m} \begin{bmatrix} \frac{1}{2} & 1 & \dots & 1 \\ 0 & \dots & \dots & \dots \\ \dots & 0 & \frac{1}{2} & 1 \\ 0 & \dots & 0 & \frac{1}{2} \end{bmatrix}_{m \times m} \quad (3)$$

is called the operational matrix for integration of the block pulse function.

If the transform matrix Φ is unitary, i.e. $\Phi^{-1} = \Phi^T$, then eq.(2) can be rewritten as

$$Q_\Phi = \Phi \cdot Q_B \cdot \Phi^T \quad (4)$$

In this connection, it is to be mentioned that recently a new method to derive the operational matrices of integration and differentiation for all orthogonal functions in a unified framework has been proposed by Wu et al[29].

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. The orthogonal set of Haar functions are defined in the interval $[0, 1)$ by

$$h_0(t) = \frac{1}{\sqrt{m}}$$

$$h_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{\frac{j}{2}}, & \frac{k-1}{2^j} \leq t < \frac{k-\frac{1}{2}}{2^j} \\ -2^{\frac{j}{2}}, & \frac{k-\frac{1}{2}}{2^j} \leq t < \frac{k}{2^j} \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

where $i = 0, 1, 2, m-1$, $m = 2^M$ and M is a positive integer. j and k represent the integer decomposition of the index i , i.e. $i = 2^j + k - 1$, $j \geq 0$ and $1 \leq k < 2^j + 1$.

This set of functions is complete, since for every $f \in L^2([0, 1))$, the sequence $\{h_i(t)\}$ is complete if $\int h_i f = 0$ implies $f = 0$ almost everywhere. The first curve of Fig. 1 is that $h_0(t) = \frac{1}{\sqrt{m}}$ during the whole interval $[0, 1)$. It is called the scaling function. The second curve $h_1(t)$ is the fundamental square wave, or the mother wavelet which also spans the whole interval $[0, 1)$. All the other subsequent curves are generated from $h_1(t)$ with two operations: translation and dilation. $h_2(t)$ is obtained from $h_1(t)$ with dilation, i.e. $h_1(t)$ is compressed from the whole interval $[0, 1)$ to the half interval $[0, 1/2]$ to generate $h_2(t)$. $h_3(t)$ is the same as $h_2(t)$ but shifted (translated) to the right by $\frac{1}{2}$.

Any function $f(t) \in L^2([0, 1))$ can be expanded into Haar wavelets by

$$y(t) = c_0 h_0(t) + c_1 h_1(t) + c_2 h_2(t) + \dots \quad (6)$$

where $c_j = \int_0^1 y(t) h_j(t) dt$.

If $y(t)$ is approximated as piecewise constant during each subinterval, eq.(6) will be terminated at finite terms, i.e. $y(t) = \sum_{i=0}^{m-1} c_i h_i(t)$ or in the matrix form

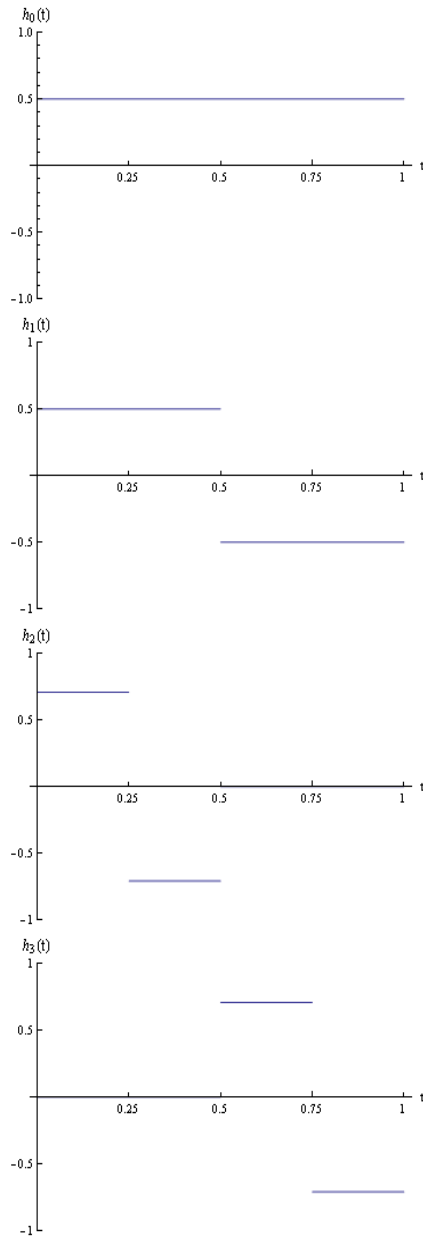


Fig. 1 Haar Wavelet functions with $m=4$

$$\mathbf{Y}^T = \mathbf{C}^T \cdot H \quad (7)$$

where \mathbf{Y} is the discrete form of the continuous function, $y(t)$ and \mathbf{C} is called the coefficient vector of \mathbf{Y} which can be calculated from $\mathbf{C}^T = \mathbf{Y}^T \cdot H^{-1}$. \mathbf{Y} and \mathbf{C} are both column vectors, and H is the Haar wavelet matrix of dimension $m = 2^M$, M is a positive integer and is defined by

$$H = [h_0^T, h_1^T, \dots, h_{m-1}^T]^T$$

i.e.

$$H = \begin{bmatrix} h_0^T \\ h_1^T \\ \dots \\ h_{m-1}^T \end{bmatrix} = \begin{bmatrix} h_{0,0} & h_{0,1} & \dots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \dots & h_{1,m-1} \\ \dots & \dots & \dots & \dots \\ h_{m-1,0} & h_{m-1,1} & \dots & h_{m-1,m-1} \end{bmatrix} \quad (8)$$

where $h_0^T, h_1^T, \dots, h_{m-1}^T$ are the discrete form of the Haar wavelet bases; the discrete values are taken from the continuous curves $h_0(t), h_1(t), \dots, h_{m-1}(t)$, respectively.

Similarly, a two-dimensional function $y(x, t) \in L^2([0, 1], [0, 1])$ can be expanded into Haar Wavelets by

$$y(x, t) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} c_{ij} h_i(x) h_j(t) \quad (9)$$

where $c_{ij} = \int_0^1 y(x, t) h_i(x) dx \int_0^1 y(x, t) h_j(t) dt$. Eq.(9) can be written into the discrete form (in matrix form) by

$$Y(x, t) = H^T(x).C.H(t) \quad (10)$$

where $C = \begin{bmatrix} c_{0,0} & c_{0,1} & \dots & c_{0,m-1} \\ c_{1,0} & c_{1,1} & \dots & c_{1,m-1} \\ \dots & \dots & \dots & \dots \\ c_{m-1,0} & c_{m-1,1} & \dots & c_{m-1,m-1} \end{bmatrix}$ is the coefficient matrix of $Y(x, t)$.

The generalized operational matrices of Haar Wavelets can be derived from the following equation [30, 31]

$$Q_H^\alpha = H.Q_B^\alpha.H^T \quad (11)$$

where Q_B^α is the generalized operational matrix of the Block Pulse function for integration with the order $\alpha (\alpha \in R)$ and Q_H^α is the operational matrix with fractional order α .

3. THE WAVELET OPERATIONAL METHOD

The integration of order $\alpha (\alpha \in R)$ of $Y(x, t) = H^T(x).C.H(t)$ with respect to variable t can be expressed as

$$\begin{aligned} J_t^\alpha Y &= J_t^\alpha (H^T(x).C.H(t)) = H^T(x).C.J_t^\alpha H(t) \\ &= H^T(x).C.Q_H^\alpha H(t) = H^T.C.Q_H^\alpha H \end{aligned} \quad (12)$$

Similarly, the integration of order $\beta (\beta \in R)$ of $Y(x, t) = H^T(x).C.H(t)$ with respect to variable x is given by

$$\begin{aligned} J_x^\beta Y &= J_x^\beta (H^T(x).C.H(t)) = J_x^\beta (H^T(x)).C.H(t) = (J_x^\beta H(x))^T.C.H(t) \\ &= (Q_H^\beta H(x))^T.C.H(t) = H^T(x).(Q_H^\beta)^T.C.H(t) = H^T.(Q_H^\beta)^T.C.H \end{aligned} \quad (13)$$

In general, with the help of eq.(12) and (13) the double integration to the function $Y(x, t)$ of order $\alpha(\alpha \in R)$ with respect to variable t and $\beta(\beta \in R)$ with respect to variable x is given by

$$J_t^\alpha J_x^\beta Y = H^T . (Q_H^\beta)^T . C . Q_H^\alpha . H \quad (14)$$

Eqs.(12)-(14) are used for numerical solution of fractional partial differential equation by Haar wavelet operational method [30,31].

4. ILLUSTRATION

We use the Haar Wavelet operational method to solve the following diffusion equation

$${}^C D_t^{1/2} y(x, t) + \frac{\partial^2 y(x, t)}{\partial x^2} = 0, x, t \geq 0 \quad (15)$$

with the initial conditions

$$y(x, 0) = x^2, y(0, t) = -4\sqrt{\frac{t}{\pi}}$$

and

$$y_x(0, t) = 0 \quad (16)$$

Here, in eq.(15) ${}^C D_t^{1/2}$ stands for Caputo's fractional derivative of order $\alpha[1]$.

Applying $J_t^{1/2}$ to both sides of eq.(15) , we obtain

$$\begin{aligned} y(x, t) - y(x, 0) + J_t^{1/2} \left(\frac{\partial^2 y(x, t)}{\partial x^2} \right) &= 0 \\ y(x, t) - 2 \int_0^x \int_0^x dx dx + J_t^{1/2} \left(\frac{\partial^2 y(x, t)}{\partial x^2} \right) &= 0, \text{ since} \\ y(x, 0) = x^2 = 2 \int_0^x \int_0^x dx dx & \quad (17) \end{aligned}$$

Integrating eq.(17) with respect to variable x with the initial conditions, we obtain

$$\begin{aligned} \int_0^x \int_0^x y(x, t) dx dx - 2 \int_0^x \int_0^x \int_0^x \int_0^x 1. dx dx dx dx \\ + J_t^{1/2} (y(x, t) - y(0, t) - xy_x(0, t)) = 0 \\ \int_0^x \int_0^x y(x, t) dx dx - 2 \int_0^x \int_0^x \int_0^x \int_0^x 1. dx dx dx dx + J_t^{1/2} (y(x, t)) + 2t = 0, \end{aligned}$$

since

$$y(0, t) = -4\sqrt{\frac{t}{\pi}}$$

and

$$y_x(0, t) = 0$$

$$\int_0^x \int_0^x y(x, t) dx dx - 2 \int_0^x \int_0^x \int_0^x \int_0^x 1. dx dx dx dx + J_t^{1/2} y(x, t) + 2 \int_0^t 1. dt = 0 \quad (18)$$

Using formulae Eqs.(12)-(14) alongwith Haar Wavelet operational matrices, the matrix form of eq. (18) can be written as

$$H^T \cdot (Q_H^2)^T \cdot C.H + H^T \cdot C.Q_H^{1/2} \cdot H = 2H^T \cdot (Q_H^4)^T \cdot E.H - 2H^T \cdot E.Q_H.H \quad (19)$$

where E is given by $E = H \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & & & \\ 1 & 1 & \dots & 1 \end{bmatrix} \cdot H^T$. By multiplying H^T to the right side and H to the left side of each term in eq.(19), yields

$$(Q_H^2)^T \cdot C + C.Q_H^{1/2} = 2(Q_H^4)^T \cdot E - 2E.Q_H \quad (20)$$

which is a matrix equation can be solved by suitable Mathematical Software packages to obtain the coefficient matrix C . In present analysis, Mathematical Software MATHEMATICA 7 has been used to obtain solution matrix for C .

Then the discrete form of the function $y(x, t)$ is given by

$$Y(x, t) = H^T(x).C.H(t) \quad (21)$$

$$Y_{16 \times 16} =$$

-0.331187	-0.720146	-0.863418	-1.05802	-1.1952	-1.32261	-1.43736	-1.54412	-1.64381	-1.73785	-1.82705	-1.91211	-1.99354	-2.07178	-2.14717	-2.22001
-0.324291	-0.710892	-0.876438	-1.04981	-1.18755	-1.31473	-1.42957	-1.53629	-1.636	-1.73004	-1.81924	-1.9043	-1.98573	-2.06397	-2.13936	-2.2122
-0.310481	-0.69243	-0.862432	-1.03343	-1.17225	-1.29898	-1.414	-1.52065	-1.62038	-1.71442	-1.80362	-1.88868	-1.97011	-2.04835	-2.12374	-2.19658
-0.289718	-0.664849	-0.841313	-1.00892	-1.14926	-1.27537	-1.39063	-1.4972	-1.59696	-1.69098	-1.78019	-1.86524	-1.94668	-2.02492	-2.10031	-2.17314
-0.261948	-0.62828	-0.812955	-0.976371	-1.11853	-1.24393	-1.35946	-1.46593	-1.56572	-1.65973	-1.74895	-1.834	-1.91543	-1.99367	-2.06906	-2.1419
-0.227097	-0.582894	-0.777196	-0.935878	-1.08001	-1.20467	-1.32047	-1.42685	-1.52668	-1.62067	-1.70989	-1.79494	-1.87638	-1.95461	-2.03001	-2.10284
-0.18508	-0.528893	-0.733848	-0.887558	-1.03365	-1.15763	-1.27365	-1.37997	-1.47981	-1.57381	-1.66302	-1.74808	-1.82951	-1.90775	-1.98314	-2.05598
-0.135796	-0.466507	-0.682701	-0.831533	-0.979378	-1.10283	-1.219	-1.32528	-1.42513	-1.51913	-1.60835	-1.6934	-1.77483	-1.85307	-1.92846	-2.0013
-0.0791322	-0.395989	-0.623534	-0.767927	-0.917139	-1.0403	-1.15649	-1.26281	-1.36263	-1.45664	-1.54585	-1.63091	-1.71234	-1.79058	-1.86597	-1.93881
-0.0149683	-0.317604	-0.556123	-0.696855	-0.846879	-0.970057	-1.08613	-1.19253	-1.29232	-1.38635	-1.47555	-1.56061	-1.64204	-1.72028	-1.79567	-1.86851
0.0568229	-0.231624	-0.480253	-0.618423	-0.768557	-0.892107	-1.00791	-1.11446	-1.21418	-1.30824	-1.39744	-1.4825	-1.56393	-1.64217	-1.71756	-1.79039
0.136373	-0.13832	-0.395724	-0.532712	-0.68215	-0.806455	-0.921839	-1.0286	-1.12823	-1.22233	-1.31151	-1.39657	-1.478	-1.55624	-1.63164	-1.70447
0.223813	-0.0379539	-0.302361	-0.439778	-0.587649	-0.713093	-0.827917	-0.934925	-1.03447	-1.1286	-1.21777	-1.30284	-1.38427	-1.46251	-1.5379	-1.61074
0.319275	0.0692298	-0.200025	-0.339646	-0.485068	-0.612002	-0.726159	-0.833443	-0.932893	-1.02706	-1.11622	-1.20129	-1.28272	-1.36096	-1.43635	-1.50919
0.422884	0.18301	-0.0886166	-0.232303	-0.374444	-0.503151	-0.616582	-0.724141	-0.82351	-0.917699	-1.00686	-1.09193	-1.17336	-1.2516	-1.32699	-1.39983
0.53476	0.303198	0.0319135	-0.117698	-0.255832	-0.3865	-0.499206	-0.60701	-0.706324	-0.800524	-0.889688	-0.974761	-1.05619	-1.13443	-1.20982	-1.28266

The plots of $Y_{64 \times 64}(x, t)$ is shown in Figs. 2.

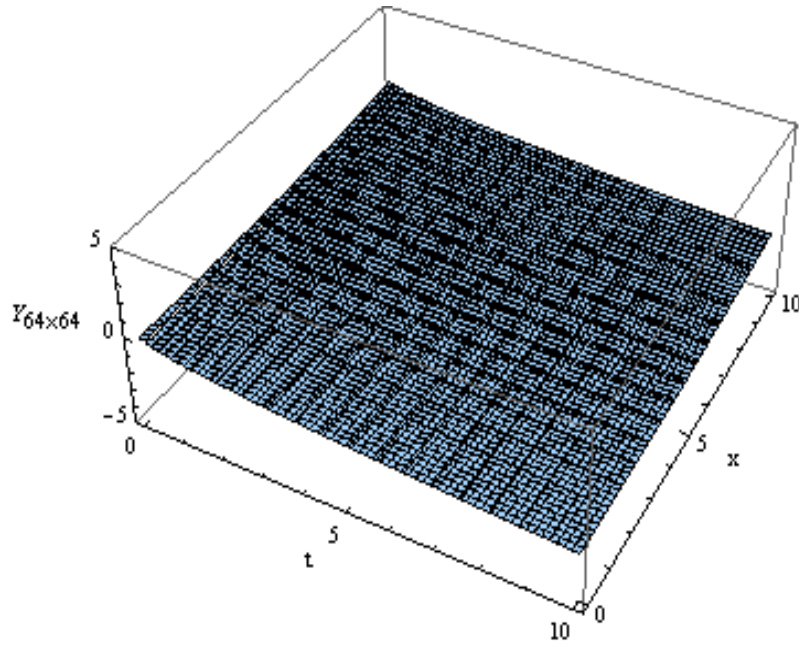
Figs. 2(b) shows the comparison of exact solution $y(x, t)$ and the approximate solution $Y_{16 \times 16}(x, t)$ at $t = 0$. In these figures the black curve indicates the approximate solution $Y_{16 \times 16}(x, t)$ and the dashed curve indicates the exact solution $y(x, t)$ at $t = 0$ respectively.

The exact solution of eq.(15) with initial conditions eq.(16) obtained by Adomian Decomposition method is

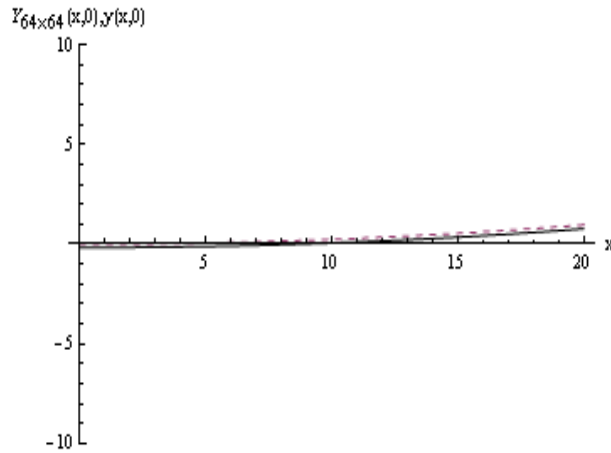
$$y(x, t) = x^2 - 4\sqrt{\frac{t}{\pi}} \quad (22)$$

It is shown in Fig. 3.

Figs. 1-3 have been drawn by the mathematical software MATHEMATICA 7.



(a)

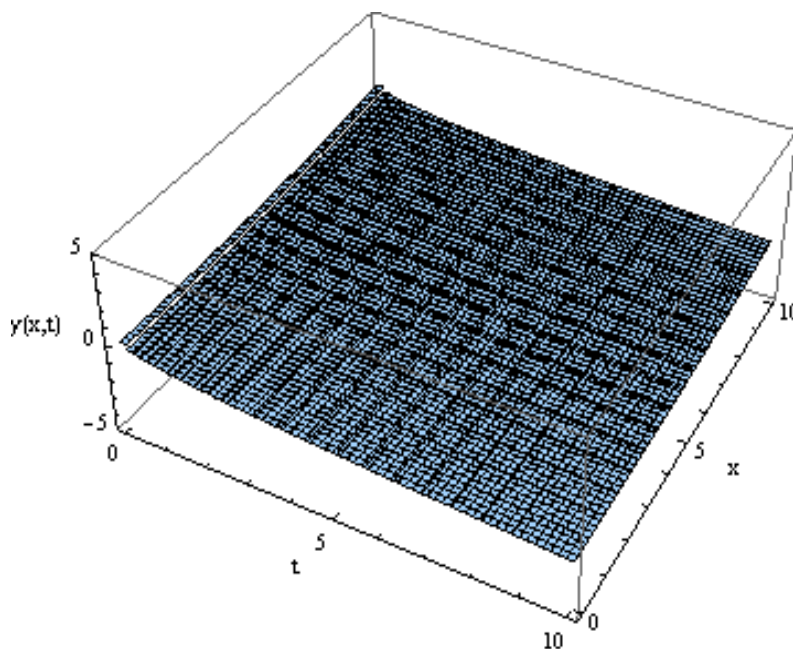


(b)

FIGURE 1. (a) The numerical solution of eq. (15) with the initial conditions eq.(16) in the case of $m = 64$, (b) The comparison of exact solution $y(x, t)$ and the approximate solution $Y_{64 \times 64}(x, t)$ at $t = 0$

5. ERROR ANALYSIS

In the present analysis a table has been created citing the Absolute error in approximating $Y_{64 \times 64}(x, t)$ which is the numerical solution of eq. (15) alongwith initial condition eq.(16) obtained by the Wavelet operational method. Table 1 shows these results are in good agreement with each other.



(c)

FIGURE 2. The exact solution $y(x, t)$ of eq. (15) with the initial conditions eq.(16)

Absolute Error= $ y(x, t) - Y_{64 \times 64}(x, t) $	$t = 0$	$t = 0.125$	$t = 0.25$	$t = 0.375$	$t = 0.5$	$t = 0.625$	$t = 0.75$	$t = 0.875$
$x = 0$	0.166096	0.0245879	0.0174578	0.0142642	0.0123531	0.0110462	0.0100802	0.00932872
$x = 0.125$	0.16616	0.0226419	0.0155048	0.0123114	0.0104002	0.00909327	0.00812726	0.0073758
$x = 0.25$	0.169487	0.0206978	0.013552	0.0103588	0.00844774	0.00714077	0.00617472	0.00542324
$x = 0.375$	0.17521	0.018735	0.0115993	0.00840665	0.00649563	0.00518864	0.00422255	0.00347102
$x = 0.5$	0.182073	0.0167442	0.00964615	0.00645465	0.00454383	0.00323685	0.00227072	0.00151914
$x = 0.625$	0.188631	0.014742	0.00769231	0.00450268	0.00259226	0.00128535	0.000319203	0.000432435
$x = 0.75$	0.193488	0.012766	0.00573718	0.00255053	0.000640816	0.000665913	0.00163204	0.00238372
$x = 0.875$	0.195531	0.0108463	0.00378008	0.000597953	0.00131063	0.00261702	0.00358307	0.00433475

Table1 Absolute error in approximating $Y_{64 \times 64}(x, t)$ in comparison to exact solution $y(x, t)$

6. CONCLUSION

In this paper, a fractional partial differential equation has been solved by a wavelet operational method. It is based on the operational matrices of orthogonal functions. Advantages of this wavelet operational method include (1) it is much simpler than the conventional numerical method for fractional differential Equations; (2) the computation is computer oriented; and (3) the step size used could be large and the result obtained is quite satisfactory.

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REFERENCES

- [1] I. Podlubny, 1999, *Fractional Differential Equations*, Academic Press, San Diego, California, USA.
- [2] S. Saha Ray and R. K. Bera, 2005, "Analytical solution of a dynamic system containing fractional derivative of order one-half by adomian decomposition method," *Trans. ASME J. Appl. Mech.*, **72**, pp. 290–5.
- [3] S. Saha Ray and R. K. Bera, 2005, "An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method," *Appl. Math. Comput.*, **167**, pp. 561–71.
- [4] S. Saha Ray and R. K. Bera, 2006, "Analytical solution of a fractional diffusion equation by Adomian decomposition method," *Appl. Math. Comput.*, **174**, pp. 329–36.
- [5] S. Saha Ray, 2007, "Exact solutions for time-fractional diffusion-wave equations by decomposition method," *Phys. Scri.*, **75**, pp. 53–61.
- [6] S. Saha Ray, 2008, "A new approach for the application of Adomian decomposition method for the solution of fractional space diffusion equation with insulated ends," *Appl. Math. Comput.*, **202**, pp. 544–9.
- [7] K. Diethelm, N.J. Ford, A.D. Freed, 2002, "A predictor–corrector approach for the numerical solution of fractional differential equations," *Nonlinear Dyn.*, 29, pp. 3–22.
- [8] K. Diethelm, N.J. Ford, 2002, "Numerical solution of the Bagley–Torvik equation," *Bit Numer. Math.*, 42 (3), pp. 490–507.
- [9] K. Diethelm, A.D. Ford, 1999, "On the solution of nonlinear fractional-order differential equations used in the modeling of viscoplasticity," in: F. Keil, W. Mackens, H. Vo, J. Werther (Eds.), *Scientific Computing in Chemical Engineering II. Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties*, Springer-Verlag, Heidelberg, pp. 217–224.
- [10] K. Diethelm, N.J. Ford, 2001, "The numerical solution of linear and nonlinear fractional differential equations involving fractional derivatives of several orders," *Numerical Analysis Report*, vol. 379, Manchester Centre for Computational Mathematics, Manchester, England.
- [11] C. Chen, and F. Liu, and K., Burrage, 2008, "Finite difference methods and a fourier analysis for the fractional reaction–subdiffusion equation," *Applied Mathematics and Computation*, **198**(2), pp. 754–769.
- [12] S. Chen, and F. Liu, 2008, "ADI-Euler and extrapolation methods for the two-dimensional fractional advection–dispersion equation," *Journal of Applied Mathematics and Computing*,

- 26**(1-2), pp. 295-311.
- [13] P. Zhuang, and F. Liu, 2007, "Implicit difference approximation for the two-dimensional space-time fractional diffusion equation," *Journal of Applied Mathematics and Computing*, **25**(1-2), pp. 269-282.
- [14] P. Zhuang, and F. Liu, 2007, "Finite difference approximation for two-dimensional time fractional diffusion equation," *Journal of Algorithms & Computational Technology*, **1**(1), pp. 1-15.
- [15] F. Liu and P. Zhuang, 2006, "Implicit Difference Approximation for the Time Fractional Diffusion Equation," *Journal of Applied Mathematics and Computing: International Journal*, **22**(3), pp. 87-99.
- [16] F. Liu, V. Anh, and I. Turner, 2004, "Numerical solution of the space fractional Fokker-Planck equation," *Journal of Computational and Applied Mathematics*, **166**(1), pp. 209-219.
- [17] B. Baeumer, M. Kovcs, M.M. Meerschaert, "Numerical solutions for fractional reaction-diffusion equations," *Computers and Mathematics with Applications*, Vol. 55 (2008), pp. 2212-2226.
- [18] M.M. Meerschaert and C. Tadjeran, "Finite difference approximations for two-sided space-fractional partial differential equations," *Applied Numerical Mathematics*, Vol. 56 (2006), No. 1, pp. 80-90.
- [19] C. F., Chen, and C. H., Hsiao, 1975, "Design of piecewise constant gains for optimal control via Walsh functions," *IEEE Trans Automatic Control*, **20**(5), pp. 596-603.
- [20] C. F. Chen, Y. T. Tsay, T. T. Wu, "Walsh operational matrices for fractional calculus and their application to distributed parameter systems," *Journal of Franklin Institute* 503 (1977) 267-284.
- [21] D. S. Shih, F. C. Kung, C. M. Chao, "Laguerre series approach to the analysis of a linear control system incorporation observers," *International Journal of Control* 43 (1986) 123-128.
- [22] R. Y. Ckang, M. L. Wang, "Legendre polynomials approximation to dynamic linear state equations with initial or boundary value condition," *International Journal of Control* 40 (1984) 215-232.
- [23] P. N. Paraskevopoulos, "Chebyshev series approach to system identification analysis and optimal control," *Journal of Franklin Institute* 316 (1983) 135-157.
- [24] P. N. Paraskevopoulos, P. D. Sparis, S. G. Mouroutsos, "The Fourier series operational matrix of integration," *International Journal of System Science* 16(1985) 171-176.
- [25] C. F. Chen, C. H. Hsiao, "Haar wavelet method for solving lumped and distributed parameter systems," *IEE Proceedings - Control Theory and Applications* 144 (1997) 87-94.
- [26] A. Kilicman, Z. A. A. A. Zhou, 2007, "Kronecker operational metrics for fractional calculus and some applications," *Appl. Math. Comp.*, 187, pp. 250-265.
- [27] M. K. Bouafoura and N. B. Braiek, 2010, " $PI^\lambda D^\mu$ controller design for integer and fractional plants using piecewise orthogonal functions," *Comm. Nonlinear Sc. and Num. Simu.*, **15**(5), pp. 1267-1278.
- [28] Ü. Lepik, 2009, "Solving fractional integral equations by the Haar wavelet method," *Appl. Math. Comp.*, 214, pp. 468-478.

- [29] J. L. Wu, C. H. Chen, and C. F. Chen, 2001, "Numerical Inversion of Laplace Transform Using Haar Wavelet Operational Matrices," IEEE Trans. On circuits and systems-I: Fundamental Theory and Applications, **48**(1), pp. 120-122.
- [30] J. L. Wu, 2009, "A Wavelet Operational Method for Solving Fractional Partial Differential Equations Numerically," Appl. Math. and Comp., **214**, pp.31-40.
- [31] J. L. Wu, and C. H. Chen, 2004, "A New Operational Approach for Solving Fractional Calculus and Fractional Differential Equations Numerically," IEICE Trans. Fundamentals, **E87-A**(5), pp. 1077-82.

S. SAHA RAY

DEPARTMENT OF MATHEMATICS, NATIONAL INSTITUTE TECHNOLOGY, ROURKELA, ORISSA-769008,
INDIA

E-mail address: santanusaharay@yahoo.com, saharays@nitrl.ac.in