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A NOTE ON SOME FRACTIONAL INTEGRAL INEQUALITIES VIA HADAMARD INTEGRAL

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Abstract. In this paper, we establish certain integral inequalities for the Chebyshev functional in case of synchronous function, using the Hadamard fractional integral.

1. Introduction

Consider the functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \quad (1.1)$$

where f and g are two integrable functions which are synchronous on $[a,b]$, ($i.e(f(x)-f(y))(g(x)-g(y)) \geq 0$ for any $x, y \in [a,b]$), given in [7]. Many researchers have studied (1.1) and number of inequalities appeared in literature see [1,3,9-12]. The main objective of this paper is to establish some fractional inequalities for (1.1), using Hadamard fractional integrals. Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville, Caputo derivative, see [2,5,6,8]. The necessary background details are given in the book by A.A.Kilbas [2,p.110-118], and S.G.Samko et al. [8,p.329-332].

2. Preliminaries

In this section we give some preliminaries and basic proposition used in our subsequent discussion. Here we give some definitions of Hadamard derivative and integral as in [4, p.159-171].

Definition 2.1. *The Hadamard fractional integral of order $\alpha \in R^+$ of function $f(x)$, for all $x > 1$ is defined as*

$$_HD_{1,x}^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_1^x \ln(\frac{x}{t})^{\alpha-1}f(t)\frac{dt}{t}, \quad (2.1)$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$.

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Definition 2.2. *The Hadmard fractional derivative of order $\alpha \in [n-1, n)$, $n \in Z^+$, of function $f(x)$ is given as follows.*

$$ {}_H D^{\alpha}_{1,x} f(x) = \frac{1}{\Gamma(n-\alpha)} (x\frac{d}{dx})^n \int_1^x \ln(\frac{x}{t})^{n-\alpha-1} f(t) \frac{dt}{t}. \tag{2.2}$$

From above definitions, we see the difference between Hadamard and Riemann-Liouville fractional derivative and integrals as kernel in the Hadamard integral has the form of $\ln(\frac{x}{t})$ instead of the form of $(x-t)$, which is involves both in the Riemann-Liouville and Caputo integral. The Hadamard derivative has the operator$(x\frac{d}{dx})^n$, whose construction is well suited to the case of the half-axis and is invariant relation to dilation [8, p.330], while the Riemann-Liouville derivative has the operator $(\frac{d}{dx})^n$. We give some image formulas under the operator (2.1) and (2.2), which would be used in the derivation of our main result.

Proposition 2.1 (4). *If $0 < \alpha < 1$, the following relation hold:*

$$ {}_H D^{-\alpha}_{1,x} (\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\ln x)^{\beta+\alpha-1}, \tag{2.3}$$

$$ {}_H D^{\alpha}_{1,x} (\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\ln x)^{\beta-\alpha-1}, \tag{2.4}$$

respectively.

Next, we will introduce the weighted space $C_{\gamma,\ln}[a,b]$, $C^m_{\delta,\gamma}[a,b]$ of function f on the finite interval $[a,b]$, if $\gamma \in (0 \le Re(\gamma) < 1)$, $n-1 < \alpha \le n$, then $C_{\gamma,\ln}[a,b] := \{f(x) : \ln(\frac{x}{a})^{\gamma} f(x) \in C[a,b], \| f \|_{c_r} = \| \ln(\frac{x}{a}) f(x) \|_c\}$, $C_{0,\ln}[a,b] = C[a,b]$ and $C^m_{\delta,\ln}[a,b] := \{g(x) : (\delta^n g)(x) \in C_{\gamma,\ln}[a,b], \| g \|_{c_r,\ln} = \sum_{k=0}^{n-1} \| (\delta^k g) \|_c + \| (\delta^n g) \|_{c_{r,\ln}}\}$, $\delta = x\frac{d}{dx}$.

For the convenience of establishing the result, we give the semigroup property

$$ ({}_H D^{-\alpha}_{1,x})({}_H D^{-\beta}_{1,x}) f(x) = {}_H D^{-(\alpha+\beta)}_{1,x} f(x). \tag{2.5}$$

3. Main Result

Theorem 3.1. *Let f and g be two synchronous function on $[0,\infty[$. Then for all $t > 0$, $\alpha > 0$, we have*

$$ {}_H D^{-\alpha}_{1,t} (fg)(t) \ge \frac{\Gamma(\alpha+1)}{(\ln t)^{\alpha}} ({}_H D^{-\alpha}_{1,t} f(t))({}_H D^{-\alpha}_{1,t} g(t)). \tag{3.1}$$

Proof: Since f and g are synchronous on $[0,\infty[$ for all $\tau \ge 0$, $\rho \ge 0$, we have

$$ (f(\tau) - f(\rho))(g(\tau) - g(\rho)) \ge 0, \tag{3.2}$$

From (3.2),

$$ f(\tau)g(\tau) + f(\rho)g(\rho) \ge f(\tau)g(\rho) + f(\rho)g(\tau), \tag{3.3}$$

Multiplying both side of (3.3) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}$, which is positive because $\tau \in (0,t)$, $t > 0$. Then integrating the resulting identity with respect to τ over (1,t), we obtain

$$ \begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_1^t \ln(\frac{t}{\tau})^{\alpha-1} f(\tau)g(\tau) \frac{d\tau}{\tau} + \frac{1}{\Gamma(\alpha)} \int_1^t \ln(\frac{t}{\tau})^{\alpha-1} f(\rho)g(\rho) \frac{d\tau}{\tau} \\ &\ge \frac{1}{\Gamma(\alpha)} \int_1^t \ln(\frac{t}{\tau})^{\alpha-1} f(\tau)g(\rho) \frac{d\tau}{\tau} + \frac{1}{\Gamma(\alpha)} \int_1^t \ln(\frac{t}{\tau})^{\alpha-1} f(\rho)g(\tau) \frac{d\tau}{\tau}, \end{aligned} \tag{3.4}$$

consequently,

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}(fg)(t)+f(\rho)g(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}1\frac{d\tau}{\tau}\\&\geq \frac{g(\rho)}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}f(\tau)\frac{d\tau}{\tau}+\frac{f(\rho)}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}g(\tau)\frac{d\tau}{\tau},\end{aligned}\tag{3.5}$$

we get

$${}_HD_{1,t}^{-\alpha}(fg)(t)+f(\rho)g(\rho)_HD_{1,t}^{-\alpha}(1)\geq g(\rho)_HD_{1,t}^{-\alpha}f(t)+f(\rho)_HD_{1,t}^{-\alpha}g(t).\tag{3.6}$$

multiplying both side of (3.6) by $\frac{(\ln(\frac{t}{\rho}))^{\alpha-1}}{\rho\Gamma(\alpha)}$, which is positive because $\rho\in(0,t)$, $t>0$. Then integrating the resulting identity with respect to ρ over (1,t), we obtain

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}(fg)(t)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}1\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}(1)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}f(\rho)g(\rho)\frac{d\rho}{\rho}\\&\geq_H D_{1,t}^{-\alpha}f(t)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}g(\rho)\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}g(t)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}f(\rho)\frac{d\rho}{\rho},\end{aligned}\tag{3.7}$$

hence

$$\begin{aligned}{}_HD_{1,t}^{-\alpha}(fg)(t)_HD_{1,t}^{-\alpha}(1)+_H D_{1,t}^{-\alpha}(1)_HD_{1,t}^{-\alpha}(fg)(t)\geq_H D_{1,t}^{-\alpha}f(t)_HD_{1,t}^{-\alpha}g(t)\\+_H D_{1,t}^{-\alpha}g(t)_HD_{1,t}^{-\alpha}f(t),\end{aligned}\tag{3.8}$$

we get

$${}_HD_{1,t}^{-\alpha}(1)\left[2_HD_{1,t}^{-\alpha}(fg)(t)\right]\geq 2_HD_{1,t}^{-\alpha}f(t)_HD_{1,t}^{-\alpha}g(t),\tag{3.9}$$

This ends the proof of Theorem 3.1.

Theorem 3.2. *Let f and g be two synchronous function on $[0,\infty[$, then for all $t>0$, $\alpha>0$, $\beta>0$ we have*

$$\begin{aligned}\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_HD_{1,t}^{-\alpha}(fg)(t)+\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_HD_{1,t}^{-\beta}(fg)(t)\geq_H D_{1,t}^{-\alpha}f(t)_HD_{1,t}^{-\beta}g(t)\\+_H D_{1,t}^{-\alpha}g(t)_HD_{1,t}^{-\beta}f(t).\end{aligned}\tag{3.10}$$

Proof: To prove above Theorem multiplying equation (3.6) by $\frac{(\ln(\frac{t}{\rho}))^{\beta-1}}{\rho\Gamma(\beta)}$, which is positive because $\rho\in(0,t)$, $t>0$. Then integrating resulting identity with respective ρ over 1 to t, we obtain

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}(fg)(t)\frac{1}{\Gamma(\beta)}\int_1^t (\ln(\frac{t}{\rho}))^{\beta-1}\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}(1)\frac{1}{\Gamma(\beta)}\int_1^t (\ln(\frac{t}{\rho}))^{\beta-1}f(\rho)g(\rho)\frac{d\rho}{\rho}\\&\geq_H D_{1,t}^{-\alpha}f(t)\frac{1}{\Gamma(\beta)}\int_1^t (\ln(\frac{t}{\rho}))^{\beta-1}g(\rho)\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}g(t)\frac{1}{\Gamma(\beta)}\int_1^t (\ln(\frac{t}{\rho}))^{\beta-1}f(\rho)\frac{d\rho}{\rho},\end{aligned}\tag{3.11}$$

and this ends the proof of Theorem 3.2.

Remark 3.1. *Applying Theorem 3.2 for $\alpha=\beta$, we obtain Theorem 3.1.*

Theorem 3.3. *Let $(f_i)_{i=1,2,...n}$ be positive increasing function on $[0,\infty[$, then for all $t>0$, $\alpha>0$, we have*

$$ {}_HD_{1,t}^{-\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left[{}_HD_{1,t}^{-\alpha}(1)\right]^{1-n} . \prod_{i=1}^{n} {}_HD_{1,t}^{-\alpha} f_i(t). \tag{3.12} $$

Proof: We prove this Theorem by induction. Clearly, for $n=1$, ${}_HD_{1,t}^{-\alpha} f_1(t) \geq {}_HD_{1,t}^{-\alpha} f_1(t)$, for all $t>0$, $\alpha>0$.
for $n=2$, applying equation (3.1), we obtain

$$ {}_HD_{1,t}^{-\alpha}(f_1f_2) \geq \left[{}_HD_{1,t}^{-\alpha}(1)\right]^{-1} {}_HD_{1,t}^{-\alpha} f_1(t) {}_HD_{1,t}^{-\alpha} f_2(t). \tag{3.13} $$

Suppose that by induction hypothesis

$$ {}_HD_{1,t}^{-\alpha}(\prod_{i=1}^{n-1} f_i)(t) \geq \left[{}_HD_{1,t}^{-\alpha}(1)\right]^{2-n} \prod_{i=1}^{n-1} {}_HD_{1,t}^{-\alpha} f_i(t) \quad for\ all\ t>0, \quad \alpha>0. \tag{3.14} $$

Now, since $(f_i)_{i=1,2,...n}$ are positive increasing function, then $(\prod_{i=1}^{n-1} f_i)(t)$ is an increasing function, therefore we can apply Theorem 3.1 to the function $\prod_{i=1}^{n-1} f_i = g$, $f_n = f$, we obtain

$$ \begin{aligned} {}_HD_{1,t}^{-\alpha}\prod_{i=1}^{n}(f_i)(t) &\geq \left({}_HD_{1,t}^{-\alpha}\prod_{i=1}^{n-1} f_if_n\right)(t) \geq_H D_{1,t}^{-\alpha}(g.f)(t) \\ &\geq \left[{}_HD_{1,t}^{-\alpha}(1)\right]_H^{-1} D_{1,t}^{-\alpha} g(t) {}_HD_{1,t}^{-\alpha} f(t) \\ &\geq \left[{}_HD_{1,t}^{-\alpha}(1)\right]_H^{-1} D_{1,t}^{-\alpha}(\prod_{i=1}^{n-1} f_i)(t) {}_HD_{1,t}^{-\alpha} f_n \\ &\geq \left[{}_HD_{1,t}^{-\alpha}(1)\right]^{-1} \left[{}_HD_{1,t}^{-\alpha}(1)\right]^{2-n} (\prod_{i=1}^{n-1} {}_HD_{1,t}^{-\alpha} f_i)(t) {}_HD_{1,t}^{-\alpha} f_n \\ &\geq \left[{}_HD_{1,t}^{-\alpha}(1)\right]^{1-n} \prod_{i=1}^{n} {}_HD_{1,t}^{-\alpha} f_i(t). \end{aligned} \tag{3.15} $$

This completes the proof of Theorem 3.3.

References

[1] G.A.Anastassiou, *Fractional Differentiation Inequalities,* Springer, New York, 2009.
[2] A.A.Kilbas, H.M.Srivastava and J.J.Trujillo, *Theory and Application of Fractional Differential Equations,* Elsevier, Amersterdam, 2006.
[3] B.G.Pachpatte, *A note on Chebyshev-Gruss type inequalities for diffrerential function,* Tamsui Oxf. J. Math. Sci., 22(10)(2006), 29-36.
[4] D.Baleanu, J.A.T.Machado and C.J.Luo, *Fractional Dynamic and Control,* Springer, 2012, pp.159-171.
[5] I.Podlubny, *Fractional Differential Equations,* Academic Press, New York, 1999.
[6] K.S.Miller and B.Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley, New York, 1993.
[7] P.L.Chebyshev, *Sur les expressions approximatives des integrales definies par les autres prise entre les memes limites,* Proc. Sco. Charkov, 2(1882), 93-98.
[8] S.G.Somko, A.A.Kilbas and O.I.Marichev, *Fractional Integral and Derivative Theory and Application,* Gordon and Breach, Switzerland, 1993.
[9] S.M.Malamud, *Some complements to the Jensen and Chebyshev inequalities and a problem of W.Walter,* Proc. Amer. Math. Soc., 129, 9 (2001), 2671-2678.

[10] S.Marinkoric, P.Rajkovic and M.Stankovic, *The inequalities for some q-integrals,* Comput. Math. Appl., 56(2008), 2490-2498.
[11] S. Belarbi and Z. Dahmani, *On some new fractional integral inequality,* J. Inequal. Pure and Appl. Math., 10(3)(2009), Art.86, 5 pp.
[12] S.S.Dargomir and C.E.Pearce, *Selected topics in Hermit-Hadamard inequality,* Monographs: http://rgmia.vu.edu.au/amonographs/hermite-hadmard.html, Victoria University, 2000.

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