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CERTAIN NEW CLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS ASSOCIATED WITH SăLăGEAN OPERATOR WITH VARYING ARGUMENTS

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Abstract. We introduce certain two new classes of analytic and univalent functions associated with *Sălăgean* operator with varying arguments. Moreover, we obtain coefficient estimates, distortion theorems and extreme points for these classes.

1. Introduction, Definitions and Preliminaries

Suppose $\mathfrak{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which is analytic and univalent in Δ where $\Delta := \{z \in \mathbb{C} : |z| < 1\}$.

Definition 1.1. [15] We define a differential operator that is called *Sălăgean* operator as follows

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = z f'(z), \tag{1.3}$$

$$D^n f(z) = D(D^{n-1} f(z)). \tag{1.4}$$

Then from (1.2), (1.3) and (1.4) we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} (k)^n a_k z^k.$$

We refer the interested readers to Sekine [16], Aouf et al. ([3] and [4]) , Hossen et al. [8] and Aouf [1] for more applications of the operator D^{n+m}.

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Let $q-MN(n,m,\sigma,\delta)$ denote the subclass of $\mathfrak{A}$ consisting of functions $f(z)$ of the form (1.1) and satisfy the following inequality

$$Re\left\{\frac{D^{n+m}f(z)}{D^{n}f(z)}-\sigma\right\}> q\left|\frac{D^{n+m}f(z)}{D^{n}f(z)}-\delta\right|, \tag{1.5}$$

where $(0\leq\sigma<\delta\leq 1;\ q(1-\delta)<(1-\sigma);\ z\in\Delta)$, $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ and $m\in\mathbb{N}$. We note that Darwish [6] studied the subclass $q-MN(n,m,\sigma,\delta)$ with $-1\leq\sigma<1$, $q\geq 0$ and $\delta=1$.

We also let $q-XZW(n,m,\sigma,\delta)$ denote the subclass of $\mathfrak{A}$ consisting of functions $f(z)$ of the form (1.1) and satisfy the following inequality

$$Re\left\{\frac{D^{n+m+1}f(z)}{D^{n+1}f(z)}-\sigma\right\}> q\left|\frac{D^{n+m+1}f(z)}{D^{n+1}f(z)}-\delta\right|, \tag{1.6}$$

where $(0\leq\sigma<\delta\leq 1;\ q(1-\delta)<(1-\sigma);\ z\in\Delta)$.

With $n=0$ and $m=1$ the class $q-MN(0,1,\sigma,\delta)$ denote the class of q uniformly starlike functions of order σ and type δ and the class $q-XZW(0,1,\sigma,\delta)$ denote the class of $q-$uniformly starlike functions of order σ and type δ.

By giving values for n, m, σ and δ, we obtain the following subclasses studied by various authors:

(i) $q-MN(0,1,\sigma,1)$=$MD(q,\sigma)$ and $q-XZW(0,1,\sigma,1)$=$RD(q,\sigma)$ (see [15, 17]),

(ii) $1-MN(0,1,\sigma,1)$=$M_j(\sigma)$ and $1-XZW(0,1,\sigma,1)$=$XZW(\sigma)$ (see [13]),

(iii)$q-MN(0,1,0,1)$=$q-MN$ and $q-XZW(0,1,0,1)$=$q-XZW$ (see [9, 10]),

(iv)$1-MN(0,1,0,1)$=M_j and $1-XZW(0,1,0,1)$=XZW (see [13, 7, 14]).

Also, we notice that

$$(i)\ \ 1-MN(n,m,\sigma,\delta)$$
$$=\left\{f(z)\in\mathfrak{A}: Re\left\{\frac{D^{n+m}f(z)}{D^{n}f(z)}-\sigma\right\}> \left|\frac{D^{n+m}f(z)}{D^{n}f(z)}-\delta\right|\right\},$$
$$(ii)\ 1-XZW(n,m,\sigma,\delta)$$
$$=\left\{f(z)\in\mathfrak{A}: Re\left\{\frac{D^{n+m+1}f(z)}{D^{n+1}f(z)}-\sigma\right\}> \left|\frac{D^{n+m+1}f(z)}{D^{n+1}f(z)}-\delta\right|\right\},$$

which are uniformly starlike functions of order σ and type δ with $n=0$ and $m=1$ and uniformly convex functions of order σ and type δ with $n=1$ and $m=1$ respectively.

Definition 1.2. (see [18]) A function $f(z)$ of the form (1.1) is said to be in the class $U(\theta_k)$ for all $k\geq 2$. Moreover if there exist a real number s i.e $\theta_k+(k-1)s\equiv\pi($ mod $2\pi)$, then $f(z)$ is said to be in the class $U(\theta_k,s)$. Suppose U denote the union of $U(\theta_k,s)$ with all possible sequences $\{\theta_k\}$ and all possible real numbers s.

Let $q-UMN(n,m,\sigma,\delta)$ be the subclass of U consisting of functions $f(z) \in q-MN(n,m,\sigma,\delta)$. Also, we let $q-UXZW(n,m,\sigma,\delta)$ be the subclass of U consisting of functions $f(z) \in q-XZW(n,m,\sigma,\delta)$.

In this paper we obtain coefficient estimates for the classes $q-MN(n,m,\sigma,\delta)$, $q-XZW(n,m,\sigma,\delta)$, $q-UMN(n,m,\sigma,\delta)$ and $q-UXZW(n,m,\sigma,\delta)$, we also obtain distortion theorems and extreme points for the classes $q-UMN(n,m,\sigma,\delta)$ and $q-UXZW(n,m,\sigma,\delta)$ (cf. [2, 11, 12]).

2. Coefficient Estimates

Theorem 2.1. *Let $f(z) \in \mathfrak{A}$ which is defined by* (1.1)*. Then $f(z)$ be in the class $q-MN(n,m,\sigma,\delta)$ if*

$$\sum_{k=2}^{\infty} k^n \Big[q(k^m-\delta)+k^m-\sigma\Big]|a_k| \le 1-\sigma-q(1-\delta).$$

Proof. To prove $f(z) \in q-MN(n,m,\sigma,\delta)$, then we need to obtain the condition (1.5) which can be written as

$$Re\Big\{(1+qe^{i\theta})\frac{D^{n+m}f(z)}{D^n f(z)} - q\delta e^{i\theta}\Big\} > \sigma$$

or

$$Re\Big\{\frac{A(z)}{B(z)}\Big\} > \sigma, \tag{2.1}$$

where $A(z) = (1+qe^{i\theta})D^{n+m}f(z) - q\delta e^{i\theta}D^n f(z)$ and $B(z) = D^n f(z)$, then the inequality (2.1) is equivalent to

$$\Big|A(z)+(1-\sigma)B(z)\Big| - \Big|A(z)-(1+\sigma)B(z)\Big| \ge 0.$$

We have

$$\begin{aligned}\Big|A(z)+(1-\sigma)B(z)\Big| &= \Big|[2-\sigma+qe^{i\theta}(1-\delta)]z + \sum_{k=2}^{\infty} k^n \Big[k^m+qe^{i\theta}(k^m-\delta)+1-\sigma\Big]a_k z^k\Big| \\ &\ge [2-\sigma-q(1-\delta)]|z| - \sum_{k=2}^{\infty} k^n \Big\{q(k^m-\delta)+k^m+1-\sigma\Big\}|a_k||z|^k \\ &\ge [2-\sigma-q(1-\delta)] - \sum_{k=2}^{\infty} k^n \Big\{q(k^m-\delta)+k^m+1-\sigma\Big\}|a_k|,\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\Big|A(z)-(1+\sigma)B(z)\Big| &= \Big|[qe^{i\theta}(1-\delta)-\sigma]z + \sum_{k=2}^{\infty} k^n \Big[k^m+qe^{i\theta}(k^m-\delta)-1-\sigma\Big]a_k z^k\Big| \\ &\le [\sigma+q(1-\delta)]|z| + \sum_{k=2}^{\infty} k^n \Big\{q(k^m-\delta)+k^m-\sigma-1\Big\}|a_k||z|^k \\ &\le [\sigma+q(1-\delta)] + \sum_{k=2}^{\infty} k^n \Big\{q(k^m-\delta)+k^m-\sigma-1\Big\}|a_k|,\end{aligned} \tag{2.3}$$

thus we obtain from (2.2) and (2.3) the inequality

$$\Big|A(z)+(1-\sigma)B(z)\Big|-\Big|A(z)-(1+\sigma)B(z)\Big|\geq$$
$$2[1-\sigma-q(1-\delta)]-2\sum_{k=2}^{\infty}k^{n}\Big\{q(k^{m}-\delta)+k^{m}-\sigma\Big\}|a_k|.$$

Then

$$|A(z)+(1-\sigma)B(z)\Big|-\Big|A(z)-(1+\sigma)B(z)|\geq 0 \text{ if}$$
$$2[1-\sigma-q(1-\delta)]-2\sum_{k=2}^{\infty}k^{n}\Big\{q(k^{m}-\delta)+k^{m}-\sigma\Big\}|a_k|\geq 0$$

or

$$\sum_{k=2}^{\infty}k^{n}\Big\{q(k^{m}-\delta)+k^{m}-\sigma\Big\}|a_k|\leq 1-\sigma-q(1-\delta).$$

This ends the proof of Theorem 2.1. □

Theorem 2.2. *Let* $f(z)\in\mathfrak{A}$ *which is defined by* (1.1). *Then* $f(z)$ *be in the class* $q-XZW(n,m,\sigma,\delta)$ *if*

$$\sum_{k=2}^{\infty}k^{n+1}\Big[q(k^{m}-\delta)+k^{m}-\sigma\Big]|a_k|\leq 1-\sigma-q(1-\delta).$$

Proof. The proof comes directly from Theorem 2.1.
Since we have $f(z)\in q-XZW(n,m,\sigma,\delta)$ if and only if $Df(z)\in q-MN(n,m,\sigma,\delta)$ from (1.5) and (1.6). □

Putting $n=0$, $m=1$ and $\delta=1$ in Theorem 2.1 and Theorem 2.2, we obtained the results which is found by Shams et al. [17, Theorems 2.1, 2.2, resp].

Theorem 2.3. *Let* $f(z)\in\mathfrak{A}$ *which is defined by* (1.1) *be in the class* $q\text{–}UMN(n,m,\sigma,\delta)$ *if and only if*

$$\sum_{k=2}^{\infty}k^{n}\Big[q(k^{m}-\delta)+k^{m}-\sigma\Big]|a_k|<1-\sigma-q(1-\delta). \tag{2.4}$$

Proof. With the aid of Theorem 2.1, we only need to prove that if $f(z)\in q-UMN(n,m,\sigma,\delta)$, then $f(z)$ satisfies (2.4).
Suppose $f(z)\in q-UMN(n,m,\sigma,\delta)$, then we have from (1.5) Since we have $f(z)\in q-XZW(n,m,\sigma,\delta)$ if and only if $Df(z)\in q-MN(n,m,\sigma,\delta)$ from (1.5) and (1.6).

$$Re\Bigg\{\frac{D^{n+m}f(z)}{D^{n}f(z)}-\sigma\Bigg\}>q\Big|\frac{D^{n+m}f(z)}{D^{n}f(z)}-\delta\Big|,$$

then we obtain

$$Re\Bigg\{\frac{(1-\sigma)+\sum_{k=2}^{\infty}k^{n}[k^{m}-\sigma]a_k z^{k-1}}{1+\sum_{k=2}^{\infty}k^{n}a_k z^{k-1}}\Bigg\}$$
$$>q\Bigg|\frac{(1-\delta)+\sum_{k=2}^{\infty}k^{n}[k^{m}-\delta]a_k z^{k-1}}{1+\sum_{k=2}^{\infty}k^{n}a_k z^{k-1}}\Bigg|. \tag{2.5}$$

Since $Re\{z\} \leq |z|$ and $f(z) \in U$, $f(z)$ is in the class $U(\theta_k, s)$ for some sequences θ_k and a real number s i.e

$$\theta_k + (k-1)s \equiv \pi \pmod{2\pi}.$$

Setting $z = re^{is}$, then (2.5) can be written as

$$\left\{ \frac{(1-\sigma) - \sum_{k=2}^{\infty} k^n [k^m - \sigma] |a_k| r^{k-1}}{1 - \sum_{k=2}^{\infty} k^n |a_k| r^{k-1}} \right\} > q \left\{ \frac{(1-\delta) + \sum_{k=2}^{\infty} k^n [k^m - \delta] |a_k| r^{k-1}}{1 - \sum_{k=2}^{\infty} k^n |a_k| r^{k-1}} \right\}.$$

Suppose $r \to 1$, then we obtain

$$\sum_{k=2}^{\infty} k^n \Big[q(k^m - \delta) + k^m - \sigma \Big] |a_k| < 1 - \sigma - q(1-\delta).$$

Thus the proof of Theorem 2.3 is complete. □

Corollary 2.1. *If* $f(z) \in q - UMN(n, m, \sigma, \delta)$*, then*

$$|a_k| \leq \frac{1 - \sigma - q(1-\delta)}{k^n \Big[q(k^m - \delta) + k^m - \sigma \Big]} \quad (k \geq 2).$$

The equality is obtained for the function

$$f(z) = z + \frac{1 - \sigma - q(1-\delta)}{k^n \Big[q(k^m - \delta) + k^m - \sigma \Big]} e^{i\theta_k} z^k \quad (k \geq 2,\ z \in \triangle).$$

Using the same method of Theorem 2.3, we obtain the following theorem.

Theorem 2.4. *Let* $f(z) \in \mathfrak{A}$ *which is defined by* (1.1) *be in the class* $q\text{–}UXZW(n, m, \sigma, \delta)$ *if and only if*

$$\sum_{k=2}^{\infty} k^{n+1} \Big[q(k^m - \delta) + k^m - \sigma \Big] |a_k| < 1 - \sigma - q(1-\delta).$$

Corollary 2.2. *If* $f(z) \in q - UXZW(n, m, \sigma, \delta)$*, then*

$$|a_k| \leq \frac{1 - \sigma - q(1-\delta)}{k^{n+1} \Big[q(k^m - \delta) + k^m - \sigma \Big]} \quad (k \geq 2). \tag{2.6}$$

The equality is obtained for the function

$$f(z) = z + \frac{1 - \sigma - q(1-\delta)}{k^{n+1} \Big[q(k^m - \delta) + k^m - \sigma \Big]} e^{i\theta_k} z^k \quad (k \geq 2,\ z \in \triangle)$$

is sharp for the inequality (2.6).

3. Distortion Theorems

Theorem 3.1. *If $f(z) \in \mathfrak{A}$ which is defined by* (1.1) *is in the class $q-UMN(n,m,\sigma,\delta)$, then*

$$r - \frac{1-\sigma-q(1-\delta)}{2^n[q(2^m-\delta)+2^m-\sigma]}r^2 \leq |f(z)| \leq r + \frac{1-\sigma-q(1-\delta)}{2^n[q(2^m-\delta)+2^m-\sigma]}r^2$$

with $|z| = r$ $(0 < r < 1)$. The result is sharp.

Proof. We use the technique used by Silverman [18]. With the aid of Theorem 2.3, Since

$$\psi(k) = k^n\Big[q(k^m-\delta)+k^m-\sigma\Big]$$

is an increasing function of k $(k \geq 2)$, thus we have

$$\psi(2)\sum_{k=2}^{\infty}|a_k| \leq \sum_{k=2}^{\infty}\psi(k)|a_k| \leq 1-\sigma-q(1-\delta),$$

that is equivalent to

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{1-\sigma-q(1-\delta)}{\psi(2)}.$$

Then we obtain

$$|f(z)| = |z + \sum_{k=2}^{\infty}a_k z^k| \leq r + r^2\sum_{k=2}^{\infty}|a_k| \quad (0 < |z| = r < 1)$$
$$\leq r + \frac{1-\sigma-q(1-\delta)}{2^n[q(2^m-\delta)+2^m-\sigma]}r^2.$$

We also obtain

$$|f(z)| = |z + \sum_{k=2}^{\infty}a_k z^k| \geq r - r^2\sum_{k=2}^{\infty}|a_k|$$
$$\geq r - \frac{1-\sigma-q(1-\delta)}{2^n[q(2^m-\delta)+2^m-\sigma]}r^2.$$

The proof is complete.

We also notice that The result is sharp with

$$f(z) = z + \frac{1-\sigma-q(1-\delta)}{2^n\Big[q(2^m-\delta)+2^m-\sigma\Big]}e^{i\theta_2}z^2, \ (z = re^{i\theta} \text{ and } 0 < r < 1). \tag{3.1}$$

□

Corollary 3.1. *By Theorem 3.1, we notice that $f(z)$ lies in a disk with center at the origin and radius r_1 given by*

$$r_1 = 1 + \frac{1-\sigma-q(1-\delta)}{2^n\Big[q(2^m-\delta)+2^m-\sigma\Big]}.$$

Theorem 3.2. *If $f(z) \in \mathfrak{A}$ which is defined by* (1.1) *is in the class $q-UMN(n,m,\sigma,\delta)$, then*

$$1 - \frac{[1-\sigma-q(1-\delta)]}{2^{n-1}[q(2^m-\delta)+2^m-\sigma]}r \leq |f'(z)| \leq 1 + \frac{[1-\sigma-q(1-\delta)]}{2^{n-1}[q(2^m-\delta)+2^m-\sigma]}r$$

with $|z| = r$ $(0 < r < 1)$. *The result is sharp.*

Proof. Similarly since

$$\frac{\psi(k)}{k} = k^{n-1}\Big[q(k^m - \delta) + k^m - \sigma\Big]$$

is an increasing function of k $(k \geq 2)$, thus with the aid of Theorem 2.3, we have

$$\frac{\psi(2)}{2}\sum_{k=2}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty}\psi(k)|a_k| \leq 1 - \sigma - q(1-\delta),$$

that is equivalent to

$$\sum_{k=2}^{\infty} k|a_k| \leq \frac{2[1-\sigma-q(1-\delta)]}{\psi(2)}.$$

Then we obtain

$$\begin{aligned}|f'(z)| &= |1 + \sum_{k=2}^{\infty} k a_k z^{k-1}| \leq 1 + r\sum_{k=2}^{\infty} k|a_k| \quad (0 < |z| = r < 1)\\ &\leq 1 + \frac{1-\sigma-q(1-\delta)}{2^{n-1}[q(2^m-\delta)+2^m-\sigma]}r.\end{aligned}$$

We also obtain

$$\begin{aligned}|f'(z)| &= |1 + \sum_{k=2}^{\infty} k a_k z^{k-1}| \geq 1 - r\sum_{k=2}^{\infty} k|a_k|\\ &\geq 1 - \frac{1-\sigma-q(1-\delta)}{2^{n-1}[q(2^m-\delta)+2^m-\sigma]}r.\end{aligned}$$

The proof is complete.

We also notice that The result is sharp with the function $f(z)$ given by (3.1). □

Corollary 3.2. *By Theorem 3.2, we notice that $f'(z)$ lies in a disk with center at the origin and radius r_2 given by*

$$r_2 = 1 + \frac{1-\sigma-q(1-\delta)}{2^{n-1}\Big[q(2^m-\delta)+2^m-\sigma\Big]}.$$

Using the same method of Theorems 3.1 and 3.2, we obtain the following theorems for the class $q - UXZW(n, m, \sigma, \delta)$.

Theorem 3.3. *If $f(z) \in \mathfrak{A}$ which is defined by* (1.1) *is in the class $q{-}UXZW(n, m, \sigma, \delta)$, then*

$$r - \frac{1-\sigma-q(1-\delta)}{2^{n+1}[q(2^m-\delta)+2^m-\sigma]}r^2 \leq |f(z)| \leq r + \frac{1-\sigma-q(1-\delta)}{2^{n+1}[q(2^m-\delta)+2^m-\sigma]}r^2 \tag{3.2}$$

with $|z| = r$ $(0 < r < 1)$. *The result is sharp with*

$$f(z) = z + \frac{1-\sigma-q(1-\delta)}{2^{n+1}\Big[q(2^m-\delta)+2^m-\sigma\Big]}e^{i\theta_2}z^2, \ (z = re^{i\theta} \text{ and } 0 < r < 1). \tag{3.3}$$

Theorem 3.4. *If $f(z) \in \mathfrak{A}$ which is defined by* (1.1) *is in the class $q-UXZW(n,m,\sigma,\delta)$, then*

$$1-\frac{1-\sigma-q(1-\delta)}{2^n[q(2^m-\delta)+2^m-\sigma]}r \leq |f'(z)| \leq 1+\frac{1-\sigma-q(1-\delta)}{2^n[q(2^m-\delta)+2^m-\sigma]}r$$

with $|z|=r$ $(0<r<1)$. The result is sharp with $f(z)$ given by (3.3).

4. Extreme Points

Theorem 4.1. *Let $f(z) \in \mathfrak{A}$ which is defined by* (1.1) *be in the class $q-UMN(n,m,\sigma,\delta)$, with $\arg a_k=\theta_k$ i.e $\theta_k+(k-1)s \equiv \pi \pmod{2\pi}$. Suppose*

$$f_1(z)=z$$

$$f_k(z)=z+\frac{1-\sigma-q(1-\delta)}{k^n\Big[q(k^m-\delta)+k^m-\sigma\Big]}e^{i\theta_k}z^k \quad (k\geq 2,\ z\in\triangle).$$

Then $f(z)\in q-UMN(n,m,\sigma,\delta)$ if and only if $f(z)$ can be obtained in the form $f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)$ s.t $\lambda_k\geq 0$ and $\sum_{k=1}^{\infty}\lambda_k=1$.

Proof. We let $f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)$ with $\lambda_k\geq 0$ and $\sum_{k=1}^{\infty}\lambda_k=1$, then we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}\lambda_k f_k(z) \\ &= \lambda_1 f_1(z)+\sum_{k=2}^{\infty}\lambda_k f_k(z) \\ &= \big(1-\sum_{k=2}^{\infty}\lambda_k\big)z+\sum_{k=2}^{\infty}\lambda_k\left\{z+\frac{1-\sigma-q(1-\delta)}{k^n\Big[q(k^m-\delta)+k^m-\sigma\Big]}e^{i\theta_k}z^k\right\} \\ &= z+\sum_{k=2}^{\infty}\frac{1-\sigma-q(1-\delta)}{k^n\Big[q(k^m-\delta)+k^m-\sigma\Big]}\lambda_k e^{i\theta_k}z^k. \end{aligned}$$

So we have

$$\begin{aligned} &\sum_{k=2}^{\infty}k^n\Big[q(k^m-\delta)+k^m-\sigma\Big]\left|\frac{1-\sigma-q(1-\delta)}{k^n\Big[q(k^m-\delta)+k^m-\sigma\Big]}\lambda_k e^{i\theta_k}\right| \\ &= \sum_{k=2}^{\infty}k^n\Big[q(k^m-\delta)+k^m-\sigma\Big]\frac{1-\sigma-q(1-\delta)}{k^n\Big[q(k^m-\delta)+k^m-\sigma\Big]}\lambda_k \\ &= \sum_{k=2}^{\infty}[1-\sigma-q(1-\delta)]\lambda_k \\ &\leq 1-\sigma-q(1-\delta). \end{aligned}$$

Then $f(z)\in q-UMN(n,m,\sigma,\delta)$ by Theorem 2.3.

Conversely suppose $f(z) \in q-UMN(n,m,\sigma,\delta)$ and define

$$\lambda_k = \frac{k^n\Big[q(k^m-\delta)+k^m-\sigma\Big]}{1-\sigma-q(1-\delta)}|a_k|, \quad (k \geq 2)$$

where $\lambda_1 = 1-\sum_{k=1}^{\infty}\lambda_k$. Then

$$\sum_{k=1}^{\infty}\lambda_k f_k(z) = z+\sum_{k=2}^{\infty}a_k z^k = f(z).$$

The proof is complete. □

By using the same technique of Theorem 4.1, we obtain the following theorem.

Theorem 4.2. *Let* $f(z) \in \mathfrak{A}$ *which is defined by* (1.1) *be in the class* $q-UXZW(n,m,\sigma,\delta)$, *where* $\arg a_k = \theta_k$ *i.e* $\theta_k+(k-1)s \equiv \pi \pmod{2\pi}$. *Suppose*

$$f_1(z) = z$$

$$f_k(z) = z+\frac{1-\sigma-q(1-\delta)}{k^{n+1}\Big[q(k^m-\delta)+k^m-\sigma\Big]}e^{i\theta_k}z^k \quad (k \geq 2, z \in \triangle).$$

Then $f(z) \in q-UXZW(n,m,\sigma,\delta)$ *if and only if* $f(z)$ *can be obtained in the form* $f(z) = \sum_{k=1}^{\infty}\lambda_k f_k(z)$ *s.t* $\lambda_k \geq 0$ *and* $\sum_{k=1}^{\infty}\lambda_k = 1$.

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