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EXISTENCE OF LATIN SQUARE DESIGNS ARISING FROM CLASSICAL GRAPH PARAMETERS

B. CHALUVARAJU AND SHAIKH AMEER BASHA

Abstract. However, the Latin square (LS) designs have been studied extensively with some classes of regular graphs, and their connections with classical graph parameters were not explored. This article determines the existence and non-existence of the Latin square designs with two association schemes. Also, discusses the uniqueness of the polynomials obtained from the matrices of the associates of LS-designs emerging from the total number of minimum covering, maximum independent, and minimum dominating sets of certain classes of regular graphs such as Hypercube, Paley graph, Torus graph, Clebsch graph, Shrikhande graph. Further, we generalized the parameters of LS-designs of Circulant graphs, Complete multipartite graphs, and some Nilpotent Cayley graphs.

1. Introduction

Throughout this paper, the graph $G = (V, E)$, is simple and finite. In general, we use, $p = |V|$ and $q = |E|$ to denote the number of vertices and edges of a graph G, respectively. The number of edges adjacent to a vertex is called the degree of a vertex; the minimum degree is denoted by $\delta(G)$ and the maximum degree is denoted by $\Delta(G)$. For graph-theoretical terminology and notation not defined here, we follow ([6] and [11]).
A square array of side s is arranged from the treatments (objects, vertices) $\nu = p = s^2$. Let the association scheme of the design be $s \times s$ array. The blocks are formed by all possible distinct g elements from the treatments laying in each row and each column of the array. From one row or column, it is possible to form C^s_g distinct g-plets, where C^s_g represents the number of combinations of s things taken g at a time, g and s being restricted in this application by $2 \leq g \leq s$. In this manner, it is possible to form $b = 2sC^s_g$ blocks, each of size g and each of the $p = s^2$ treatments will appear once in each of $h = 2C^{s-1}_{g-1}$ blocks.
Two treatments lying in the same row or the same column of the association scheme are first associates, while two treatments not lying in the same row or the same column are second associates. Thus, $n_1 = 2(s-1)$ and $n_2 = (s-1)^2$. Further,

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two treatments are first associates which occur together in C_{g-2}^{s-2} blocks, while two treatments that are second associates do not occur together in any block. Hence, $\lambda_1 = C_{g-2}^{s-2}$ and $\lambda_2 = 0$.

Cox [8], in his analysis, utilized the Latin Square Experimental structure, strategy on three varieties of wheat developed with different varieties of fertilizer types considers the prerequisites for the analysis, which incorporate a lot of test units regularly called plots, various treatments or combinations applied to various test units and treatments are duplicated. The comparison between the test unit, treatment combination, and factor levels depends on information recorded after the treatments have an opportunity to influence the testing material. This information contains at least one variate each comprising of one value for each plot. The variate might be persistent (e.g., weight, stature) or discrete (e.g., tallies, scores), or determined (acquired through arithmetical calculations). From this research, Cox reasoned that this design; Assisted in controlling the impact of inconvenient factors, by gathering test units into blocks having a similar number of treatment levels as the factor of interest. Decreasing the error by obstructing in two ways discover noteworthy outcomes for treatments almost certain. For more details, we refer to ([5], [7], [9] and [19]). Due to the existence and non-existence of the LS-Design for the following graph parameters, we have considered all the families of graphs with order $p = s^2$ and are not randomly selected.

2. Covering Sets (CS)

A set X of graph G is called a vertex covering set if every edge of G is incident to at least one vertex in X. The minimum cardinality among all the covering sets is a vertex covering number $\alpha(G)$. A minimum covering set X of G with $|X| = \alpha(G)$ is called a α-set of G.

Theorem 2.1. *Let G be a connected graph with $p = s^2$. Then the collection of all α-sets forms an LS-Design with parameters $b = 2$, $g = 2$, $h = 1$, $\lambda_1 = 2$, and $\lambda_2 = 0$, if and only if $G \cong C_4(= Q_2)$.*

Proof. Let $G \cong C_4$ be a graph with $p = s^2$ vertices. Then $\alpha(G) = 2$. Therefore, the collection of all α-sets are given as blocks $\{v_1, v_3\}$, $\{v_2, v_4\}$. The total number of blocks $b = 2$, the size of the block $g = 2$, the number of repetition of the elements of the blocks $h = 1$ and the associates are given by $\lambda_1 = 2$ and $\lambda_2 = 0$.
Conversely, suppose $G \cong K_4$, then $\alpha(K_4) = 3$. Therefore, the collection of all α-sets are $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_1\}$, $\{v_4, v_1, v_2\}$. These α-sets are given by blocks $b = 4$, $g = 3$, $h = 3$. By the definition of LS-Design with $2 \leq g \leq s$, which is a contradiction. Hence, the collection of α-sets does not form an LS-Design. □

3. Independent Sets (IS)

A set Y is an independent set of G, if no two vertices in Y are adjacent. The maximum cardinality of independent set Y is called a vertex independence number $\beta(G)$. The maximum independent set Y with $|Y| = \beta(G)$ is called a β-set of G.

Theorem 3.1. *Let G be a connected graph with order 4. Then the collection of all β-sets forms LS-Design with parameters $b = 2$, $g = 2$, $h = 1$, $\lambda_1 = 2$ and $\lambda_2 = 0$, if and only if $G \cong C_4(= Q_2)$.*

Proof. Let G be a connected graph with order 4. Then the vertex set $V = \{v_1, v_2, v_3, v_4\}$ forms a cycle C_4. This implies that the set $Y = \{v_1, v_3\}$ is a β-set of G and $\{v_2, v_4\}$ is an α-set of G. We know that $\alpha(G) + \beta(G) = p$. Therefore, $\alpha(C_4) = \beta(C_4) = 2$. Thus, from the proof of Theorem 2.1, the desired LS-Design with parameters follows. □

Theorem 3.2. *Let G be a connected graph with order 9. Then the collection of all β-sets forms an LS-Design, if and only if $G \cong G_1$ or G_2,*

(i) $G_1 = C_9(1, 3)$ *be a circulant graph with the parameters* $b = 9$, $g = 3$, $h = 3$, $\lambda_1 = 1$ *and* $\lambda_2 = 0$.

(ii) $G_2 = C_9(1, 2, 4)$ *be a circulant graph with the parameters* $b = 3$, $g = 3$, $h = 1$, $\lambda_1 = 1$ *and* $\lambda_2 = 0$.

Proof. Let G be a circulant graph with $p = 9; s = 3$.

(i) If $G_1 \cong C_9(1, 3)$ is a circulant graph, then $\beta(C_9(1, 3)) = 3$. Therefore, the collection of all β-sets are $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$, $\{v_3, v_5, v_7\}$, $\{v_4, v_6, v_8\}$, $\{v_5, v_7, v_9\}$, $\{v_6, v_8, v_1\}$, $\{v_7, v_9, v_2\}$, $\{v_8, v_1, v_3\}$, $\{v_9, v_2, v_4\}$. These β-sets are given by blocks $b = 9$, the size of the block $g = 3$, the number of repetition of the elements of the blocks $h = 3$ and the associates are $\lambda_1 = 1$ and $\lambda_2 = 0$.

(ii) If $G_2 \cong C_9(1, 2, 4)$ is a circulant graph, then, $\beta(C_9(1, 2, 4)) = 3$. Therefore, the collection of all β-sets are $\{v_1, v_4, v_7\}$, $\{v_2, v_5, v_8\}$, $\{v_3, v_6, v_9\}$. These β-sets are given by blocks $b = 3$, the size of the blocks $g = 3$, the number of repetition of the elements of the blocks $h = 1$ and the associates are $\lambda_1 = 1$ and $\lambda_2 = 0$.

Conversely, suppose $G \cong C_9$, then $\beta(C_9) = 3$. Therefore, the collection of all β-sets are $\{v_1, v_4, v_7\}$, $\{v_2, v_5, v_8\}$, $\{v_3, v_6, v_9\}$. These β-sets are given by blocks $b = 3$, the size of the blocks $g = 3$, the number of repetition of the elements of the blocks $h = 1$. By the definition of LS-Design with $2 \le g \le s$, which is a contradiction. Hence, the collection of β-sets does not forms an LS-Design. □

To prove our next result we use the following definition.
The square lattice graphs $L_2(n)$ are strongly regular graphs with the parameters $(n^2, 2(n-1), n-2, 2)$ which are unique for all $n \neq 4$. When $n = 4$ we obtain two non-isomorphic strongly regular graphs with parameters $(16, 6, 2, 2)$. The graph with these parameters is called as Shrikhande graph. For more details, see ([2], [17], [20] and [21]).

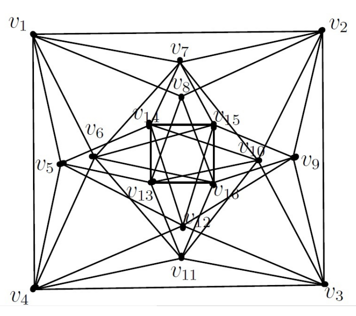

FIGURE 1. Shrikhande graph

Theorem 3.3. *Let G_3 be a connected graph with order 16. Then the collection of all β-sets forms an LS-Design if and only if G_3 is a Shrikhande graph with the parameters $b = 12$, $g = 4$, $h = 3$, $\lambda_1 = 0$ and $\lambda_2 = 3$.*

Proof. Let G_3 be a Shrikhande graph with $p = 16$ vertices. Then, $\beta(G_3) = g = 4$. Therefore, there are $\binom{16}{4}$ possibilities of choosing β-sets. The total number of β-sets as blocks $b = 12$, the size of the blocks $g = 4$, the number of repetition of the elements of the blocks $h = 3$ and the first and second associates are $\lambda_1 = 0$ and $\lambda_2 = 3$.
Conversely, suppose $G_3 \cong K_{16}$, then $\beta(K_{16}) = 1$. Therefore, the total number of blocks $b = 16$, the size of the blocks $g = 1$, the number of repetition of the elements of the blocks $h = 1$. By the definition of LS-Design with $2 \leq g \leq s$, which is a contradiction. Hence, the collection of β-sets does not forms an LS-Design. □

To prove our next results we use the following definition.
The Cayley graph associated with the group (Z_n, N) and its symmetric subset N of non zero nilpotent elements in the ring $(Z_n, \oplus, \odot)$ is the graph, whose vertex set V is $Z_n = \{\overline{0}, \overline{1}, \overline{2}, \cdots \overline{n-1}\}$ and the edge set $E = \{(x, y) : x, y \in Z_n \, and \, either \, x - y \in N \, or \, y - x \in N\}$. This graph is called as Nilpotent Cayley graph of the ring $(Z_n, \oplus, \odot)$ and it is represented as $G(Z_n, N)$. For more details, see ([1] and [16]).

Observation 3.1.

(i) *If $n = \prod_{i=0}^{t} x_i^{a_i}$, where $x_1 < x_2 < \ldots < x_t$ are primes and $a_i \geq 1$, $1 \leq i \leq t$ are integers, then the Nilpotent Cayley graph $G(Z_n, N)$ can be decomposed into m disjoint complements, each of which is a compete graph, where $m = x_1 < x_2 < \ldots < x_t$.*

(ii) *If $n = 4$, then the girth and the circumference of the graph $G(Z_n, N)$ are undefined.*

(iii) *If $n = x_1 x_2 \ldots x_t$, where $x_1 < x_2 < \ldots < x_t$ are primes, then the graph $G(Z_n, N)$ has only vertices and no edges.*

(iv) *The graph $G(Z_n, N)$ is $(\prod_{i=0}^{t} x_i^{a_i - 1} - 1)$-regular and the number of edges in $G(Z_n, N)$ is given by $\frac{n}{2}(\prod_{i=0}^{t} x_i^{a_i - 1} - 1)$.*

(v) *If $n = 2^2 x_1 x_2 \ldots x_t$, where $2 < x_1 < x_2 < \ldots < x_t$ are primes, then the graph $G(Z_n, N)$ is a bipartite graph and thus has no triangles.*

From the above observations we can infer that each component of the Nilpotent Cayley graph $G(Z_n, N)$ is either isomorphic to K_p or C_p or K_{p_1, p_2} or Circulant graph $G(n, S)$ or some other families of graphs.

Theorem 3.4. *Let G be a connected graph with p vertices. Then the collection of all β-sets forms an LS-Design with parameters of the following graphs:*

(i) $b = np$, $g = 2$, $h = s$, $\lambda_1 = \begin{cases} s-2, & for \, p \text{ is even}, \\ s-1, & \text{otherwise}. \end{cases}$

and $\lambda_2 = \begin{cases} s-2, & for \, p \text{ is odd} \\ s-3, & \text{otherwise}. \end{cases}$ *; if* $G \cong C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)$; $p = s^2 \geq 4$.

(ii) $b = m$; $m \geq 3$, $g = p_1$, $h = 1$, $\lambda_1 = 1$ *and* $\lambda_2 = 0$; *if* $G \cong K_{p_1, p_2, \cdots, p_m}$; $p_1 = p_2 = \ldots = p_m$ *and* $m > 2$.

(iii) $b = s^t$; $t \geq 2$, $g = s$, $h = s^{t-1}$, $\lambda_1 = 1$ *and* $\lambda_2 = s^{t-1} - 1$; *if* $G \cong G(Z_n, N)$.

Proof.

(i) Let G be a circulant graph $C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)$ with $p = s^2; p = 4n$ or $4n + 1$ or $4n + 2$; $n \geq 1$ and Y is the β-set of G. Then $\beta\left(C_p\left(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor\right)\right) = 2$. Therefore, there are $\binom{p}{2}$ possibilities of choosing β-sets, then we obtain $\binom{p}{2} \times \binom{p}{2} \times \cdots \times \binom{p}{2}$, n times of β-sets. Thus, the total number of β-sets as blocks $b = np$, the size of the blocks $g = 2$, the number of repetition of the elements of the blocks $h = s$ the first and second associates are $\lambda_1 = \begin{cases} s-2, & for\, p\ is\ even \\ s-1, & otherwise \end{cases}$
and $\lambda_2 = \begin{cases} s-2, & for\, p\ is\ odd \\ s-3, & otherwise. \end{cases}$

(ii) Let $K_{p_1, p_2, \cdots, p_m}$; $p_1 = p_2 = \cdots = p_m$ be the complete regular multi-partite graph with $m \geq 3$. Then $\beta(K_{p_1, p_2, \cdots, p_m}) = p_1$. Therefore, the collection of total number of β-sets is considered as blocks. Thus, the total number of β-sets as blocks $b = m$, the size of the blocks $g = p_1$, the number of repetition of the elements of the blocks $h = 1$ the first and second associates are $\lambda_1 = 1$ and $\lambda_2 = 0$.

(iii) Let $G(Z_n, N)$ be a Nilpotent Cayley graph with vertex set $Z_n = \{\overline{0}, \overline{1}, \overline{2}, \cdots, \overline{n-1}\}$ such that $Z_n = s^2$, $2 \leq s$ has m; $m \geq 1$ number of disjoint complete sub graphs with Z_{n_i} where $i \geq 1$. Then, the collection of all β-sets of $G(Z_n, N)$ are obtained from $\binom{p}{s}$ possibilities of β-sets, given by $\binom{n_i}{1} \times \binom{n_i}{1} \times \cdots \times \binom{n_i}{1}$, m times of β-sets. Therefore, the total number of β-sets as blocks $b = s^t$ where $t \geq 2$, the size of the blocks $g = s$, the number of repetition of the elements of the blocks $h = s^{t-1}$ the first and second associates are $\lambda_1 = 1$ and $\lambda_2 = s^{t-1} - 1$. □

4. Dominating Sets (DS)

A set $D \subseteq V$ is said to be a dominating set of a graph G, if every vertex in $V - D$ is adjacent to some vertex in D, and is the minimum cardinality of a dominating set D is the domination number $\gamma(G)$. The minimum cardinality of a dominating set D with $|D| = \gamma(G)$ is called a γ-set of G. The comprehensive details of domination-related parameters and their applications are discussed in ([13] and [14]).

Theorem 4.1. *Let G be a connected graph of order 4. Then the collection of all γ-sets of G forms the LS-Design with parameters $b = 6$, $g = 2$, $h = 3$, $\lambda_1 = 2$ and $\lambda_2 = 1$, if and only if $G \cong C_4 \cong Q_2$.*

Proof. Let $G \cong Q_2$ with $\gamma(Q_2) = 2$. Then, the collection of all γ-sets are given by $\{v_1, v_2\}$,$\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_1\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$. Therefore, the total number of γ-sets considered as blocks $b = 6$, the size of the blocks $g = 2$, the number of repetition of the elements of the blocks $h = 3$ the first and second associates are $\lambda_1 = 2$ and $\lambda_2 = 1$.
Conversely, suppose $G \cong K_4$, then $\gamma(K_4) = 1$. Therefore, the collection of all γ-sets are $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}$. These γ-sets are given by blocks $b = 4$, $g = 1$, $h = 1$. By the definition of LS-Design with $2 \leq g \leq s$, which is a contradiction. Hence, the collection of γ-sets does not form an LS-Design. □

To prove the next results we use the following definitions.
The Cartesian product of two graphs G and H is denoted by $G \times H$. The graph $V(G \times H) = V(G) \times V(H)$, as vertex set and $\{(v_i, v_j) : v_i \in G, v_j \in H\}$ as the edge

set. The Cartesian product of two cycles $C_{p_1} \times C_{p_2}$ is a Torus graph with $p_1, p_2 \geq 3$ vertices. Torus graph $C_3 \times C_3$ is also known as generalized quadrangle $GQ(2,1)$ or $(2,3)$-Hamming graph or $(3,3)$-Rook graph or 9-Paley graph or conference graph. For more details, we refer to ([10] and [15]).

Theorem 4.2. *Let G be a connected r-regular graph of order 9. If $r = 2, 4, 6$, then the collection of all γ-sets forms the LS-Design with parameters shown in Table 1.*

Graphs of order 9	Parameters of Latin square design				
	b	g	h	λ_1	λ_2
G_4	3	3	1	1	0
$G_4(a)$	9	2	2	1	1
$G_4(b)$	45	3	15	1	14
G_5	18	2	4	2	2

TABLE 1. Parameters of LS-Design of graphs with 9.

Proof. Let D be a γ-set of a connected r-regular graph G of order 9. Then, $\gamma(G_4) = \gamma(G_4(a)) = \gamma(G_4(b)) = 3$, we have

Case 1. If $G_4 \cong C_9$ is a cycle of order 9, then $\gamma(C_9) = 3$. Therefore, the collection of all γ-sets are $\{v_1, v_4, v_7\}$, $\{v_2, v_5, v_8\}$, $\{v_3, v_6, v_9\}$. These γ-sets are given by blocks $b = 3$, the size of the blocks $g = 3$, the number of repetition of the elements of the blocks $h = 1$, the first and second associates are $\lambda_1 = 1$ and $\lambda_2 = 0$.

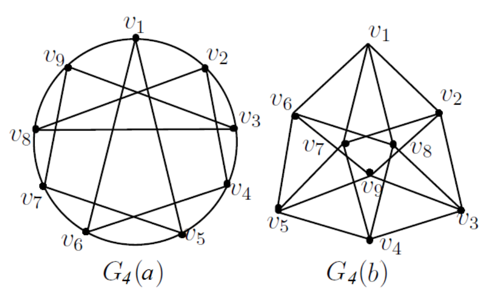

FIGURE 2. 4-regular graphs

Case 2. If $G \cong G_4(a)$ is a connected r-regular graph of order 9, then the collection of all γ-sets are given by blocks $b = 3$ and the size of the block $g = 3$. Therefore, the number of repetition of the elements of the blocks $h = 1$, the first and second associates are $\lambda_1 = 1$ and $\lambda_2 = 0$.

Case 3. If $G \cong G_4(b)$ is a Torus graph $C_3 \times C_3$, with $s = 3$, then the collection of all γ-sets are given by blocks $b = 45$ and the size of the blocks $g = 3$. Therefore, the number of repetition of the elements of the blocks $h = 15$ and the associates are $\lambda_1 = 1$ and $\lambda_2 = 14$. **Case 4.** If $G \cong G_5 \cong C_9(1, 2, 4)$ is a graph of order 9, which implies $s = 3$, then vertex v_1 dominates $\{v_2, v_3, v_5, v_6, v_8, v_9\}$ and the remaining vertices v_4, v_7 are dominated by another vertex as there is no adjacency between them. Thus, $\gamma(G_5) = 2$. Therefore, there are $\binom{9}{2}$ possibilities of choosing γ-sets and these γ-sets are given by blocks $b = 18$, the size of the blocks $g = 2$, the number of

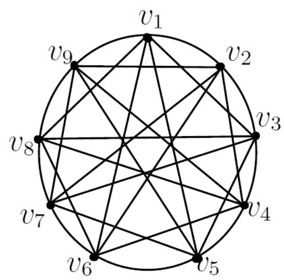

FIGURE 3. 6-regular graph

repetition of the elements of the blocks $h = 4$, the first and second associates are $\lambda_1 = 2$ and $\lambda_2 = 2$. □

Association scheme of graphs of order 16

To prove our next results we use the definition.

Clebsch graph is a strongly regular Quintic (5-regular) graph with 16 vertices and 40 edges having the parameters $(16, 5, 0, 2)$. For more details, see ([2] and [20]).

Theorem 4.3. *Let G be a graph with order 16. Then the collection of all γ-sets forms an LS-Design for the following graphs*

(i) $b = 16$, $g = 4$, $h = 4$, $\lambda_1 = 0$ *and* $\lambda_2 = 2$; *if* $G \cong Q_4$.
(ii) $b = 64$, $g = 4$, $h = 16$, $\lambda_1 = 0$ *and* $\lambda_2 = 16$; *if G is a Clebsch graph.*
(iii) $b = 32$, $g = 3$, $h = 6$, $\lambda_1 = 0$ *and* $\lambda_2 = 6$; *if G is a Shrikhande graph.*
(iv) $b = 8$, $g = 4$, $h = 2$, $\lambda_1 = 0$ *and* $\lambda_2 = 2$; *if* $G \cong C_4 \times C_4$.

Proof.

(i) Let $G \cong Q_4$ be a hypercube with $p = s^2 = 16$ vertices, and $\gamma(Q_4) = 4$. Then there are $\binom{16}{4}$ possibilities of choosing γ-sets. Therefore, these γ-sets are given by blocks $b = 16$, the size of the blocks $g = 4$, the number of repetition of the elements of the blocks $h = 4$, the first and second associates are $\lambda_1 = 0$ and $\lambda_2 = 2$.

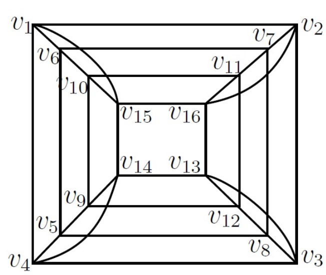

FIGURE 4. Hypercube Q_4

(ii) Let G be the Clebsch graph with $p = s^2 = 16$ vertices, and $\gamma(G) = 4$. Then there are $\binom{16}{4}$ possibilities of choosing γ-sets. These γ-sets are given by blocks $b = 64$, the size of the blocks $g = 4$, the number of repetition of the elements of the blocks $h = 16$, the first and second associates are $\lambda_1 = 0$ and $\lambda_2 = 16$.

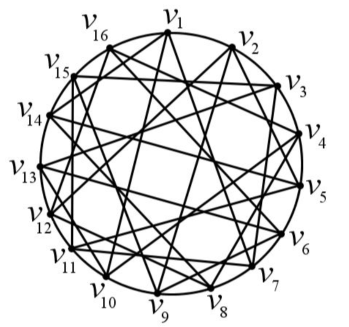

FIGURE 5. Clebsch graph

(iii) Let G be a Shrikhande graph with $p = s^2 = 16$ vertices, and $\gamma(G) = 3$. Then there are $\binom{16}{3}$ possibilities of choosing γ-sets. By considering the total number of γ-sets as blocks, we have 32 blocks. Therefore, these γ-sets are given by blocks $b = 32$, the size of the blocks $g = 3$, the number of repetition of the elements of the blocks $h = 6$, the first and second associates are $\lambda_1 = 0$ and $\lambda_2 = 6$.

(iv) Let G be a Torus graph with $p = s^2 = 16$ vertices and $\gamma(C_4 \times C_4) = 4$. Then $\binom{16}{4}$ possibilities of choosing γ-sets. By considering the total number of γ-sets as blocks, we obtain 8 blocks. Therefore, these γ-sets are given by blocks $b = 8$, the size of the blocks $g = 4$, the number of repetition of the elements of the blocks $h = 2$, the first and second associates are $\lambda_1 = 0$ and $\lambda_2 = 2$. □

Theorem 4.4. *Let G be a Torus graph with order 25. Then, the collection of all γ-sets of forms an LS-Design with the parameters $b = 10$, $g = 5$, $h = 2$, $\lambda_1 = 0$ and $\lambda_2 = 2$.*

Proof. Let $G \cong C_5 \times C_5$ be a Torus graph with $p = s^2 = 5^2 = 25$ and $\gamma(C_5 \times C_5) = 5$. Then by considering the total number of γ-sets as blocks, these γ-sets are given by blocks $b = 10$, the size of the blocks $g = 5$, the number of repetition of the elements of the blocks $h = 2$, the first and second associates are $\lambda_1 = 0$ and $\lambda_2 = 2$. □

Observation 4.1. *Let $G = C_{p_1} \times C_{p_2}$ with $p_1 = p_2 \geq 6$. Then the collection of all γ-sets does not form an LS-Design.*

Here, we generalize the parameters of LS-Design for certain classes of graphs.

Theorem 4.5. *Let G be a graph. Then the collection of all γ-sets forms the LS-Design for the following graphs:*

(i) $b = s^2$, $g = 2$, $h = s$, $\lambda_1 = \frac{s}{2}$ *and* $\lambda_2 = \frac{s}{2}(s-1)$*; if* $G \cong C_p(1, 3, \ldots, \lfloor \frac{p}{2} \rfloor)$.

(ii) $b = m$ *with* $m > 2$, $g = p_1$, $h = 1$, $\lambda_1 = 1$ *and* $\lambda_2 = 0$*; if* $G \cong K_{p_1, p_2, \cdots, p_m}$*;* $p_1 = p_2 = \cdots = p_m$ *and* $m > 2$.

(iii) $b = s^t$ *with* $t \geq 2$, $g = s$, $h = s^{t-1}$, $\lambda_1 = 1$ *and* $\lambda_2 = s^{t-1} - 1$*; if* $G \cong G(Z_n, N)$.

Proof.

(i) Let $C_p(1,3,\ldots,\lfloor\frac{p}{2}\rfloor)$ be a circulant graph with odd jump size. Then, $\gamma(C_p(1,3,\ldots,\lfloor\frac{p}{2}\rfloor)) = 2$. Therefore, the collection of total number of γ-sets is considered as blocks $b = s^2$, the size of the blocks $g = 2$, the number of repetition of the elements of the blocks $h = s$, the first and second associates are $\lambda_1 = \frac{s}{2}$ and $\lambda_2 = \frac{s}{2}(s-1)$.

(ii) Let $K_{p_1,p_2,\cdots,p_m}$; $p_1 = p_2 = \cdots = p_m$ be a complete regular multipartite graph with $m > 2$. Then, $\gamma(K_{p_1,p_2,\cdots,p_m}) = p_1$. Therefore, the collection of total number of γ-sets is considered as blocks $b = m$; $m > 2$, the size of the blocks $g = p_1$, the number of repetition of the elements of the blocks $h = 1$, the first and second associates are $\lambda_1 = 1$ and $\lambda_2 = 0$.

(iii) Let $G(Z_n, N)$ be a Nilpotent Cayley graph with vertex set $Z_n = \{\overline{0},\overline{1},\overline{2},\cdots,\overline{n-1}\}$ such that $Z_n = s^2$, $s \geq 2$ has a m number of disjoint complete sub graphs with Z_{n_i} with $i \geq 1$. Then $\gamma(G(Z_n, N)) = s$. Therefore, there are $\binom{p}{s}$ possibilities of choosing γ-sets, and $\binom{n_i}{1} \times \binom{n_i}{1} \times \cdots \times \binom{n_i}{1}$, m times of γ-sets. Thus, the collection of total number of γ-sets is considered as blocks $b = s^t$; $t \geq 2$, the size of the blocks $g = s$, the number of repetition of the elements of the blocks $h = s^{t-1}$, the first and second associates are $\lambda_1 = 1$ and $\lambda_2 = s^{t-1} - 1$. □

Non-existence of LS-Design

The non-existence of LS-Design from the α-sets, β-sets and γ-sets of a graph is given by the condition $2 \leq g \leq s$, where g is the cardinality of the block. For more details, we refer to ([4], [5] and [7]).

Theorem 4.6. *The collection of α-sets, β-sets, and γ-sets of an irregular graph with $p \geq 3$ vertices do not form an LS-Design.*

Proof. Let G be an irregular graph with $p \geq 3$. Then, it is clear that $deg(v_i) \neq deg(v_j)$; $0 \leq i,j \leq p-1$. Therefore, the elements of the α-sets, β-sets and γ-sets of the graph have unequal replications. Hence, an irregular graph does not form an LS-Design. □

Observation 4.2. *Some of the regular graphs do not form the LS-Design from the α-sets, β-sets and γ-sets of graphs. For example, the Complete graph K_p with $p \geq 2$, we have $\alpha(K_p) = p-1$ and $\beta(K_p) = \gamma(K_p) = 1$.*

5. Matrix representation and their polynomials of CS, IS and DS

A graph polynomial $\mathscr{P}$ is a graph invariant whose values are polynomials, which is characterized by matrices P^1 and P^2 obtained from the parameters of the second kind.

The graph with 2×2 association schemes

By Theorem 2.1, Theorem 3.1 and Theorem 4.1, we have the association scheme of the LS-Design arsing from α-set, β-set and γ-set of a hypercube Q_2 are given by an array of 2×2. Therefore, two vertices lying in the same row or column of the association scheme are the first associates, while two vertices not lying in the same row or column are the second associates. Hence, the matrices P^1 and P^2 give the correct values of second kind parameters of graphs with order 4. Then,

$$P^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \textit{and} \ P^2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, polynomials of the matrices P^1 and P^2 are $\mathscr{P}(P^1,x) = x^2 - 1$ with roots $x = \pm 1$ and $\mathscr{P}(P^2,x) = x^2 - 2x$ with roots $x = 2$ or $x = 0$. Therefore, the polynomials are not unique.

The graph with 3×3 association schemes

By Theorem 3.2 and Theorem 4.2, we have the association scheme of the LS-Design arsing from β-set of G_1 and G_2, and γ-set of G_4, $G_4(a)$, $G_4(b)$ and G_5 are given by an array of 3×3. Then, the matrices P^1 and P^2 give the correct values of second kind parameters of graphs of order 9. Therefore,

$$P^1 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \ and \ P^2 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

Hence, the polynomials of the matrices P^1 and P^2 are $\mathscr{P}(P^1,x) = \mathscr{P}(P^2,x) = x^2 - 3x - 2$ with roots $x = \frac{1}{2}\left(3 \pm \sqrt{17}\right)$. Therefore the roots are unique.

The graph with 4×4 association schemes

By Theorem 3.3 and Theorem 4.3, we have the association scheme of the LS-Design arsing from β-set of a Shrikhnde graph $L_2(4)$ and γ-set of a Clebsch graph G and a Hypercube Q_4, a Shrikhnde graph $L_2(4)$ and a product graph $C_4 \times C_4$ are given by an array of 4×4. Then the matrices P^1 and P^2 give the correct values for the parameters of the second kind. Then,

$$P^1 = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} \text{ and } P^2 = \begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix}.$$

Hence, the polynomials of the matrices P^1 and P^2 are $\mathscr{P}(P^1,x) = x^2 - 8x + 3$ with roots $x = 4 \pm \sqrt{13}$ and $\mathscr{P}(P^2,x) = x^2 - 6x - 8$ with roots $x = 3 \pm \sqrt{17}$. Therefore, the polynomials are not unique.

The graph with 5×5 association schemes

By Theorem 4.4, we have the association scheme of the LS-Design arsing from γ-set of a Torus graph $C_5 \times C_5$ is given by an array of 5×5. Hence, the matrices P^1 and P^2 give the correct values of second kind parameters of graphs with order 25. Then,

$$P^1 = \begin{pmatrix} 3 & 4 \\ 4 & 12 \end{pmatrix} \text{ and } P^2 = \begin{pmatrix} 2 & 6 \\ 6 & 9 \end{pmatrix}.$$

Hence, the polynomials of the matrices P^1 and P^2 are $\mathscr{P}(P^1,x) = x^2 - 15x + 20$ with roots $x = \frac{1}{2}\left(15 \pm \sqrt{145}\right)$ and $\mathscr{P}(P^2,x) = x^2 - 11x - 8$ with roots $x = \frac{1}{2}\left(11 \pm \sqrt{89}\right)$. Therefore the polynomials are not unique.

The graph with $s \times s$ association schemes

By Theorem 3.4 and Theorem 4.5, we have the association scheme of the LS-Design arsing from β-set of a circulant graph $C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)$ with $p = s^2; p = 4n$ or $4n + 1$ or $4n + 2$; $n \geq 1$, a complete regular multipartite graph $K_{p_1,p_2,\cdots,p_m}$; $p_1 = p_2 = \cdots = p_m$; $m > 2$ and a Nilpotent Cayley graph $G(Z_n, N)$, and γ-set of a circulant graph with odd jump size, a complete regular multipartite graph $K_{p_1,p_2,\cdots,p_m}$; $p_1 = p_2 = \cdots = p_m$ with $m > 2$ and a Nilpotent Cayley graph $G(Z_n, N)$ are given by an array of $s \times s$. Therefore, two vertices lying in the same row or the same column of the association scheme are the first associate, while two vertices not lying in the same row or the same column are second associates.

$$\begin{vmatrix} v_1 & v_2 & v_3 & \cdots & v_s \\ v_{(s+1)} & v_{(s+2)} & v_{(s+3)} & \cdots & v_{2s} \\ v_{(2s+1)} & v_{(2s+2)} & v_{(2s+3)} & \cdots & v_{3s} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ v_{[(s-2)s+1]} & v_{[(s-2)s+2]} & v_{[(s-2)s+3]} & \cdots & v_{(s-1)s} \\ v_{[(s-1)s+1]} & v_{[(s-1)s+2]} & v_{[(s-1)s+3]} & \cdots & v_{s^2} \end{vmatrix}$$

Association scheme for the graphs with an array of $s \times s$.

Therefore, the matrices P^1 and P^2 give the correct values for the parameters of the second kind. Then,

$$P^1 = \begin{pmatrix} s-2 & s-1 \\ s-1 & (s-1)(s-2) \end{pmatrix} \text{ and}$$
$$P^2 = \begin{pmatrix} 2 & 2(s-2) \\ 2(s-2) & (s-2)^2 \end{pmatrix}$$

where $2 \leq g \leq s$. Hence, the polynomial is characterized by matrices P^1 and P^2 obtained from the parameters of the second kind. Therefore, the polynomials of the matrices P^1 and P^2 are
$\mathscr{P}(P^1, x) = s^3 - s^2x - 6s^2 + 2sx + 10s + x^2 - 5$ with roots
$x = \frac{1}{2}\left(s^2 \pm \sqrt{s^4 - 8s^3 + 28s^2 - 40s + 20} - 2s\right)$ and
$\mathscr{P}(P^2, x) = s^2(-x) - 2s^2 + 4sx + 8s + x^2 - 6x - 8$ with roots
$x = \frac{1}{2}\left(s^2 \pm \sqrt{s^4 - 8s^3 + 36s^2 - 80s + 68} - 4s + 6\right)$.

6. Conclusion and open problems

In this article, we determine the existence of the Latin square designs with two association schemes and also, discussed the uniqueness of the polynomials obtained from the matrices of the associates emerging from the total number of minimum covering, maximum independent, and minimum dominating sets of certain classes of regular graphs such as Hypercube, Paley graph, Torus graph, Clebsch graph, Shrikhande graph. Further, we generalized the parameters of LS-designs of circulant graphs Complete multipartite graphs, and some Nilpotent Cayley graphs. Finally, we discussed the non-existence of LS-designs and we pose some open problems.

Open Problem 1. Find the parameters of Latin Square designs emerging from minimum dominating sets of hypercube Q_n, $n \geq 5$.

Open Problem 2. Find the parameters of the mutually orthogonal Latin Square designs from the minimum dominating sets of some regular graphs.

Open Problem 3. Compare the parameters of the Latin square designs and the mutually orthogonal Latin Square designs emerging from minimum dominating sets of certain regular graphs.

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B. Chaluvaraju
Department of Mathematics, Bangalore University,, Janabharathi Campus, Bengaluru-560 056,, Karnataka, INDIA
Email address: bchaluvaraju@gmail.com

Shaikh Ameer Basha
Department of Mathematics, Bangalore University,, Janabharathi Campus, Bengaluru-560 056,, Karnataka, INDIA
Email address: shaikhameerbasha@gmail.com