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Exact and More Accurate Solution for Boundary Value Problems using Closed-Form and Numerical Methods

M. G. Ibrahim[a], Hussein A. Soliman[b], Lamyaa E. Elmasry[c]

(a),(b),(c) Basic and Applied Science Department, International Academy of Engineering and Media Science, Cairo, Egypt

*Corresponding author: Email address: dr_mohamedgamal@iams.edu.eg

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ABSTRACT

It is a new algorithm to find the exact solution of boundary value problems using the closed-form method using Laplace transforms method and a more accurate numerical solution of boundary value problems using fourth-order finite difference method (FOFDM). It is known that the Laplace transforms method gives a closed-form for initial value problems but in the present study, we were able to use it to find the exact solution of boundary value problems which is the novelty of the present study. Also, high-accurate numerical methods to solve linear boundary value problems which approximate to exact solutions. The uniqueness, convergence, and stability of the new technique (FOFDM) are verified and tested by comparisons with a closed-form method.

Keywords: BVP, Laplace Transform, Finite Difference Method

1 INTRODUCTION

Two-point boundary value problems have received considerable attention due to their importance in many areas of sciences and engineering. These types of differential equations arise very frequently in fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process, and geophysics [30].

The most popular technique for solving differential equations numerically is the Runge-Kutta method. Three new Runge–Kutta methods are presented for numerical integration of systems of linear inhomogeneous ordinary differential equations (ODEs) with constant coefficients. Such ODEs arise in the numerical solution of partial differential equations governing linear wave phenomena [31]. A novel second-order prediction differential model is designed, and numerical solutions of this novel model are presented using the integrated strength of the Adams and explicit Runge–Kutta schemes [32]. A new special two-derivative Runge-Kutta type (STDRKT) methods involving the fourth derivative of the solution for solving third-order ordinary differential equations [33].

Finite difference method proposed for the solution of two-point boundary value problems has been widely applied [24-26]. They used the finite difference method (FDM) of second-order accuracy to solve the nonlinear system of differential equations. they observed that the velocity reached the steady-state faster than temperature and nanoparticles concentration. Attia et al. [34] studied the effects of the-Drcian Forchheimer and Hall current resistances on the unsteady flow and heat transfer between two porous plates. they solved the governing partial differential equations, numerically, by the finite difference method FDM. Joule and viscous dissipations are considered in the energy equation. Ewis [35] used a second-order accurate finite difference method to solve the governing equations of natural convection of non- Newtonian (RivlinEricksen) fluid flow and heat transfer under the influences of non-Darcy resistance force, constant pressure gradient, dissipation, and radiation.

Various analytical and numerical techniques proposed for the solution of differential equations are available in the literature; some of these are Differential Transform Method [1-6], Rung-Kutta 4th Order Method [7], Bernoulli Polynomials [8], Cubic Spline Method [9], Sinc Collocation Method [10], Modified Picard Technique [11], Block Method [12-14], Adomian Decomposition Method [15-20], Homotopy Perturbation Method [21-23].

This work presented a new technique to solve boundary value problems using Laplace Transform and (FOFDM). The new technique presents an exact solution for boundary value problems. The other technique approximating to exact solution is (FOFDM), which has a very strong accuracy that approaches the exact solution. The novelty of the present paper is the new technique using Laplace

Transform and a more accurate solution using (FOFDM). The uniqueness, convergence, and stability of the new technique are verified and tested by comparisons with the exact solution. The provided comparisons highlight the effectiveness of the new approach using Laplace Transform and (FOFDM), which is convergent, stable, and highly accurate.

2 MODELLING

Problem 1: Consider the following problem of a two-point boundary value

$\varphi''(\eta) + 3\varphi'(\eta) + 2\varphi(\eta) = 1,$

The appropriate boundary conditions are described as follows:

$\varphi(0) = 1$ and $\varphi(1) = 0$.

Exact Solution using Laplace Transform

Take the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms of derivatives:

$$\mathcal{L}\{\varphi''(\eta)\} + 3\mathcal{L}\{\varphi'(\eta)\} + 2\mathcal{L}\{\varphi(\eta)\} = \mathcal{L}\{1\}$$

$$s^2\mathcal{L}\{\varphi\} - s\,\varphi(0) - \varphi'(0) + 3[s\,\mathcal{L}\{\varphi\} - \varphi(0)] + 2\mathcal{L}\{\varphi\} = \frac{1}{s} \quad (1')$$

Let $\varphi'(0) = \alpha$ and from the given boundary conditions $\varphi(0) = 0$, then by substitution in Eq. $(1')$ and rearranging gives:

$$\mathcal{L}\{\varphi\} = \frac{1+3s+\alpha\, s+s^2}{s(s^2+3s+2)} \quad (2')$$

$$\varphi = \mathcal{L}^{-1}\left(\frac{1+3s+\alpha\, s+s^2}{s(s^2+3s+2)}\right) \quad (3')$$

When we convert $\frac{1+3s+\alpha\, s+s^2}{s(s^2+3s+2)}$ to a partial fraction, we obtain,

$$\frac{1+3s+\alpha\, s+s^2}{s(s^2+3s+2)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} \quad (4')$$

By resolving Eq. $(4')$, we get

$$A = \frac{1}{2},\ B = \frac{-(2\,\alpha+1)}{2}, \text{ and } C = \alpha + 1. \quad (5')$$

When we substitution by $(4')$ and $(5')$ in Eq. $(3')$ gives,

$$\varphi = \mathcal{L}^{-1}\left(\frac{1}{2\,s} - \frac{2\,\alpha+1}{2(s+2)} + \frac{\alpha+1}{(s+1)}\right) \quad (6')$$

The inverse Laplace equation gets,

$$\varphi = \frac{1}{2} - \frac{2\,\alpha+1}{2}e^{-2\eta} + (\alpha+1)e^{-\eta} \quad (7')$$

And by using $\varphi(1) = 0$ in Eq. $(7')$, we get

$\alpha = \frac{(1-2e-e^2)}{2(-1+e)}$, then by substitution in Eq. $(7')$,

$$\varphi(\eta) = \frac{1}{2}e^{-2\eta}\left(-1 + e^{2\eta} - \frac{(1-2e-e^2)}{(-1+e)} + 2e^{\eta}\left(1 + \frac{(1-2e-e^2)}{2(-1+e)}\right)\right) \quad (8')$$

Then, Eq. ($8'$) in the simplest form:

$$\varphi(\eta) = \frac{e^{-2\eta}(e+e^2-e^{\eta}-e^{2\eta}-e^{2+\eta}+e^{1+2\eta})}{2(-1+e)} \quad (9')$$

3 MODEL CONTROL

The second and fourth-order finite difference method

The finite domain $(0 < \varphi < 1)$ of the solution is divided into $(n-1)$ subintervals such that the mesh size is$\left(h = \frac{1}{n-1}\right)$ with counter $i = 1,2,3,\ldots\ldots,n$. The linearized linear ordinary differential equation of (Problem 1) is transformed into a system of algebraic equations using the fourth-order difference schemes. The following (second and fourth-order) schemes are obtained by Taylor's expansions about $\varphi_i = (i-1)h$. At $n = 1001 \rightarrow h = \frac{1}{1000} = 0.001$, second and fourth-order difference schemes ($10'$-$21'$) should be applied to equation (Problem 1) to minimize round-off errors in computations [27-29].

Forward finite-difference formulas for first and second:

- ✓ $\varphi' = \frac{1}{2h}(-3\varphi_i + 4\varphi_{i+1} - \varphi_{i+2}) + O(h^2)$ ($10'$)
- ✓ $\varphi'' = \frac{1}{h^2}(2\varphi_i - 5\,\varphi_{i+1} + 4\varphi_{i+2} - \varphi_{i+3}) + O(h^2)$ ($11'$)
- ✓ $\varphi' = \frac{1}{12h}(-25\varphi_i + 48\varphi_{i+1} - 36\varphi_{i+2} + 16\varphi_{i+3} - 3\varphi_{i+4}) + O(h^4)$ ($12'$)
- ✓ $\varphi'' = \frac{1}{12h^2}(45\varphi_i - 154\,\varphi_{i+1} + 214\varphi_{i+2} - 156\varphi_{i+3} + 61\varphi_{i+4} - 10\varphi_{i+5}) + O(h^4)$ ($13'$)

Backward finite-difference formulas for first and second derivative:

- ✓ $\varphi' = \frac{1}{2h}(3\varphi_i - 4\varphi_{i-1} + \varphi_{i-2}) + O(h^2)$ ($14'$)
- ✓ $\varphi'' = \frac{1}{h^2}(2\varphi_i - 5\,\varphi_{i-1} + 4\varphi_{i-2} - \varphi_{i-3}) + O(h^2)$ ($15'$)
- ✓ $\varphi' = \frac{1}{12h}(25\varphi_i - 48\varphi_{i-1} + 36\varphi_{i-2} - 16\varphi_{i-3} + 3\varphi_{i-4}) + O(h^4)$ ($16'$)
- ✓ $\varphi'' = \frac{1}{12h^2}(45\varphi_i - 154\,\varphi_{i-1} + 214\varphi_{i-2} - 156\varphi_{i-3} + 61\varphi_{i-4} - 10\varphi_{i-5}) + O(h^4)$ ($17'$)

Central finite-difference formulas for first and second derivative:

- ✓ $\varphi' = \frac{1}{2h}(\varphi_{i+1} - \varphi_{i-1}) + O(h^2)$ ($18'$)
- ✓ $\varphi'' = \frac{1}{h^2}(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}) + O(h^2)$ ($19'$)
- ✓ $\varphi' = \frac{1}{12h}(-\varphi_{i+2} + 8\varphi_{i+1} - 8\varphi_{i-1} + \varphi_{i-2}) + O(h^4)$ ($20'$)
- ✓ $\varphi'' = \frac{1}{12h^2}(-\,\varphi_{i+2} + 16\varphi_{i+1} - 30\varphi_i + 16\varphi_{i-1} - \varphi_{i-2}) + O(h^4)$ ($21'$)

Problem 2:

Consider the following problem of a two-point boundary value

$\theta''(\eta) - \theta(\eta) + 4\eta e^{\eta} = 0,$

The appropriate boundary conditions are described as follows:

$\theta(0) = 0$ and $\theta(1) = 1.$

Exact Solution using Laplace Transform

Take the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms of derivatives:

$\mathcal{L}\{\theta''(\eta)\} - \mathcal{L}\{\theta(\eta)\} + 4\mathcal{L}\{\eta e^{\eta}\} = 0$

$$s^2\mathcal{L}\{\theta\} - s\,\theta(0) - \theta'(0) - \mathcal{L}\{\theta\} + \frac{4}{(s-1)^2} = 0 \tag{1}$$

Let $\theta'(0) = \delta$ and from the given boundary conditions $\theta(0) = 0$, then by substitution in Eq. (1) and rearranging gives:

$$\mathcal{L}\{\theta\} = \frac{-4+\delta-2\delta s+\delta s^2}{(s-1)^2(s^2-1)} \tag{2}$$

$$\theta = \mathcal{L}^{-1}\left(\frac{-4+\delta-2\delta s+\delta s^2}{(s-1)^2(s^2-1)}\right) \tag{3}$$

When we convert $\frac{-4+\delta-2\delta s+\delta s^2}{(s-1)^2(s^2-1)}$ to a partial fraction, we obtain,

$$\frac{-4+\delta-2\delta s+\delta s^2}{(s-1)^2(s^2-1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{s+1} \tag{4}$$

By resolving Eq. (4), we get

$$A = \frac{\delta-1}{2},\ B = 1,\ C = -2, \text{ and } D = \frac{1-\delta}{2}. \tag{5}$$

When substitution by (4) and (5) in Eq. (3) gives,

$$\theta = \mathcal{L}^{-1}\left(\frac{\delta-1}{2(s-1)} + \frac{1}{(s-1)^2} + \frac{-2}{(s-1)^3} + \frac{1-\delta}{2(s+1)}\right) \tag{6}$$

The inverse Laplace equation gets,

$$\theta = \frac{\delta-1}{2}e^{\eta} + \eta e^{\eta} - 2\eta^2 e^{\eta} + \frac{1-\delta}{2}e^{-\eta} = \frac{1}{2}e^{-\eta}(1 - \delta + e^{2\eta}(-1 + \delta + 2\eta - 2\eta^2)) \tag{7}$$

And by using $\theta(1) = 1$ in Eq. (7), we get

$\delta = \frac{-1+2e+e^2}{-1+e^2}$, then by substitution in Eq. (7)

$$\theta = \frac{1}{2}e^{-\eta}(1 - \frac{-1+2e+e^2}{-1+e^2} + e^{2\eta}(-1 + \frac{-1+2e+e^2}{-1+e^2} + 2\eta - 2\eta^2)) \tag{8}$$

Then, Eq. (8) in the simplest form

$$\theta = \frac{e^{-\eta}(-e+e^{1+2\eta}+e^{2\eta}(-1+\eta)\eta-e^{2+2\eta}(-1+\eta)\eta)}{-1+e^2} \tag{9}$$

4 NUMERICAL SOLUTION

The second and fourth-order finite difference method

The finite domain $(0 < \theta < 1)$ of the solution is divided into $(n-1)$ subintervals such that the mesh size is$\left(h = \frac{1}{n-1}\right)$ with counter $i = 1, 2, 3, \ldots\ldots, n$. The linearized linear ordinary differential equation of (Problem 1) is transformed into a system of algebraic equations using the fourth-order difference schemes. The following (second and fourth-order) schemes are obtained by Taylor's expansions about $\theta_i = (i-1)h$. At $n = 1001 \rightarrow h = \frac{1}{1000} = 0.001$, second and fourth-order difference schemes (10-21) should be applied to equation (Problem 1) to minimize round-off errors in computations [27-29].

Forward finite-difference formulas for first and second:

- ✓ $\theta' = \frac{1}{2h}(-3\theta_i + 4\theta_{i+1} - \theta_{i+2}) + O(h^2)$ (10)
- ✓ $\theta'' = \frac{1}{h^2}(2\theta_i - 5\,\theta_{i+1} + 4\theta_{i+2} - \theta_{i+3}) + O(h^2)$ (11)
- ✓ $\theta' = \frac{1}{12h}(-25\theta_i + 48\theta_{i+1} - 36\theta_{i+2} + 16\theta_{i+3} - 3\theta_{i+4}) + O(h^4)$ (12)
- ✓ $\theta'' = \frac{1}{12h^2}(45\theta_i - 154\,\theta_{i+1} + 214\theta_{i+2} - 156\theta_{i+3} + 61\theta_{i+4} - 10\theta_{i+5}) + O(h^4)$ (13)

Backward finite-difference formulas for first and second derivative:

- ✓ $\theta' = \frac{1}{2h}(3\theta_i - 4\theta_{i-1} + \theta_{i-2}) + O(h^2)$ (14)
- ✓ $\theta'' = \frac{1}{h^2}(2\theta_i - 5\,\theta_{i-1} + 4\theta_{i-2} - \theta_{i-3}) + O(h^2)$ (15)
- ✓ $\theta' = \frac{1}{12h}(25\theta_i - 48\theta_{i-1} + 36\theta_{i-2} - 16\theta_{i-3} + 3\theta_{i-4}) + O(h^4)$ (16)
- ✓ $\theta'' = \frac{1}{12h^2}(45\theta_i - 154\,\theta_{i-1} + 214\theta_{i-2} - 156\theta_{i-3} + 61\theta_{i-4} - 10\theta_{i-5}) + O(h^4)$ (17)

Central finite-difference formulas for first and second derivative:

- ✓ $\theta' = \frac{1}{2h}(\theta_{i+1} - \theta_{i-1}) + O(h^2)$ (18)
- ✓ $\theta'' = \frac{1}{h^2}(\,\theta_{i+1} - 2\theta_i + \theta_{i-1}) + O(h^2)$ (19)
- ✓ $\theta' = \frac{1}{12h}(-\theta_{i+2} + 8\theta_{i+1} - 8\theta_{i-1} + \theta_{i-2}) + O(h^4)$ (20)
- ✓ $\varphi'' = \frac{1}{12h^2}(-\,\theta_{i+2} + 16\theta_{i+1} - 30\theta_i + 16\theta_{i-1} - \theta_{i-2}) + O(h^4)$ (21)

5 SIMULATION RESULTS

To demonstrate the accuracy of the new technique ((FOFDM), the current results are presented as tables. The convergence and stability of the new technique are tested by comparison with an with a closed-form method. These results in tables (1-2) show that the (FOFDM) is a very effective and powerful method. Tables (1-2) show absolute error 1 (A.E.$_1$) between exact solution (Laplace Transform) and (FOFDM). Also, tables (1-2) show absolute error 2 (A.E.$_2$) between exact solution (Laplace Transform) and (SOFDM). From the results shown in the tables (1-2), it became clear to us the following:

I. These results show the uniqueness, convergence, and stability of the new technique by comparisons with the fourth-order accurate finite-difference solution (FOFDM) and exact solution.
II. The results of a new method (FOFDM) are very close to the results of the exact method more than the results of (SOFDM).
III. Absolute error 1 (A.E.$_1$) is observed as less than absolute error 2 (A.E.$_2$), which confirms what was mentioned in (I).
IV. (FOFDM) is really an excellent agreement with the exact method.

Table 1: Comparison (Exact) with (FOFDM) and (SOFDM) for problem (1).

η	Exact	FOFDM	SOFDM	A.E.$_1$	A.E.$_2$
0	1	1	1	0	0
0.1	0.699168879416016	0.69916887941840	0.62971880235996	$-2.3*10^{-12}$	$6.9*10^{-2}$
0.2	0.4728720308970179	0.47287203090030	0.36179517895014	$-3.2*10^{-12}$	$1.1*10^{-1}$
0.3	0.305695161042993	0.30569516104626	0.17432159427595	$-3.2*10^{-12}$	$1.3*10^{-1}$
0.4	0.1851992578818620	0.18519925788477	0.04968078219687	$-2.9*10^{-12}$	$1.3*10^{-1}$
0.5	0.1013640257081071	0.10136402571037	-0.02626231546449	$-2.2*10^{-12}$	$1.2*10^{-2}$
0.6	0.0461338484947412	0.04613384849639	-0.06481924369983	$-1.6*10^{-12}$	$1.1*10^{-2}$
0.7	0.01304754049420133	0.01304754049530	-0.07501471308152	$-1.0*10^{-12}$	$8.8*10^{-2}$
0.8	-0.0030634452640600	-0.00306344526342	-0.06402362912203	$-6.3*10^{-13}$	$6.0*10^{-2}$
0.9	-0.0063208877003827	-0.00632088770010	-0.03752676076915	$-2.7*10^{-13}$	$3.1*10^{-2}$
1	0	0	0	0	0

Table 2: Comparison (Exact) with (FOFDM) and (SOFD) for problem (2).

η	Exact	FOFDM	SOFDM	A.E.1	A.E. 2
0	0	0	0	0	0
0.1	0.184699086065509	0.18475596764058	0.1850377439891	-5.6*10^{-5}	-3.3*10^{-4}
0.2	0.366744895735082	0.36685111001490	0.3673772638166	-1.0*10^{-4}	-6.3$10^{-4}$
0.3	0.542592187701892	0.54273901143483	0.5434663270244	-1.4*10^{-4}	-8.7*10^{-4}
0.4	0.707554527675984	0.70773171479718	0.7086094341953	-1.7*10^{-4}	-1.0*10^{-4}
0.5	0.855589759660069	0.85578514124969	0.8567529824432	-1.9*10^{-4}	-1.1$10^{-3}$
0.6	0.979048586552162	0.97924761029620	0.9802334859247	-1.9*10^{-4}	-1.1*10^{-3}
0.7	1.068380692250951	1.068565891212193	1.0694832783615	-1.8*10^{-4}	-1.1$10^{-3}$
0.8	1.111792028600031	1.111942408476771	1.1126873126804	-1.5*10^{-4}	-8.9*10^{-4}
0.9	1.094845970850082	1.094936301926755	1.0953837520562	-9.0*10^{-5}	-5.3*10^{-4}
1	1	1	1	0	0

6 CONCLUSION:

This paper presented a new technique to find the exact solution for boundary value problems using Laplace Transform and a more accurate solution using finite difference method. Numerical algorithm methods are very high accuracy with a closed-form method. The novelty of the present method is that it is using Laplace Transform to solve boundary value problems. numerical methods and then uses Laplace transform method to find approximate the exact solution. The uniqueness, convergence, and stability of the new numerical technique are verified and tested by comparisons with a closed-form method and second-order finite difference (SOFDM) solution. The provided comparisons highlight the effectiveness of the new approach, which is convergent, stable, and highly accurate.

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