Journal Of Al-Azhar University Engineering Sector

Vol. 14, No. 50, January, 2019, 109-120

SOLUTION OF GENERAL FORM OF FUZZY SECOND ORDER BOUNDARY VALUE PROBLEM

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ABSTRACT
In this paper; we obtained a new method for solving the homogenous second order linear differential equation with boundary condition in fuzzy environment under strong generalized Hukuhara differentiability and illustrated this by several examples.

ملخص:
ان المعادلات التفاضلية المبهمة أصبحت سريعة الانتشار و التطور و ذلك نتيجة لكثره استخدامها فى التطبيقات الهندسيه.
فى هذه الورقه البحثيه قمنا بتحضير طريقه حل تحليليه لحل المعادلات التفاضلية المبهمه (الفازيه) من الدرجه الثانيه و نقطتين حدوديتين و لكن فى بيئه مبهمه . و طريقه الحل هذه معتمده على طريه هيوكوهارا التقاضليه العامه و قمنا بتوضيح هذا من خلال امثله رقميه.

KEYWORDS: Fuzzy Set Theory, Fuzzy Analysis, Fuzzy Differential Equations.

1. INTRODUCTION

Fuzzy differential equation (FDE) has been rapidly developing in recent years, and has attracted many researchers. The use of FDE is a smart way to model dynamic systems under uncertain information [25].The nation of fuzzy derivative was first induced by (Zadeh and Chang) [9], it was followed up by (Dubois and prade) [10], also other process has been discussed by (Puri and Ralescu and Goetschel and Voxman) [20, 12]. The concept of differential equation in fuzzy environment concepts was formulated by (Kaleva) [17], using of Hukuhara or generalized derivatives the solution turns fuzzier as time goes by [11]. But (Bede) found that a large class of BVPs has no solution if hukuhara derivative is used [3], so to overcome this, the concept of generalized derivative was developed [2, 7]. (Khastan and Nieto) found solutions of a large class of BVPs using the generalized derivative [18].
(Stefanini and Bede) by the concept of generalization of hukuhara difference for compact convex set and introduced generalized hukuhara differentiability for fuzzy valued function, the demonstrated that [2,23]. Recently, (Gasilov) solve the fuzzy initial value problem by a new technique (linear transformation) [13] and (Barros) solve fuzzy differential equation by fuzzification of the derivative operator [6].
Due to the wide applications of the second order fuzzy differential equation, it is considered as the most important between all fuzzy differential equations, So, Many researchers have worked on the second FDE, (Wang and Gue) [28] solve second order by adomian, (Gasilov) [13, 14] solve by linear transformation, (Ahmadi) apply fuzzy Laplace transform [20], (jamoshidi and Avazpour) found way by shooting method [16], while (Rabiei) solved by improved runge kutta [22], Finally (Mondal and ray) solved in fuzzy environment analytically [26].

2. BASIC CONCEPTS

In this section, we will illustrate the fundamental concepts and facts related to fuzzy differential equations. According to Zadeh [25], a fuzzy set is a generalization of a classical set that allows the membership function to take any value in the unit interval [0, 1].

Definition 2.1[1-4] Let U a nonempty universe and fuzzy set (A) in U is a function $A: U \to [0,1]$,Where $\mu(x)$ is the degree of membership of$x\ in\ A$, when $\mu(x)$ goes closer to 1, the x is more considered to belong to$A$, but when it goes closer to 0, the x is less considered to belong to$A$. $A = \{(x,\mu(x)), x \in X\}$

Definition 2.2 [1-4]Let A be a fuzzy set in$U$, the support of A is the crisp set in all elements in U with non-zero membership in$A$.$Supp(A) = \{x \in A | A(x) > 0\}$

Definition2.3[1-4]Let A be a fuzzy set in$U$, the core of A is the crisp set in all elements in U with membership in$A$ equals 1.$Core(A) = \{x \in A | A(x) = 1\}$

Definition 2.4[1-4] Let A be a fuzzy set in$R$. A Is called a fuzzy interval if:

(i) A is normal: there exists $x_0 \in R$ then $A(x_0) = 1$

(ii) A is Convex: for all $x, y \in R\ and\ 0 \leq \lambda \leq 1$, it holds that

$$A(\lambda x + (1-\lambda)y) \geq \min(A(x), A(y));$$

(iii)A is upper semi-continuous

$$A(x_0) \geq \lim_{x \to x_0^{\pm}} A(x);$$

(iv) $[A]^0 = \overline{supp(A)}$ is compact subset of R

Definition 2.5:[12,23]Let A be a fuzzy set then, $\alpha - cut\ of\ A$Is the crisp set of $[A]^\alpha$that contains all elements with membership greater than or equalα.Where;$\alpha \in]0,1][A]^\alpha = \{x \in R | A(x) \geq \alpha\}$.$[A]^\alpha = [a_1^\alpha, a_2^\alpha]$.Where a_1^α is lower and a_2^α is upper.

Definition 2.6 [3,4]The fuzzy number is an extension of a regular number but, it does not refer to a single value. It refers to a connected set of possible values, where each possible value has its own membership from 0 to 1. Thus fuzzy number is a convex and normal fuzzy set.

Definition 2.7: generalized trapezoidal fuzzy number($GTrFN$) [17]

$\overline{A}_{GTFN}$is a subset of IFN in R with following membership:

$\overline{A}_{GTFN} = (a_1, a_2, a_3, a_4, \omega)$;

$$\mu(x) = \begin{cases} \omega \frac{x-a_1}{a_2-a_1} & if\ a_1 \leq x \leq a_2 \\ \omega & if\ a_2 \leq x \leq a_3 \\ \omega \frac{a_4-x}{a_4-a_3} & if\ a_3 \leq x \leq a_4 \\ 0 & otherwise \end{cases};$$

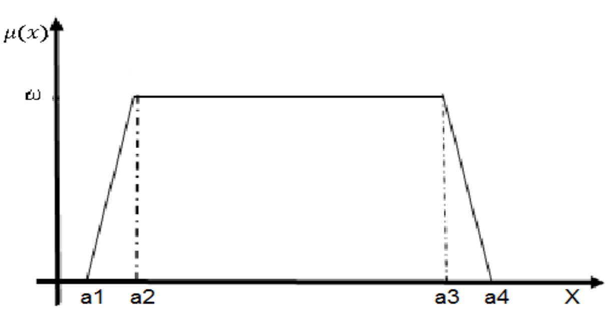

Fig. 1: $GTrFN$ representation

At $\omega = 1$, we can get **trapezoidal fuzzy number**.

Definition 2.8 [3,17]: Generalized triangle fuzzy number $GTrFN$

We can say that this number is a special case of generalized trapezoidal fuzzy number when the core becomes point not interval.

$\overline{A}_{GTFN} = (a_1, a_2, a_3, \omega)$;

$$\mu(x) = \begin{cases} \omega \frac{x-a_1}{a_2-a_1} & if\ a_1 \le x \le a_2 \\ \omega & if\ a_2 = x \\ \omega \frac{a_3-x}{a_3-a_2} & if\ a_2 \le x \le a_3 \\ 0 & \end{cases};$$

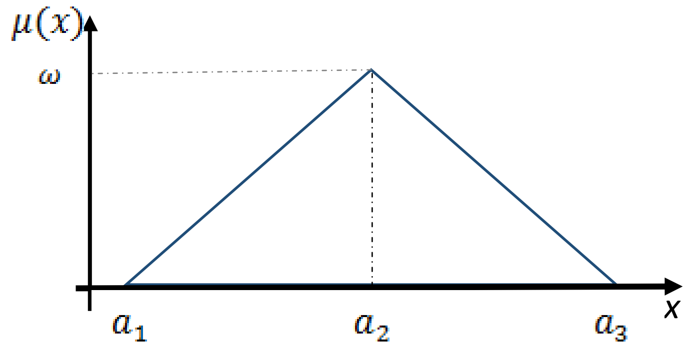

Fig. 2: triangle fuzzy number

At $\omega = 1$, we can get **triangle fuzzy number**.

Definition 2.9: [11], distance between two fuzzy intervals

Let *A* and *B* are two fuzzy intervals then,Hausedorff distance [11] between $[A]^\alpha and [b]^\alpha$ is; $d_H([A]^\alpha, [B]^\alpha) = \max\{|a_1^\alpha - b_1^\alpha|, |a_2^\alpha - b_2^\alpha|\}$By using Hausedorff distance, it is easily to find the distance between two fuzzy intervals which can be written as following $D(A, B) = sup_{\alpha\in[0,1]} d_H([A]^\alpha, [B]^\alpha)$.

Definition 2.10: [10], Let $x, y\ and\ z$ are fuzzy numbers, and there exists $x = y + z$ Then, Hukuhara difference is

$z = x \ominus y = [z_1^\alpha, z_2^\alpha] = [x_1^\alpha, x_2^\alpha] \ominus [y_1^\alpha, y_2^\alpha] = \quad [x_1^\alpha - y_1^\alpha, x_2^\alpha - y_2^\alpha]$,Where $x \ominus y \neq x + (-y)$.

Then, Generalized Hukuhara difference is $z = x \ominus_{gh} y = [z_1^\alpha, z_2^\alpha]$

$then;\ z_1^\alpha = \min[(x_1^\alpha - y_1^\alpha), (x_2^\alpha - y_2^\alpha)], z_2^\alpha = \max[(x_1^\alpha - y_1^\alpha), (x_2^\alpha - y_2^\alpha)]$.

Definition 2.11: [4-7,23], The generalized hukuhara first derivative of a fuzzy parametric function is defined as; $f'(t_0) = \lim_{h\to 0} \frac{f(t_0+h)\ominus_{gh} f(t_0)}{h}$,From the definition, we have two classes:

(i)-differentiable at t_0 $\quad [f'(t_0)]_\alpha = [f_1'(t_0,\alpha), f_2'(t_0,\alpha)]$

(ii)-differentiable at t_0 $\quad [f'(t_0)]_\alpha = [f_2'(t_0,\alpha), f_1'(t_0,\alpha)]$

Where, f_1 is the lower and f_2 is the upper.

Definition 2.12 [23, 24], The generalized hukuhara second derivative of fuzzy function is defined as; $f''(t_0) = \lim_{h\to 0} \frac{f'(t_0+h)\ominus_{gh} f'(t_0)}{h}$

According to the Definition 12, we have the following classes:

$f'(t_0)$is(i)--differentiable if:

$$f''(t_0) = \begin{cases} [f_1''(t_0,\alpha), f_2''(t_0,\alpha)] \text{ if } f \text{ is } (i)-differentiable \\ class(1,1) \\ [f_2''(t_0,\alpha), f_1''(t_0,\alpha)] \text{ if } f \text{ is } (ii)-differentiable \\ class(2,2) \end{cases}$$

$f'(t_0)$ Is (ii)-differentiable if:

$$f''(t_0) = \begin{cases} [f_2''(t_0,\alpha), f_1''(t_0,\alpha)] \text{ if } f \text{ is } (i)-differentiable \\ class(1,2) \\ [f_1''(t_0,\alpha), f_2''(t_0,\alpha)] \text{ if } f \text{ is } (ii)-differentiable \\ class(2,1) \end{cases}$$

Definition 2.13: [26] Let$[x_1(t,\alpha), x_2(t,\alpha)]$ be solution of any fuzzy differential is called a strong solution, if $\frac{dx_1(t,\alpha)}{d\alpha} > 0, \frac{dx_2(t,\alpha)}{d\alpha} < 0 \ \forall \alpha \in [0,\omega], x_1(t) \leq x_2(t)$ Otherwise it is called a weak solution.

3. Fuzzy boundary value problems (FBVP)

In this section, the study concerns with the fuzzy effect and using the generalized Hukuhara differentiability on the following FBVP [24]:

$$\tilde{y}''(x) = p\tilde{y}'(x) + q\tilde{y}(x) \qquad (1)$$

According to the following fuzzy boundary conditions

$$\tilde{y}(x_0) = \tilde{a}, \qquad \tilde{y}(x_b) = \tilde{b}, \qquad (2)$$

Where p and q are constants, $\tilde{a}$and$\tilde{b}$are two generalized trapezoidal fuzzy number represented as:

$\tilde{a} = (a_1, a_2, a_3, a_4; \omega)$ And$\tilde{b} = (b_1, b_2, b_3, b_4; \omega)$.

The lower and upper values of $\tilde{a}$ and $\tilde{b}$ are given according to Table 1:

	Lower values	Upper values
$\tilde{a} = [\underline{a}, \overline{a}]_\alpha$	$\underline{a} = a_1 + \frac{\alpha(a_2 - a_1)}{\omega}$	$\overline{a} = a_4 - \frac{\alpha(a_4 - a_3)}{\omega}$
$\tilde{b} = [\underline{b}, \overline{b}]_\alpha$	$\underline{b} = b_1 + \frac{\alpha(b_2 - b_1)}{\omega}$	$\overline{b} = b_4 - \frac{\alpha(b_4 - b_3)}{\omega}$

$table\ 1: the\ lower\ and\ upper\ values$

4. SOLUTION OF FUZZY BOUNDARY VALUE PROBLEMS

Class (1, 1)

The FBVP(1) can be written as:

$$\overline{y}''(x,\alpha) = p.\overline{y}'(x,\alpha) + q.\overline{y}(x,\alpha), \qquad \overline{y}(x_0,\alpha) = \overline{a}, \quad \overline{y}(x_b,\alpha) = \overline{b}, \qquad (3)$$

And

$$\underline{y}''(x,\alpha) = p.\underline{y}'(x,\alpha) + q.\underline{y}(x,\alpha), \qquad \underline{y}(x_0,\alpha) = \underline{a}, \quad \underline{y}(x_b,\alpha) = \underline{b}. \qquad (4)$$

In order to get the solution of (3, 4), we write these equations in the following system:

$$\begin{bmatrix} r \\ u \\ z \\ w \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & q & p \end{bmatrix} \begin{bmatrix} r \\ u \\ z \\ w \end{bmatrix},$$

(5)

Where$\underline{y} = r$ and$\overline{y} = z$.

So the lower and upper solutions are given by:

$$\underline{y}(x,\alpha) = c_2 e^{\frac{d_1 x}{2}} + c_4 e^{\frac{d_3 x}{2}},$$

(6)

$$\overline{y}(x,\alpha) = c_1 e^{\frac{d_1 x}{2}} + c_3 e^{\frac{d_3 x}{2}},$$

(7)

Where, d_1, d_2, d_3 and d_4 are constants given by:

$d_1 = p + \sqrt{p^2 + 4q},$ $\qquad d_2 = p + \sqrt{p^2 - 4q},$

$d_3 = p - \sqrt{p^2 + 4q}\,,$ $\qquad d_4 = p - \sqrt{p^2 - 4q}$

Andthe constantsc_1, c_2, c_3 and c_4 can be obtained by applying the fuzzy boundary condition given by Eq. (3, 4).

By using the value of $c, \underline{a}\,, \overline{a}, \underline{b}$ and$\overline{b}$ then:

$$\underline{y}(x,\alpha) = \frac{\underline{a}\left(e^{\frac{d_1 x_b + d_3 x}{2}} - e^{\frac{d_3 x_b + d_1 x}{2}}\right) + \underline{b}\left(e^{\frac{d_3 x_0 + d_1 x}{2}} - e^{\frac{d_1 x_0 + d_3 x}{2}}\right)}{\left(e^{\frac{d_1 x_b + d_3 x_0}{2}} - e^{\frac{d_3 x_b + d_1 x_0}{2}}\right)} \tag{8}$$

$$\overline{y}(x,\alpha) = \frac{\overline{a}\left(e^{\frac{d_1 x_b + d_3 x}{2}} - e^{\frac{d_3 x_b + d_1 x}{2}}\right) + \overline{b}\left(e^{\frac{d_3 x_0 + d_1 x}{2}} - e^{\frac{d_1 x_0 + d_3 x}{2}}\right)}{\left(e^{\frac{d_1 x_b + d_3 x_0}{2}} - e^{\frac{d_3 x_b + d_1 x_0}{2}}\right)} \tag{9}$$

Where, $\underline{a}\,, \overline{a}, \underline{b}$ and$\overline{b}$ values from table 1.

Class (1, 2)

$$\underline{y}''(x,\alpha) = p.\overline{y}'(x,\alpha) + q.\overline{y}(x,\alpha), \overline{y}(x_0,\alpha) = \overline{a},\ \ \overline{y}(x_b,\alpha) = \overline{b}, \tag{10}$$

$$\overline{y}''(x,\alpha) = p.\underline{y}'(x,\alpha) + q.\underline{y}(x,\alpha), \underline{y}(x_0,\alpha) = \underline{a},\ \ \underline{y}(x_b,\alpha) = \underline{b}, \tag{11}$$

The general solution:

$$\underline{y}(x,\alpha) = c_1 e^{\frac{-x}{2}d_2} + c_2 e^{\frac{-x}{2}d_4} + c_3 e^{\frac{x}{2}d_3} + c_4 e^{\frac{x}{2}d_1}, \tag{12}$$

$$\overline{y}(x,\alpha) = c_1 e^{\frac{-x}{2}d_2}\left(1 - d_4\left(\frac{p}{q}\right)\right) + c_2 e^{\frac{-x}{2}d_4}\left(1 - d_2\left(\frac{p}{q}\right)\right) + c_3 e^{\frac{x}{2}d_3} + c_4 e^{\frac{x}{2}d_1} \tag{13}$$

Class (2, 1)

$$\overline{y}''(x,\alpha) = p.\overline{y}'(x,\alpha) + q.\underline{y}(x,\alpha), \overline{y}(x_0,\alpha) = \overline{a},\ \ \overline{y}(x_b,\alpha) = \overline{b}, \tag{14}$$

$$\underline{y}''(x,\alpha) = p.\underline{y}'(x,\alpha) + q.\bar{y}(x,\alpha), \underline{y}(x_0,\alpha) = \underline{a},\ \underline{y}(x_b,\alpha) = \underline{b}, \quad (15)$$

The general solution:

$$\underline{y}(x,\alpha) = c_1 e^{\frac{x}{2}d_2} + c_2 e^{\frac{x}{2}d_1} + c_3 e^{\frac{x}{2}d_4} + c_4 e^{\frac{x}{2}d_3} \quad (16)$$

$$\bar{y}(x,\alpha) = -c_1 e^{\frac{x}{2}d_2} + c_2 e^{\frac{x}{2}d_1} - c_3 e^{\frac{x}{2}d_4} + c_4 e^{\frac{x}{2}d_3} \quad (17)$$

Class (2, 2)

$$\bar{y}''(x,\alpha) = p.\underline{y}'(x,\alpha) + q.\bar{y}(x,\alpha), \bar{y}(x_0,\alpha) = \bar{a},\ \bar{y}(x_b,\alpha) = \bar{b}, \quad (18)$$

$$\underline{y}''(x,\alpha) = p.\bar{y}'(x,\alpha) + q.\underline{y}(x,\alpha), \underline{y}(x_0,\alpha) = \underline{a},\ \underline{y}(x_b,\alpha) = \underline{b}, \quad (19)$$

The general solution:

$$\underline{y}(x,\alpha) = c_1 e^{\frac{-x}{2}d_3} + c_2 e^{\frac{x}{2}d_3} + c_3 e^{\frac{-x}{2}d_1} + c_4 e^{\frac{x}{2}d_1}, \quad (20)$$

$$\bar{y}(x,\alpha) = -c_1 e^{\frac{-x}{2}d_3} + c_2 e^{\frac{x}{2}d_3} - c_3 e^{\frac{-x}{2}d_1} + c_4 e^{\frac{x}{2}d_1}, \quad (21)$$

Example 4.1.

Consider the following FBVP

$$\tilde{y}''(t) = 5\tilde{y}'(t) + 4\tilde{y}(t),\ \tilde{y}(0) = (0.8,1,1.1,1.3;0.7),\ \tilde{y}(1) = (2.6,2.8,3,3.4;0.7).$$

Class (1, 1)

The general solution is given by

$$\underline{y}(t,\alpha) = (0.48\alpha + 7.3)10^{-3} e^{\frac{(5+\sqrt{41})t}{2}} + (0.285\alpha + 0.79) e^{\frac{(5-\sqrt{41})t}{2}},$$

$$\bar{y}(t,\alpha) = (9.2 - 1.4\alpha)10^{-3} e^{\frac{(5+\sqrt{41})t}{2}} + (1.29 - 0.284\alpha) e^{\frac{(5-\sqrt{41})t}{2}}.$$

Also, for different values of α, we plotted the lower and upper solutions in Fig. 3, and we listed the lower and upper solutions for $t = 0.5$ in Table 2.

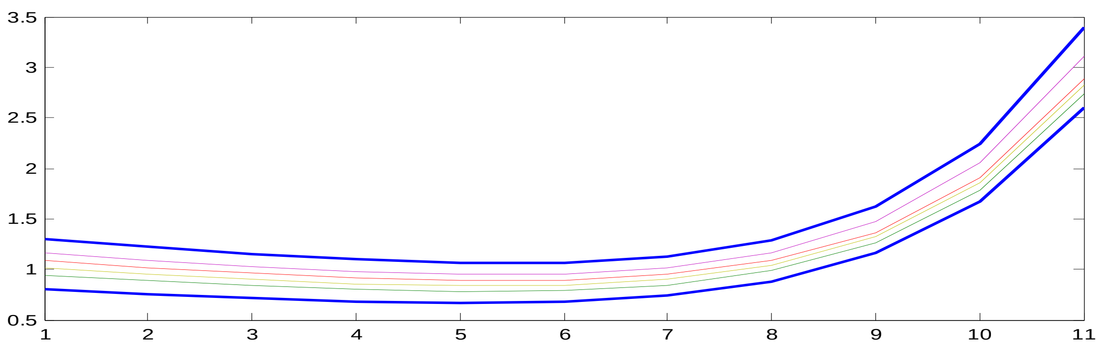

Fig. 3:The lower and upper solution class (1,1) of Example 1for $\alpha = 0$,0.5,0.7

Class (1, 2)

The general solution is given by

$$\underline{y}(t,\alpha) =$$

$$(0.7-0.74\alpha)e^{-4t}+(0.242\alpha-0.226)e^{-t}+(0.78\alpha+0.312)e^{\frac{(5-\sqrt{41})t}{2}}+(0.008-0.0059\alpha)e^{\frac{(5+\sqrt{41})t}{2}}$$

,

$$\bar{y}(t,\alpha) = (0.7-0.74\alpha)e^{-4t}\left(-\left({}^{3}/_{2}\right)\right)+(0.242\alpha-0.226)e^{-t}(-9)+(0.78\alpha+0.312)e^{\frac{(5-\sqrt{41})t}{2}}+(0.008-0.0059\alpha)e^{\frac{(5+\sqrt{41})t}{2}}$$

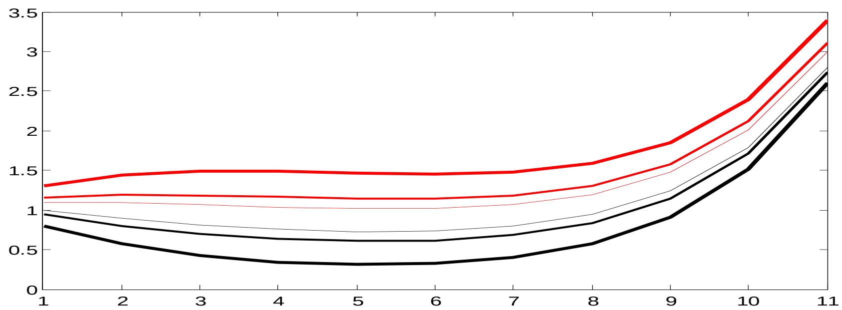

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Fig. 4:The lower and upper solution class (1,2) of Example 1for $\alpha = 0$,0.5,0.7

Class (2, 1)

The general solution is given by

$$\underline{y}(t,\alpha) = (0.0054-0.0067\alpha)e^{4t}+(0.0083-0.00047\alpha)e^{\frac{(5+\sqrt{41})t}{2}}+(0.29\alpha-0.255)e^{t}+(0.00047\alpha+1.04)e^{\frac{(5-\sqrt{41})t}{2}},$$

$$\bar{y}(t,\alpha) = -(0.0054-0.0067\alpha)e^{4t}+(0.0083-0.00047\alpha)e^{\frac{(5+\sqrt{41})t}{2}}-(0.29\alpha-0.255)e^{t}+(0.00047\alpha+1.04)e^{\frac{(5-\sqrt{41})t}{2}}$$

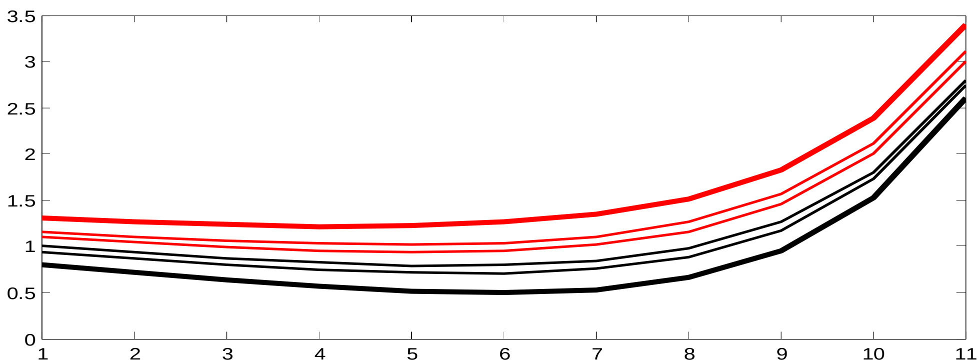

Fig. 5:The lower and upper solution class (2,1) of Example 1for $\alpha = 0$,0.5,0.7

Class (2, 2)

The general solution is given by

$$\underline{y}(t,\alpha) = (-0.071\alpha - 0.198)e^{\frac{-(5-\sqrt{41})t}{2}} + (1.042 - 0.0014\alpha)e^{\frac{(5-\sqrt{41})t}{2}}$$
$$+(0.357\alpha - 0.0517)e^{\frac{-(5+\sqrt{41})t}{2}} + (0.0014\alpha + 0.0083)e^{\frac{(5+\sqrt{41})t}{2}}, \text{ and}$$
$$\bar{y}(t,\alpha) = (0.071\alpha + 0.198)e^{\frac{-(5-\sqrt{41})t}{2}} + (1.042 - 0.0014\alpha)e^{\frac{(5-\sqrt{41})t}{2}}$$
$$-(0.357\alpha - 0.0517)e^{\frac{-(5+\sqrt{41})t}{2}} + (0.0014\alpha + 0.0083)e^{\frac{(5+\sqrt{41})t}{2}}$$

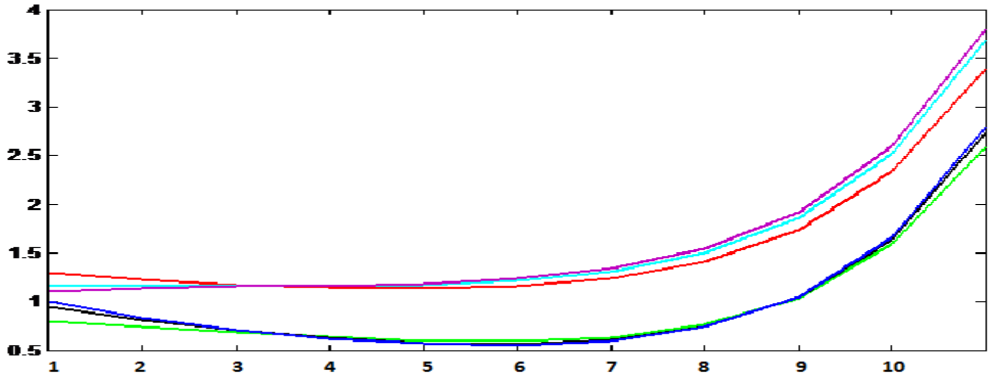

Fig. 6:The lower and upper solution class (2,2) of Example 1for $\alpha = 0$,0.5,0.7

From Fig. 6, we notice the intersection between the solution and this indicates that this class represents a weak solution which means that the inner band may become outer band at different t so, we can check the variation with α in Table 2.

Table 2: The lower and upper solutions of Example 1 for $t = 0.5$

	Class (1, 1)		Class (1, 2)		Class (2, 1)		Class (2, 2)	
α	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$
0	0.6857	1.0684	0.3239	1.4576	0.4958	1.2583	0.5925	1.1616
0.1	0.7066	1.0459	0.3827	1.3946	0.5383	1.2142	0.5868	1.1720
0.2	0.7275	1.0234	0.4416	1.3316	0.5807	1.1702	0.5811	1.1825
0.3	0.7484	1.0009	0.5005	1.2686	0.6232	1.2610	0.5754	1.1930
0.4	0.7694	0.9784	0.5594	1.2056	0.6656	1.0821	0.5697	1.2034
0.5	0.7903	0.9559	0.6182	1.1426	0.7081	1.0381	0.5640	1.2139
0.6	0.8112	0.9334	0.6771	1.0796	0.7506	0.9940	0.5583	1.2243
0.7	0.8321	0.9109	0.7360	1.0166	0.7930	0.9500	0.5526	1.2348

From Table 2, we conclude that $\underline{y}(t,\alpha)$ is increasing and $\bar{y}(t,\alpha)$ is decreasing

Then, the solution is a strong solution for class (1,1), (1,2) and (2,1) while the solution of class (2,2) is a weak solution.

Example 4.2

Consider the following FBVP

$\tilde{y}''(t) = -5\tilde{y}'(t) - 4\tilde{y}(t),\ \tilde{y}(0) = (0.8,1,1.4),\ \tilde{y}(1) = (2.6,3,3.1).$

Class(1, 1)

The final solution:

$$\underline{y}(t,\alpha) = (-0.427\alpha - 5.596)e^{-4t} + (0.0003\alpha - 0.0008)e^{-t} + (0.086\alpha + 1.619)e^{\frac{(5-\sqrt{41})t}{2}} + (0.540\alpha + 4.778)e^{\frac{(5+\sqrt{41})t}{2}}$$

$$\bar{y}(t,\alpha) = \left(\frac{3}{2}\right) * (-0.427\alpha - 5.596)e^{-4t} + 9 * (0.0003\alpha - 0.0008)e^{-t} - (0.086\alpha + 1.619)e^{\frac{(5-\sqrt{41})t}{2}} - (0.540\alpha + 4.778)e^{\frac{(5+\sqrt{41})t}{2}}$$

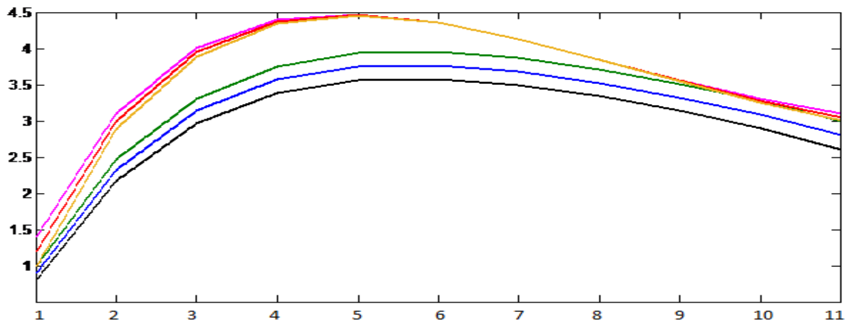

Fig. 7 the lower and upper solution class(1, 1) Example 2

We can easily notice the intersections and overlapping which starts at t = 0.5 till t = 1 and this indicates the weakness in the solution.

Class (1, 2)

$$\underline{y}(t,\alpha) = (1.134\alpha + 7.396)e^{-t} + (-0.934\alpha - 6.596)e^{-4t}$$

$$\bar{y}(t,\alpha) = (8.794 - 0.265\alpha)e^{-t} + (-0.134\alpha - 7.395)e^{-4t}$$

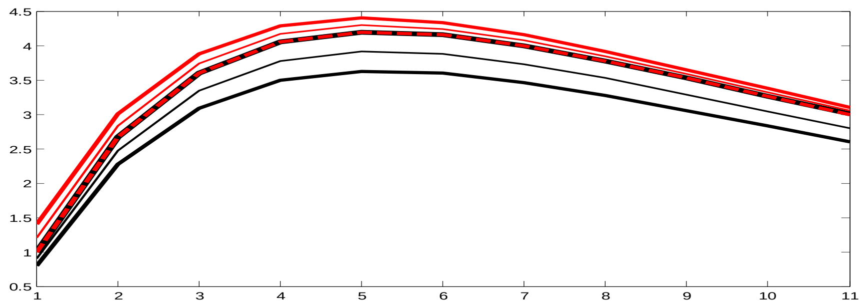

Fig. 8 the lower and upper solution class (1, 2) Example 2

The red waves represent $\bar{y}(t,\alpha)$and the black waves represent $\underline{y}(t,\alpha)$also we can see at$\alpha = 1\ \bar{y}(t,1) = \underline{y}(t,1)$the crisp solutionbut only for the triangular fuzzy conditions.

Class (2, 1)

$$\underline{y}(t,\alpha) = (0.0108 - 0.0108\alpha)e^{4t} + (0.434\alpha + 8.095)e^{-t} + (-0.534\alpha - 6.995)e^{-4t} + (0.3109\alpha - 0.3109)e^{t}$$

$$\bar{y}(t,\alpha) = -(0.0108 - 0.0108\alpha)e^{4t} + (0.434\alpha + 8.095)e^{-t} + (-0.534\alpha - 6.995)e^{-4t} - (0.3109\alpha - 0.3109)e^{t}$$

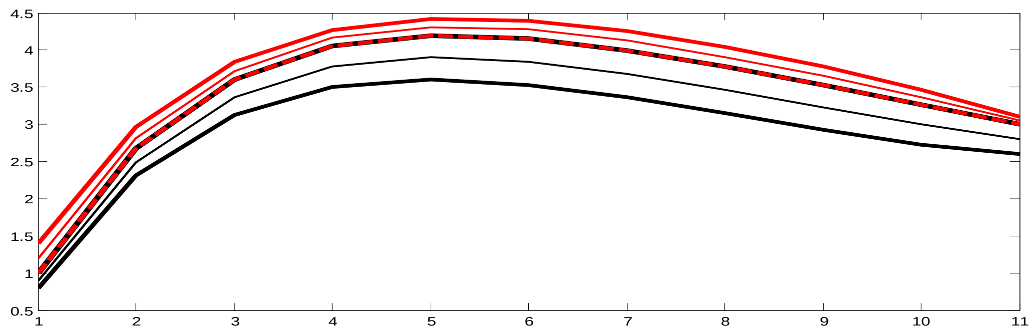

Fig. 9 the lower and upper solution class(2, 1) example 2

The red waves represent $\overline{\boldsymbol{y}}(\boldsymbol{t},\boldsymbol{\alpha})$and the black waves represent $\underline{\boldsymbol{y}}(\boldsymbol{t},\boldsymbol{\alpha})$also we can see at$\boldsymbol{\alpha} = \mathbf{1}\ \overline{\boldsymbol{y}}(\boldsymbol{t},\mathbf{1}) = \underline{\boldsymbol{y}}(\boldsymbol{t},\mathbf{1})$the crisp solutionbut only for the triangular fuzzy conditions.

Class (2, 2)

$$\underline{y}(t,\alpha) = (-0.534\alpha - 6.995)e^{-4t} + (0.176\alpha - 0.176)e^{\frac{-(5+\sqrt{41})t}{2}} + (0.434\alpha + 8.095)e^{-t} + (0.124\alpha - 0.124)e^{\frac{-(5-\sqrt{41})t}{2}}$$

$$\bar{y}(t,\alpha) = (-0.534\alpha - 6.995)e^{-4t} - (0.176\alpha - 0.176)e^{\frac{-(5+\sqrt{41})t}{2}} + (0.434\alpha + 8.095)e^{-t} - (0.124\alpha - 0.124)e^{\frac{-(5-\sqrt{41})t}{2}}$$

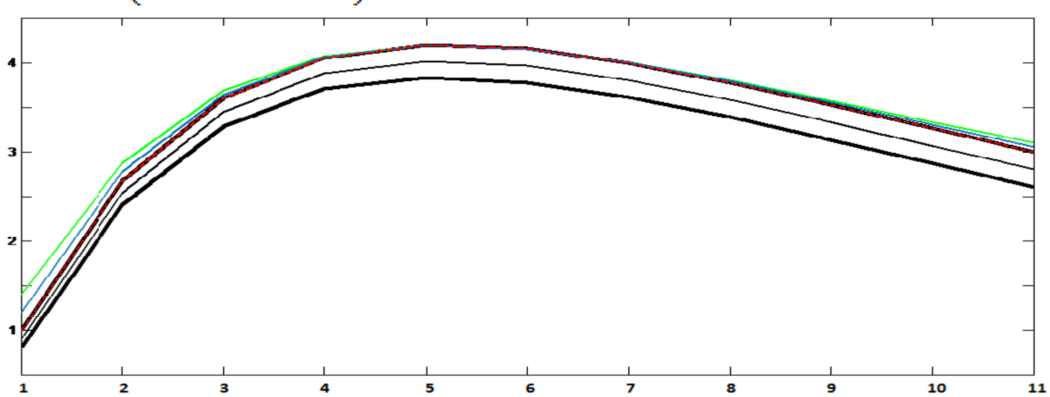

Fig. 10 the lower and upper solution class (2, 2) example 2

We can see some overlapping but it may be equal at point or so near wavebut, we can notice that waves are not intersected or by another words the lower bands don't exceed the upper bands but we can also see that the narrow region starts from t = 0.5 and 0.6.

Table 3:The lower and upper solutions of Example 2 for $t = 0.5$

	Class (1, 1)		Class (1, 2)		Class (2, 1)		Class (2, 2)	
α	$\underline{y}(0.5,\alpha)$	$\bar{y}(0.5,\alpha)$	$\underline{y}(0.5,\alpha)$	$\bar{y}(0.5,\alpha)$	$\underline{y}(0.5,\alpha)$	$\bar{y}(0.5,\alpha)$	$\underline{y}(0.5,\alpha)$	$\bar{y}(0.5,\alpha)$
0	3.5739	4.3528	3.5932	4.3336	3.5313	4.3954	3.7134	4.2134
0.2	3.6502	4.3530	3.7055	4.2977	3.6560	4.3472	3.8016	4.2016
0.4	3.7265	4.3531	3.8177	4.2619	3.7806	4.2991	3.8898	4.1898
0.6	3.8028	4.3533	3.9300	4.2261	3.9052	4.2509	3.9781	4.1781
0.8	3.8791	4.3534	4.0422	4.1903	4.0299	4.2027	4.0663	4.1663
1	3.9554	4.3536	4.1545	4.1545	4.1545	4.1545	4.1545	4.1545

From table 3 we notice that in class (2, 2), (1, 2) and (2,1) the lower solutions at $t = 0.5$ are increasing and the upper solutions are decreasing which means that these classes have a strong solution and we can find that the solutions at 1 are equal which strength the solution because of the triangle fuzzy number in conditions.
But class (1, 1) besides the overlapping shown in fig. 7, we can notice the upper solutions are slightly increasing so we can easily conclude that it is weak solution.

CONCLUSION:
In this paper; the analytical solution of a second order differential equation with fuzzy conditions under generalized hukuhara differentiability. Here fuzzy numbers are taken as generalized trapezoidal fuzzy number.
The results are very useful in the field of differential equation theoretically and in the applications, it was effective in determining the strong and weak solutions.
In the future research we will solve the same problem but with adding a fuzzy nonhomogeneous term.

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