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# SOME SUBORDINATION RESULTS FOR P-VALENT FUNCTIONS ASSOCIATED WITH DIFFERINTEGRAL OPERATOR

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Abstract. Making use of the prenciple of differential subordination, we investigate some inclusion relationships of certain subclasses of p-valent analytic functions which are defined by certain differintegral operator.

## 1. Introduction

Let $A_n(p)$ denote the class of analytic and p-valent functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \ (p, n \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

For convenience, we write $A_1(p) = A(p)$. For analytic functions $f, g$ in $U$, we say that $f$ is subordinate to $g$, written $f(z) \prec g(z)$ if there exists a Schwarz function $w$, which is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if $g$ is univalent in $U$, then we have the following equivalence, (cf., e.g.,[4], [5] see also [1]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f_i \in A_n(p)$ $(i = 1, 2)$ given by

$$f_i(z) = z^p + \sum_{k=n}^{\infty} a_{k+p,i} z^{k+p} \ (i = 1, 2; p, n \in \mathbb{N}), \tag{2}$$

the Hadamard product (or convolution) of $f_1$ and $f_2$ is defined by

$$(f_1 * f_2)(z) = z^p + \sum_{k=n}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p} = (f_2 * f_1)(z). \tag{3}$$

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In this present investigation, we shall also make use of the Gaussian hypergeometric function ${}_2F_1$ defined by

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \ (a,b,c \in \mathbb{C}; c \notin \mathbb{Z}_0 = \{0,-1,-2,...\}), \tag{4} $$

where $(d)_k$ denotes the Pochhammer symbol given in terms of the Gamma function $\Gamma$, by

$$ (d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} \begin{cases} 1 & (k=0; d \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ d(d+1)...(d+k-1) & (k \in \mathbb{N}; d \in \mathbb{C}). \end{cases} $$

We note that the series defined by (4) converges absolutely for $z \in U$ and hence ${}_2F_1$ represents an analytic function in $U$ [19 ,Ch.14]. With a view to introducing an extended fractional differintegral operator, we begin by recalling the following definitions of fractional calculus considered by Owa [9] ( see also [10] and [17] ). . *The fractional integral of order $\lambda$ $(\lambda > 0)$ is defined, for a function $f$, analytic in a simply-connected region of the complex plane containing the origin by*

$$ D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{5} $$

*where the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.*

. *Under the hypothesis of Definition 1, the fractional derivative of $f$ of order $\lambda$ $(\lambda \geqslant 0)$ is defined by*

$$ D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt & (0 \le \lambda < 1) \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z) & (n \le \lambda < n+1; n \in N_0 = N \cup \{0\}), \end{cases} \tag{6} $$

*where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.*

In [13] Patel and Mishra defined the extended fractional differintegral operator $\Omega_z^{(\lambda,p)} : A(p) \to A(p)$ for a function $f$ of the form (1.1) (with $n=1$) and for a real number $\lambda$ $(-\infty < \lambda < p+1)$ by:

$$ \begin{aligned} \Omega_z^{(\lambda,p)} f(z) &= z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)} a_{k+p} z^{k+p} \\ &= z^p \ {}_2F_1(1,p+1;p+1-\lambda;z) * f(z) \ (-\infty < \lambda < p+1; z \in U). \end{aligned} \tag{7} $$

It is easily seen from (7) that (see [13])

$$ z(\Omega_z^{(\lambda,p)} f(z))' = (p-\lambda)\Omega_z^{(\lambda+1,p)} f(z) + \lambda \Omega_z^{(\lambda,p)} f(z) \ (-\infty < \lambda < p; z \in U). \tag{8} $$

We also note that

$$ \Omega_z^{(0,p)} f(z) = f(z), \ \ \Omega_z^{(1,p)} f(z) = \frac{zf'(z)}{p}, $$

and ( in general )

$$ \Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \ (-\infty < \lambda < p+1; z \in U), \tag{9} $$

where $D_z^{\lambda} f(z)$ is, respectively, the fractional integral of $f$ of order $-\lambda$ when $-\infty < \lambda < 0$ and the fractional derivative of $f$ of order $\lambda$ when $0 \le \lambda < p+1$.

For integral values of $\lambda$, (9) becomes:

$$\Omega_z^{(j,p)} f(z) = \frac{(p-j)! z^j f^{(j)}(z)}{p!} \; (j \in N; j < p+1),$$

and

$$\begin{aligned} \Omega_z^{(-m,p)} f(z) &= \frac{p+m}{z^m} \int_0^z t^{m-1} \Omega_z^{(-m+1,p)} f(t) dt \; (m \in N) \\ &= F_{1,p} \circ F_{2,p} \circ ... \circ F_{m,p}(f)(z) \\ &= F_{1,p} \left( \frac{z^p}{1-z} \right) * F_{2,p} \left( \frac{z^p}{1-z} \right) * ... * F_{m,p} \left( \frac{z^p}{1-z} \right) * f(z), \end{aligned}$$

where $F_{\mu,p}$ is the familiar integral operator defined by (3.12) ( see Section 3) and $\circ$ denotes the usual composition of functions.

The fractional differential operator $\Omega_z^{(\lambda,p)}$ with $0 \leq \lambda < 1$ was investigated by Srivastava and Aouf [15]. More recently, Srivastava and Mishra [16] obtained several interesting properties and charactaristics for certain subclasses of p-valent analytic functions involving the differintegral operator $\Omega_z^{(\lambda,p)}$ when $-\infty < \lambda < 1$. The operator $\Omega_z^{(\lambda,1)} = \Omega_z^{\lambda}$ was introduced by Owa and Srivastava [10].

By using the extended fractional differintegral operator $\Omega_z^{(\lambda,p)}$ $(-\infty < \lambda < p)$, we introduce the following subclass of functions in $A_n(p)$.

*. For fixed parameters $A, B(-1 \leq B < A \leq 1)$ , we say that a function $f \in A_n(p)$ is in the class $S_{p,n}^{\lambda}(A,B)$ if it satisfies the following subordination condition:*

$$\frac{(\Omega_z^{(\lambda,p)} f(z))'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz} \; (z \in U; p \in \mathbb{N}). \tag{9}$$

In view of the definition of subordination, (10) is equivalent to the following condition:

$$\left| \frac{\frac{(\Omega_z^{(\lambda,p)} f(z))'}{pz^{p-1}} - p}{B\frac{(\Omega_z^{(\lambda,p)} f(z))'}{pz^{p-1}} - pA} \right| < 1 \; (z \in U).$$

For convenience, we write $S_{p,n}^{\lambda}(1 - \frac{2\eta}{p}, -1) = S_{p,n}^{\lambda}(\eta)$ $(0 \leq \eta < p)$, where $S_{p,n}^{\lambda}(\eta)$ denotes the class of functions in $A_n(p)$ satisfying the inequality :

$$Re \left\{ \frac{(\Omega_z^{(\lambda,p)} f(z))'}{z^{p-1}} \right\} > \eta \; (0 \leq \eta < p; p \in N; z \in U). \tag{10}$$

Let us consider the first-order differential subordination

$$H(\varphi(z), z\varphi'(z)) \prec h(z).$$

Then, a univalent function $q$ is called its dominant, if $\varphi(z) \prec q(z)$ for all analytic functions $\varphi$ that satisfy this differential subordination. A dominant $\widetilde{q}$ is called the best dominant, if $\widetilde{q}(z) \prec q(z)$ for all dominants $q$. For the general theory of the first-order differential subordination and its applications, we refer the reader to [1] and [5].

The object of the present paper is to obtain several inclusion relationships and other interesting properties of functions belonging to the subclass $S_{p,n}^{\lambda}(A,B)$ and $S_{p,n}^{\lambda}(\eta)$ by using the method of differential subordination.

with the initial conditions ([5]), where $a_1$, $a_2$, $b_1$, $b, c$ are positive constants.

## 2. Preliminaries

To prove our main results, we shall need the following lemmas. [2]. *Let $h(z)$ be analytic and convex (univalent) function in $U$ with $h(0) = 1$. Also let the function $\phi$ given by*

$$\phi(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + ... \tag{11}$$

*be analytic in $U$. If*

$$\phi(z) + \frac{z\phi'(z)}{\delta} \prec h(z) \quad (Re(\delta) \geqslant 0; \delta \neq 0), \tag{12}$$

*then*

$$\phi(z) \prec \psi(z) = \frac{\delta}{n} z^{-\frac{\delta}{n}} \int_0^z t^{\frac{\delta}{n}-1} h(t)dt \prec h(z), \tag{13}$$

*and $\psi$ is the best dominant of* (13).

With a view to stating a well-known result ( Lemma 2 below), we denote by $P(\delta)$ the class of functions $\Phi$ given by

$$\Phi(z) = 1 + c_1 z + c_1 z^2 + ... \tag{14}$$

which are analytic in $U$ and satisfy the following inequality :

$$Re\{\Phi(z)\} > \delta \quad (0 \leq \delta < 1).$$

[11] .*Let the function $\Phi$, given by* (15)*, be in the class $P(\delta)$. Then*

$$Re\{\Phi(\delta)\} \geqslant 2\delta - 1 + \frac{2(1-\delta)}{1+|z|} \quad (0 \leq \delta < 1).$$

[18] . *For $0 \leq \delta_1, \delta_2 < 1$,*

$$P(\delta_1) * P(\delta_2) \subset P(\delta_3) \quad (\delta_3 = 1 - 2(1-\delta_1)(1-\delta_2)).$$

*The result is the best possible.*

[7] .*Let $\phi$ be analytic in $U$ with $\phi(0) = 1$ and $\phi(z) \neq 0$ for $0 < |z| < 1$,and let $A, B \in \mathbb{C}$ with $A \neq B, |B| \leq 1$.*

*(i) Let $B \neq 0$ and $\gamma \in \mathbb{C}^*$ satisfy either $\left|\frac{\gamma(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{\gamma(A-B)}{B} + 1\right| \leq 1$.*

*If $\phi$ satisfies*

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec \frac{1+Az}{1+Bz} \;,$$

*then*

$$\phi(z) \prec (1+Bz)^{\gamma(\frac{A-B}{B})}$$

*and this is the best dominant.*

*(ii) Let $B = 0$ and $\gamma \in \mathbb{C}^*$ be such that $|\gamma A| < \pi$, and if $\phi$ satisfies*

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec 1 + Az \;,$$

*then*

$$\phi(z) \prec e^{\gamma Az}$$

*and this is the best dominant.*

[14] . *Let the function $g$ be analytic in $U$, with*

$$g(0)=1 \text{ and } Re\{g(z)\} > \frac{1}{2}\ (z\in U).$$

*Then, for any function $F$ analytic in $U$, $(g*F)(U)$ is contained in the convex hull of $F(U)$.*

[20] .*Let $\mu$ be a positive measure on the unit interval $[0,1]$ . Let $g(z,t)$ be a complex valued function defined on $U\times[0,1]$ such that $g(.,t)$ is analytic in $U$ for each $t\in[0,1]$, and such that $g(z,.)$ is $\mu$ integrable on $[0,1]$, for all $z\in U$ .In addition, suppose that $Re\{g(z,t)\}>0$, $g(-r,t)$ is real and*

$$Re\left\{\frac{1}{g(z,t)}\right\} \geqslant \frac{1}{g(-r,t)} \quad (|z|\le r<1; t\in[0,1]).$$

*If the function $G(z)$ is defined by*

$$G(z)=\int_0^1 g(z,t)d\mu(t),$$

*then*

$$Re\left\{\frac{1}{G(z)}\right\} \geqslant \frac{1}{G(-r)} \quad (|z|\le r<1).$$

Each of the identities ( asserted by Lemma 7 below ) is fairly well known [19, Ch. 14 ] for the Gauss hypergeometric function ${}_2F_1$ defined by (1.4).

[19] . *For real or complex numbers $a,b$ and $c$ $(c\neq 0,-1,-2,...)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \tfrac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z)\ \ (Re(c)>Re(b)>0); \tag{15}$$

$${}_2F_1(a,b;c;z)=(1-z)_2^{-a}F_1(a,c-b;c;\frac{z}{z-1}); \tag{16}$$

and

$${}_2F_1(a,b;c;z)={}_2\,F_1(b,a;c;z). \tag{17}$$

## 3. Main Results

Unless otherwise mentioned, we assume throughout this paper that: $\lambda<p; -1\le B<A\le 1$, $0<\alpha\le 1, z\in U$ and the powers are considered principal ones. . *Let the function $f(z)$ given by (1) satisfy the following subordination condition:*

$$(1-\alpha)\frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}}+\alpha\frac{\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}, \tag{18}$$

then

$$\frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}} \prec Q(z) \prec \frac{1+Az}{1+Bz}, \tag{19}$$

*where the function $Q$ given by*

$$Q(z)=\begin{cases} \frac{A}{B}+(1-\frac{A}{B})(1+Bz)^{-1}\ {}_2F_1(1,1;\frac{p-\lambda}{n\alpha}+1;\frac{Bz}{1+Bz}) & (B\neq 0)\\ 1+\frac{p-\lambda}{n\alpha+p-\lambda}Az & (B=0),\end{cases} \tag{20}$$

*is the best dominant of* (20). *Furtheremore,*

$$Re\left\{\frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}}\right\} > \eta\ , \tag{it22}$$

where

$$\eta = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\ {}_2F_1(1,1;\frac{p-\lambda}{n\alpha}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{p-\lambda}{n\alpha+p-\lambda}A & (B=0). \end{cases} \tag{21}$$

*The estimate in* (22) *is the best possible.*

*Proof.* Let

$$\phi(z) = \frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}}. \tag{24}$$

Then $\phi$ is of the form (12) and analytic in $U$. Applying the identity (8) in (24) and differentiating the resulting equation with respect to $z$, we get

$$(1-\alpha)\frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}} + \alpha\frac{\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'}{pz^{p-1}}$$

$$= \phi(z) + \frac{\alpha z}{p-\lambda}\phi'(z) \prec \frac{1+Az}{1+Bz}\ \ (z \in U).$$

Now, by using Lemma 1 for $\delta = \frac{p-\lambda}{\alpha}$ we deduce that

$$\frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}} \prec Q(z) = \frac{p-\lambda}{n\alpha}z^{-\frac{p-\lambda}{n\alpha}}\int_0^z t^{\frac{p-\lambda}{n\alpha}-1}(\frac{1+At}{1+Bt})dt$$

$$= \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\ {}_2F_1(1,1;\frac{p-\lambda}{n\alpha}+1;\frac{Bz}{1+Bz}) & (B \neq 0) \\ 1 + \frac{p-\lambda}{n\alpha+p-\lambda}Az & (B=0), \end{cases}$$

where we have made a change of variables followed by the use of the identities $(16),(17)$ and (18) ( with $a=1, b=\frac{p-\lambda}{n\alpha}$ and $c=b+1$). This proves the assertion (20) of Theorem 1.

Next, in order to prove the assertion (22) of Theorem 1, it sufficies to show that

$$\inf_{|z|<1}\{Re(Q(z))\} = Q(-1). \tag{25}$$

Indeed, we have for, $|z| \leq r < 1$,

$$Re\left\{\frac{1+Az}{1+Bz}\right\} \geqslant \frac{1-Ar}{1-Br}\quad .$$

Setting $G(z,s) = \dfrac{1+Asz}{1+Bsz}$ and $d\nu(s) = \frac{p-\lambda}{n\alpha}s^{\frac{p-\lambda}{n\alpha}-1}ds$ $(0 \leq s \leq 1)$, which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z) = \int_0^1 G(z,s)d\nu(s),$$

so that

$$Re\{Q(z)\} \geqslant \int_0^1 \frac{1-Asr}{1-Bsr}d\nu(s) = Q(-r)\ \ (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (22). Finally, the estimate (22) is the best possible as the function $Q(z)$ is the best dominant of (20).

Taking $\alpha = 1, A = 1 - \frac{2\eta}{p}$ $(0 \le \eta < p)$ and $B = -1$ in Theorem 1, we obtain the following corollary.

. *The following inclusion property holds true for the function class* $S^{\lambda}_{p,n}(\eta)$ :

$$S^{\lambda+1}_{p,n}(\eta) \subset S^{\lambda}_{p,n}(\beta(p,n,\lambda,\eta)) \subset S^{\lambda}_{p,n}(\eta),$$

*where*

$$\beta(p,n,\lambda,\eta) = \eta + (p-\eta)\{\ {}_2F_1(1,1;\frac{p-\lambda}{n}+1;\frac{1}{2})\}.$$

*The result is the best possible.*

Taking $\alpha = 1, \lambda = 0, A = 1 - \frac{2\eta}{p}(0 \le \eta < p)$ and $B = -1$ in Theorem 1, we obtain

.*Let the function* $f(z)$ *given by* (1) *satisfy the following inequality:*

$$Re\left\{\frac{(zf'(z))'}{p^2 z^{p-1}}\right\} > \eta \ (z \in U),$$

*then*

$$Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \eta + (1-\eta)[{}_2F_1(1,1;\frac{p}{n}+1;\frac{1}{2}) - 1].$$

*The result is the best possible.*

Taking $\alpha = 1$ in Theorem 1, we obtain

. *The following inclusion property holds true for the function class* $S^{\lambda}_{p,n}(A,B)$ :

$$S^{\lambda+1}_{p,n}(A,B) \subset S^{\lambda}_{p,n}(1-\frac{2\rho}{p},-1) \subset S^{\lambda}_{p,n}(A,B)\ (0 \le \rho < p),$$

*where*

$$\rho = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\ {}_2F_1(1,1;\frac{p-\lambda}{n}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{p-\lambda}{n+p-\lambda}A & (B = 0). \end{cases}$$

*The result is the best possible.*

. *If* $f \in S^{\lambda}_{p,n}(\eta)$ $(0 \le \eta < p)$, *then*

$$Re\left\{(1-\alpha)\frac{\left(\Omega^{(\lambda,p)}_z f(z)\right)'}{pz^{p-1}} + \alpha\frac{\left(\Omega^{(\lambda+1,p)}_z f(z)\right)'}{pz^{p-1}}\right\} > \eta\ (|z| < R), \tag{it26}$$

*where*

$$R = \left\{\frac{\sqrt{(p-\lambda)^2 + n^2\alpha^2} - \lambda\alpha}{p-\lambda}\right\}^{\frac{1}{n}}.$$

*The result is the best possible.*

*Proof.* Since $f \in S^{\lambda}_{p,n}(\eta)$, we write

$$\frac{\left(\Omega^{(\lambda,p)}_z f(z)\right)'}{pz^{p-1}} = \eta + (1-\eta)u(z) \quad (z \in U). \tag{27}$$

Then, $u$ is of the form (12) , analytic in $U$ and has a positive real part in $U$. Making use of (8) in (27) and differentiating the resulting equation with respect to $z$, we have

$$\frac{1}{1-\eta}\left\{(1-\alpha)\frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}}+\alpha\frac{\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'}{pz^{p-1}}-\eta\right\}=u(z)+\frac{\alpha}{p-\lambda}zu'(z). \tag{28}$$

Applying the following well-known estimate [3] :

$$\frac{|zu'(z)|}{Re\{u(z)\}}\leq\frac{2nr^n}{1-r^{2n}}\quad(|z|=r<1),$$

in (28), we get

$$\frac{1}{1-\eta}Re\left\{(1-\alpha)\frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}}+\alpha\frac{\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'}{pz^{p-1}}-\eta\right\}$$

$$\geqslant Re(u(z))\left(1-\frac{2\alpha nr^n}{(p-\lambda)[1-r^{2n}]}\right). \tag{29}$$

It is easily seen that the right-hand side of (29) is positive, if $r<R$, where $R$ is given by (26).

In order to show that the the bound $R$ is the best possible, we consider the function $f\in A_n(p)$ defined by

$$\frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}}=\eta+(1-\eta)\frac{1+z^n}{1-z^n}\quad(0\leq\eta<1;z\in U).$$

Noting that

$$\frac{1}{1-\eta}\left\{(1-\alpha)\frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}}+\alpha\frac{\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'}{pz^{p-1}}-\eta\right\}$$

$$=\frac{(p-\lambda)(1-z^{2n})-2\alpha nz^n}{(p-\lambda)(1-z^n)^2}=0,$$

for $z=R.\exp\{\frac{i\pi}{n}\}$. This completes the proof of Theorem 2.

Putting $\alpha=1$ in Theorem 2, we obtain the following result.

. *If $f\in S_{p,n}^{\lambda}(\eta)$ $(0\leq\eta<p)$, then $f\in S_{p,n}^{\lambda+1}(\eta)$ for $|z|<\widetilde{R}$, where*

$$\widetilde{R}=\left\{\frac{\sqrt{(p-\lambda)^2+n^2}-\lambda}{p-\lambda}\right\}^{\frac{1}{n}}.$$

*The result is the best possible.*

For a function $f\in A_n(p)$ the generalized Bernardi-Libera-Livingston integeral operator $F_{p,\delta}$ is defined by

$$F_{p,\delta}(f)(z)=\frac{\delta+p}{z^p}\int_0^z t^{\delta-1}f(t)dt$$

$$= (z^p + \sum_{k=1}^{\infty} \frac{\delta + p}{\delta + p + k} z^{p+k}) * f(z) \qquad (\delta > -p)$$

$$= z^p \ {}_2F_1(1, \delta + p; \delta + p + 1; z) * f(z). \tag{30}$$

. *Let* $f \in S^{\lambda}_{p,n}(A, B)$ *and let the function* $F_{p,\delta}$ *defined by* (30). *Then*

$$\frac{\left(\Omega_z^{(\lambda,p)} F_{p,\delta} f(z)\right)'}{pz^{p-1}} \prec K(z) \prec \frac{1 + Az}{1 + Bz}, \tag{it31}$$

*where the function* $K$ *given by*

$$K(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 + Bz)^{-1} \ {}_2F_1(1, 1; \frac{\delta+p}{n} + 1; \frac{Bz}{1+Bz}) & (B \neq 0) \\ 1 + \frac{\delta+p}{\delta+p+n} Az & (B = 0), \end{cases} \tag{it32}$$

*is the best dominant of* (31). *Furtheremore,*

$$Re \left\{ \frac{\left(\Omega_z^{(\lambda,p)} F_{p,\delta} f(z)\right)'}{pz^{p-1}} \right\} > \chi, \tag{it33}$$

*where*

$$\chi = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)^{-1} \ {}_2F_1(1, 1; \frac{\delta+p}{n} + 1; \frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{\delta+p}{\delta+p+n} A & (B = 0). \end{cases}$$

*The result is the best possible.*

*Proof.* From the identity (8) and (30) we have

$$z \left(\Omega_z^{(\lambda,p)} F_{p,\delta} f(z)\right)' = (p + \delta)\Omega_z^{(\lambda,p)} f(z) - \delta \Omega_z^{(\lambda,p)} F_{p,\delta} f(z). \tag{34}$$

Let

$$\phi(z) = \frac{(\Omega_z^{(\lambda,p)} F_{\delta,p}(f)(z))'}{pz^{p-1}}, \tag{35}$$

then $\phi$ is of the form (12) and is analytic in $U$. Using the identity (34) in (35), and differentiating the resulting equation with respect to $z$, we have

$$\frac{(\Omega_z^{(\lambda,p)} f(z))'}{pz^{p-1}} = \phi(z) + \frac{z\phi'(z)}{p + \delta} \prec \frac{1 + Az}{1 + Bz} .$$

Imploying the same technique that used in proving Theorem 1, the reminder part of the theorem can be proved.

. *We observe that*

$$\frac{(\Omega_z^{(\lambda,p)} F_{\delta,p}(f)(z))'}{pz^{p-1}} = \frac{p + \delta}{pz^{p+\delta}} \int_0^z t^{\delta} (\Omega_z^{(\lambda,p)} f(z))' dt \ (f \in A_p(n)). \tag{it36}$$

In view of (36), Theorem 3 for $A = 1 - \frac{2\eta}{p}$ $(0 \leq \eta < p; p \in N)$ and $B = -1$ yields

. *If* $\delta > 0$ *and if* $f \in A_n(p)$ *satisfies the following inequality:*

$$Re \left\{ \frac{(\Omega_z^{(\lambda,p)} f(z))'}{pz^{p-1}} \right\} > \eta \ (0 \leq \eta < 1; p \in N; z \in U),$$

*then*

$$Re\left\{\frac{p+\delta}{pz^{p+\delta}}\int_0^z t^{\delta}(\Omega_z^{(\lambda,p)}f(t))'dt\right\} > \eta+(1-\eta)\left[{}_2F_1(1,1;\frac{\delta+p}{n}+1;\frac{1}{2})-1\right]\ (z\in U).$$

*The result is the best possible.*

. *Let the function $f$ defined by (1) be in the class $A_n(p)$. Let also that $g\in A_n(p)$ satisfies the following inequality:*

$$Re\left\{\frac{\Omega_z^{(\lambda,p)}g(z)}{z^p}\right\} > 0\ \ (z\in U).$$

*If*

$$\left|\frac{\Omega_z^{(\lambda,p)}f(z)}{\Omega_z^{(\lambda,p)}g(z)}-1\right| < 1 \quad (z\in U),$$

*then*

$$Re\left\{\frac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{\Omega_z^{(\lambda,p)}f(z)}\right\} > 0\ \ (|z|<R_0),$$

*where*

$$R_0=\frac{\sqrt{9n^2+4p(p+n)}-3n}{2(p+n)}. \tag{37}$$

*Proof.* Let

$$\varphi(z)=\frac{\Omega_z^{(\lambda,p)}f(z)}{\Omega_z^{(\lambda,p)}g(z)}-1=e_nz^n+e_{n+1}z^{n+1}+...\ \ , \tag{38}$$

we note that $\varphi$ is analytic in $U$, with

$$\varphi(0)=0 \quad \text{and}\ \ |\varphi(z)|\le|z|^n\ \ .$$

Then, by applying the familiar Schwarz lemma [6], we have

$$\varphi(z)=z^n\Psi(z),$$

where, the function $\Psi$ is analytic in $U$ and $|\Psi(z)|\le 1 \quad (z\in U)$. Therefore, (3.20) leads to

$$\Omega_z^{(\lambda,p)}f(z)=\Omega_z^{(\lambda,p)}g(z)(1+z^n\Psi(z)). \tag{39}$$

Differentiating (39) logarithmically with respect to $z$, we obtain

$$\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}=\frac{z(\Omega_z^{(\lambda,p)}g(z))'}{\Omega_z^{(\lambda,p)}g(z)}+\frac{z^n\left\{n\Psi(z)+z\Psi'(z)\right\}}{1+z^n\Psi(z)}. \tag{40}$$

Letting $\chi(z)=\frac{\Omega_z^{(\lambda,p)}g(z)}{z^p}$, we see that the function $\chi$ is of the form (12), analytic in $U$, $Re\chi(z)>0$ and

$$\frac{z(\Omega_z^{(\lambda,p)}g(z))'}{\Omega_z^{(\lambda,p)}g(z)}=\frac{z\chi'(z)}{\chi(z)}+p.$$

So that , we find from (40) that

$$Re\left\{\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}\right\}\geqslant p-\left|\frac{z\chi'(z)}{\chi(z)}\right|-\left|\frac{z^n\left\{n\Psi(z)+z\Psi'(z)\right\}}{1+z^n\Psi(z)}\right|. \tag{41}$$

Now, by using the following known estimates [12] ( see also [3]) :

$$\left|\frac{\chi'(z)}{\chi(z)}\right| \leq \frac{2nr^{n-1}}{1-r^{2n}} \quad \text{and} \quad \left|\frac{n\Psi(z)+z\Psi'(z)}{1+z^n\Psi(z)}\right| \leq \frac{n}{1-r^n} \quad (|z|=r<1),$$

in (41), we have

$$Re\left\{\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}\right\} \geqslant \frac{p-3nr^n-(p+n)r^{2n}}{1-r^{2n}} \quad (|z|=r<1),$$

which is certainly positive, provided that $r<R_0$, $R_0$ given by (3.19).

. *Let* $-1 \leq B_i < A_i \leq 1 \quad (i=1,2)$. *If each of the functions* $f_i(z) \in A_n(p)$ *satisfies the following subordination condition:*

$$(1-\alpha)\frac{\Omega_z^{(\lambda,p)}f_i(z)}{z^p}+\alpha\frac{\Omega_z^{(\lambda+1,p)}f_i(z)}{z^p} \prec \frac{1+A_iz}{1+B_iz} \quad (i=1,2), \tag{it42}$$

*then*

$$(1-\alpha)\frac{\Omega_z^{(\lambda,p)}F(z)}{z^p}+\alpha\frac{\Omega_z^{(\lambda+1,p)}F(z)}{z^p} \prec \frac{1+(1-2\eta)z}{1-z}, \tag{it43}$$

*where*

$$F(z)=\Omega_z^{(\lambda,p)}(f_1*f_2)(z) \tag{it44}$$

*and*

$$\eta = 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\,{}_2F_1(1,1;\frac{p-\lambda}{\alpha}+1;\frac{1}{2})\right]. \tag{it45}$$

*The result is the best possible when* $B_1=B_2=-1$.

*Proof.* Suppose that the functions $f_i(z) \in A_n(p) \quad (i=1,2)$ satisfy the condition (42). Then by setting

$$p_i(z)=(1-\alpha)\frac{\Omega_z^{(\lambda,p)}f_i(z)}{z^p}+\alpha\,\frac{\Omega_z^{(\lambda+1,p)}f_i(z)}{z^p} \ (i=1,2), \tag{46}$$

then, we have

$$p_i(z) \in P(\delta_i) \ \ (\delta_i=\frac{1-A_i}{1-B_i},\ \ i=1,2).$$

Thus, by making use of the identity (8) in (46), we get

$$\Omega_z^{(\lambda,p)}f_i(z)=\frac{p-\lambda}{\alpha}z^{p-\frac{p-\lambda}{\alpha}}\int_0^z t^{\frac{p-\lambda}{\alpha}-1}p_i(t)dt \ \ (i=1,2), \tag{47}$$

which, in view of the definition of $F$ given by (44) and (46), yields

$$\Omega_z^{(\lambda,p)}F(z)=\frac{p-\lambda}{\alpha}z^{p-\frac{p-\lambda}{\alpha}}\int_0^z t^{\frac{p-\lambda}{\alpha}-1}P(t)dt, \tag{48}$$

where

$$\begin{aligned} P(z) &= (1-\alpha)\frac{\Omega_z^{(\lambda,p)}F(z)}{z^p}+\alpha\,\frac{\Omega_z^{(\lambda+1,p)}F(z)}{z^p} \\ &= \frac{p-\lambda}{\alpha}z^{-\frac{p-\lambda}{\alpha}}\int_0^z t^{\frac{p-\lambda}{\alpha}-1}(p_1*p_2)(t)dt. \end{aligned} \tag{49}$$

Since $\quad p_i(z) \in P(\delta_i) \quad (i=1,2)$, it follows from Lemma 3 that

$$(p_1*p_2)(z) \in P(\delta_3) \quad (\delta_3=1-2(1-\delta_1)(1-\delta_2)). \tag{50}$$

Now, by using (50) in (49) and then appealing to Lemma 2 and Lemma 3, we have

$$\begin{aligned} Re\left\{P(z)\right\} &= \frac{p-\lambda}{\alpha}\int_0^1 u^{\frac{p-\lambda}{\alpha}-1} Re\left\{(p_1 * p_2)(uz)\right\} du \\ &\geqslant \frac{p-\lambda}{\alpha}\int_0^1 u^{\frac{p-\lambda}{\alpha}-1}\left(2\delta_3 - 1 + \frac{2(1-\delta_3)}{1+u\left|z\right|}\right) du \\ &> \frac{p-\lambda}{\alpha}\int_0^1 u^{\frac{p-\lambda}{\alpha}-1}\left(2\delta_3 - 1 + \frac{2(1-\delta_3)}{1+u}\right) du \\ &= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \tfrac{p-\lambda}{\alpha}\int_0^1 u^{\frac{p-\lambda}{\alpha}-1}(1+u)^{-1} du\right] \\ &= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\,{}_2F_1(1,1;\tfrac{p-\lambda}{\alpha}+1;\frac{1}{2})\right] \\ &= \eta. \end{aligned}$$

When $B_1 = B_2 = -1$, we consider the functions $f_i(z) \in A_n(p)$ $(i = 1, 2)$ which satisfy the condition (42) of Theorem 5 and are defined by

$$\Omega_z^{(\lambda,p)} f_i(z) = \tfrac{p-\lambda}{\alpha} z^{-\frac{p-\lambda}{\alpha}} \int_0^z t^{\frac{p-\lambda}{\alpha}-1}\left(\frac{1+A_i t}{1-t}\right) dt \; (i = 1, 2).$$

Thus it follows from (39) and Lemma 2 that

$$\begin{aligned} P(z) &= \tfrac{p-\lambda}{\alpha}\int_0^1 u^{\frac{p-\lambda}{\alpha}-1}\left[1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{(1-uz)}\right] du \\ &= 1 - (1+A_1)(1+A_2) + (1+A_1)(1+A_2)(1-z)^{-1}\,{}_2F_1(1,1;\tfrac{p-\lambda}{\alpha}+1;\frac{z}{z-1}) \\ &\to 1 - (1+A_1)(1+A_2) + \frac{1}{2}(1+A_1)(1+A_2)\,{}_2F_1(1,1;\tfrac{p-\lambda}{\alpha}+1;\frac{1}{2}) \text{ as } z \to -1, \end{aligned}$$

which evidently completes the proof of Theorem 5.

Taking $A_i = 1 - 2\eta_i$ , $B_i = -1$ $(i = 1, 2)$ and $\lambda = 0$ in Theorem 5, we obtain

. *If the functions* $f_i \in A_n(p)$ $(i = 1, 2)$ *satisfy the following inequality:*

$$Re\left\{(1-\alpha)\frac{f_i(z)}{z^p} + \alpha\frac{f_i'(z)}{pz^{p-1}}\right\} > \eta_i \; (0 \leq \eta_i < 1; i = 1, 2),$$

*then*

$$Re\left\{(1-\alpha)\frac{(f_1 * f_2)(z)}{z^p} + \alpha\frac{(f_1 * f_2)'(z)}{pz^{p-1}}\right\} > \eta_0 \, ,$$

*where*

$$\eta_0 = 1 - 4(1-\eta_1)(1-\eta_2)\left[1 - \frac{1}{2}\,{}_2F_1(1,1;\frac{p}{\alpha}+1;\frac{1}{2})\right].$$

*The result is the best possible.*

. *Let the function* $f \in A_n(p)$ *and let* $g \in A_n(p)$ *satisfies the following inequality:*

$$Re\left(\frac{g(z)}{z^p}\right) > \frac{1}{2}.$$

*Then*

$$(f * g)(z) \in S_p^{\lambda}(A, B).$$

*Proof.* We have

$$\frac{\left(\Omega_z^{(\lambda,p)}(f*g)(z)\right)'}{pz^{p-1}} = \frac{\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{pz^{p-1}} * z^p g(z) \quad (z \in U).$$

Since

$$Re\left(\frac{g(z)}{z^p}\right) > \frac{1}{2} \quad (z \in U),$$

and the function

$$\frac{1+Az}{1+Bz}$$

is convex (univalent) in $U$, it follows from (10) and Lemma 5 that $(f*g)(z) \in S_p^{\lambda}(A,B)$, which completes the proof of Theorem 6.

. *Let $p > \lambda$, $\nu \in C^*$ and let $A, B \in C$ with $A \neq B$ and $|B| \leq 1$.Suppose that*

$$\left|\frac{\nu(p-\lambda)(A-B)}{B} - 1\right| \leq 1 \text{ or } \left|\frac{\nu(p-\lambda)(A-B)}{B} + 1\right| \leq 1 \text{ if } B \neq 0,$$

$$|\nu| \leq \frac{\pi}{(p-\lambda)}, \text{ if } B = 0.$$

*If $f \in A_n(p)$ with $\Omega_z^{(\lambda,p)}f(z) \neq 0$ for all $z \in U^* = U\backslash\{0\}$, then*

$$\frac{\Omega_z^{(\lambda+1,p)}f(z)}{\Omega_z^{(\lambda,p)}f(z)} \prec \frac{1+Az}{1+Bz},$$

*implies*

$$\left(\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}\right)^{\nu} \prec q_1,$$

*where*

$$q_1 = \begin{cases} (1+Bz)^{\nu(p-\lambda)(A-B)/B}, & \text{if } B \neq 0, \\ e^{\nu(p-\lambda)Az}, & \text{if } B = 0, \end{cases}$$

*is the best dominant.*

*Proof.* Let us put

$$\phi(z) = \left(\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}\right)^{\nu}, \tag{51}$$

where the power is the princiapal one.
Then $\phi$ is analytic in $U$, $\phi(0) = 1$ and $\phi(z) \neq 0$ for all $z \in U$. Taking the logarithmic derivatives in both sides of (51) and using the identity (8) we have

$$1 + \frac{z\phi'(z)}{\nu(p-\lambda)\phi(z)} = \frac{\Omega_z^{(\lambda+1,p)}f(z)}{\Omega_z^{(\lambda,p)}f(z)} \prec \frac{1+Az}{1+Bz}.$$

Now the assertions of Theorem 7 follows by using Lemma 4 with $\gamma = \nu(p-\lambda)$. This completes the proof of Theorem 7.

Putting $A = 1-2\rho$, $0 \leq \rho < 1$ and $B = -1$, in Theorem 7, we obtain the following result.

. *Assume that $p > \lambda$ and $\nu \in C^*$ satisfies either*
$|2\nu(p-\lambda)(1-\rho) - 1| \leq 1$ *or* $|2\nu(p-\lambda)(1-\rho) + 1| \leq 1$.

*If* $f \in A_n(p)$ *with* $\Omega_z^{(\lambda,p)} f(z) \neq 0$ *for* $z \in U^*$, *then*

$$Re\left\{\frac{\Omega_z^{(\lambda+1,p)} f(z)}{\Omega_z^{(\lambda,p)} f(z)}\right\} > \rho,$$

*implies*

$$\left(\frac{\Omega_z^{(\lambda,p)} f(z)}{z^p}\right)^{\nu} \prec q_2 = (1-z)^{-2\nu(p-\lambda)(1-\rho)},$$

*and* $q_2$ *is the best dominant.*

Putting $A = 1 - \frac{2\eta}{p}$ $(0 \leq \eta < p), B = -1$ and $\lambda = 0$ in Theorem 7, we have

. *Assume that* $\nu \in C^*$ *satisfies either* $|2\nu(\eta - p) - 1| \leq 1$ *or* $|2\nu(\eta - p) + 1| \leq 1$. *If* $f \in A_n(p)$ *with* $f(z) \neq 0$ *for all* $z \in U^*$, *then*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \eta\ (0 \leq \eta < p),$$

*implies*

$$\left(\frac{f(z)}{z^p}\right)^{\nu} \prec q_3 = (1-z)^{-2\nu(\eta-p)},$$

*where* $q_3$ *is the best dominant.*

. *Putting* $p = 1$ *in Corollary 8, we obtain the corresponding result obtained by Obradović et al.[8, Theorem 1 with* $b = 1 - \eta, 0 \leq \eta < 1$*].*

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