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AN APPROXIMATE SOLUTION TO SOME CLASSES OF FRACTIONAL NONLINEAR PARTIAL DIFFERENTIAL - DIFFERENCE EQUATION USING ADOMIAN DECOMPOSITION METHOD

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Abstract. It is shown how the Adomian decomposition method (ADM) applicable for nonlinear differential equations to derive their both exact and approximate solutions can be extended to fractional nonlinear partial differential - difference equations with two independent variables. The effectiveness of the ADM is illustrated through time fractional discrete Korteweg-de Vries, time fractional discrete modified Korteweg - de Vries and time fractional Toda lattice equations.

1. Introduction

The study of fractional differential equations has drawn much attention both from the mathematical and physical point of view by researchers in nonlinear phenomena in recent years. The primary reason for interest is that the exact description of most of the phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science have been governed by nonlinear equations involving fractional order derivatives [1–5]. Also in reality a physical phenomenon may depend not only the time instant but also the previous time history, which can be successfully modeled by using the theory of derivatives and integrals of fractional order [1–4]. It is well known that nonlinear differential equations are not exactly solvable in general and thus deriving their explicit solutions are of fundamental importance. In the context of nonlinear evolution equations exhibiting solitons the derivation of traveling wave or multisolitons solutions are useful. Several analytical methods have been devised to derive exact solution of nonlinear partial differential equations in general and soliton possessing equations in particular [6, 23–25]. However the study of fractional differential equations has been handicapped due to the absence of well defined analytic techniques to deal with them. Recently we have

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shown that how Lie transformation group theory provides an useful tool to analyze time fractional nonlinear partial differential equations [7].

Recently Adomian [8–10] has introduced the ADM to derive both exact and approximate solutions for deterministic and stochastic, linear and nonlinear problems. The notable advantage of ADM is that it is algorithmic and does not need linearization, weak nonlinearity assumptions, discretization or perturbation technique [8–12]. In the case of approximate solution it provides very fast convergence to the actual solution. The application of ADM has been demonstrated to a variety of problems arising from science and engineering for all types of boundary and initial conditions governed by nonlinear equations involving both integer and fractional derivatives [12–16]. We would like to mention that several authors have made comparisons with other numerical methods and found that the approximate solution obtained through ADM is in very good agreement with their exact solution [17–20].

However the application of ADM to nonlinear discrete systems governed partial differential - difference equations (PDΔEs) to derive their exact solutions have not been illustrated widely [21, 22]. To the best of our knowledge, no attempt has been made to extend the ADM to nonlinear discrete systems governed by time fractional nonlinear PDΔEs. The main objective of this article is to illustrate the effectiveness of ADM for nonlinear PDΔEs and explain how it can be extended to time fractional nonlinear PDΔEs with two independent variables and derive their approximate solutions.

The plan of the article is as follows. In section 2, to be self contained we consider discrete Korteweg-de Vries (ΔKdV) and discrete modified Korteweg-de Vries (ΔmKdV) equations and show how ADM provides an effective tool to derive both rational and one-soliton solutions. The accuracy of the derived approximate solution through ADM has also been examined. In section 3, first we introduce certain basic definitions and properties of the fractional operators which are required for establishing our results and then explain how ADM can be extended to time fractional PDΔEs with initial conditions. In section 4, we illustrate the effectiveness of the ADM to time fractional ΔKdV equation, time fractional ΔmKdV equation and time fractional Toda lattice equation and derive their approximate solutions. In Section 5, we give a brief summary of our results and concluding remarks.

2. ADM : ΔKdV and ΔmKdV

In this section we illustrate the effectiveness of the ADM to derive exact (including rational) one-soliton solutions through ΔKdV and ΔmKdV equations. For clarity we consider them separately.

2.1. **ADM : Discrete Korteweg-de Vries Equation.**

It is known that the governing equation of ΔKdV is

$$\frac{\partial u_n}{\partial t} = u_n(u_{n+1} - u_{n-1}),\ u_n = u(n,t),\ n \in \mathbb{Z}, \tag{1}$$

which can be rewritten as

$$L_t u_n = F(u_n, u_{n+1}) - G(u_n, u_{n-1}), \tag{2}$$

where $L_t = \frac{\partial}{\partial t}$, $F(u_n, u_{n+1}) = u_n u_{n+1}$, $G(u_n, u_{n-1}) = u_n u_{n-1}$. It is assumed that the inverse operator L_t^{-1} is an integral operator given by

$$L_t^{-1}(.) = \int_0^t (.) \; ds. \tag{3}$$

In the ADM, one looks for solution $u(n,t)$ of the given PDΔE expressed as an infinite series of the form

$$u(n,t) = \sum_{k=0}^{\infty} U_k(n,t), \tag{4}$$

with $U_0(n,t) = u(n,0)$ and note that the component U_k does not stand for the k^{th} lattice and it means k^{th} element in the decomposition series. Applying the inverse operator L_t^{-1} on both sides of (2), we obtain

$$u(n,t) = u(n,0) + L_t^{-1}[F(u_n, u_{n+1}) - G(u_n, u_{n-1})]. \tag{5}$$

In order to apply the ADM to (1), it is necessary to write the nonlinear terms $F(u_n, u_{n+1})$, $G(u_n, u_{n-1})$ as an infinite series of Adomian polynomials, that is

$$F(u_n, u_{n+1}) = \sum_{k=0}^{\infty} A_k, \tag{6}$$

$$G(u_n, u_{n-1}) = \sum_{k=0}^{\infty} B_k. \tag{7}$$

The explicit expressions for A_k and B_k can be determined from the following formulae [9, 10]

$$A_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} F\left(\sum_{m=0}^{\infty} U_m(n,t)\lambda^m, \sum_{m=0}^{\infty} U_m(n+1,t)\lambda^m\right)\right]_{\lambda=0}, \tag{8}$$

$$B_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} G\left(\sum_{m=0}^{\infty} U_m(n,t)\lambda^m, \sum_{m=0}^{\infty} U_m(n-1,t)\lambda^m\right)\right]_{\lambda=0}. \tag{9}$$

Substituting (4) along with (6) and (7) into the functional equation (5) yields

$$\sum_{k=0}^{\infty} U_k(n,t) = u(n,0) + L_t^{-1}\left[\sum_{k=0}^{\infty}(A_k - B_k)\right]. \tag{10}$$

(10) yields the components of the decomposition series (4) as follows

$$U_0(n,t) = u(n,0), \tag{11}$$

$$U_{k+1}(n,t) = L_t^{-1}[A_k - B_k], \; k \geq 0. \tag{12}$$

and solving them give explicit form of U_k's. Now when $k=0$ in (12), $U_1(n,t)$ reads

$$U_1(n,t) = L_t^{-1}[A_0 - B_0],$$

which can be written as

$$U_1(n,t) = \int_0^t U_{n,0}(U_{n+1,0} - U_{n-1,0}) \; ds,$$

$$U_1(n,t) = U_{n,0}\left(U_{n+1,0} - U_{n-1,0}\right) t, \tag{13}$$

here $U_{n,0} = U_0(n,t), U_{n+1,1} = U_1(n+1,t), U_{n-1,1} = U_1(n-1,t)$, etc. Now the second element in the decomposition series is

$$U_2(n,t) = L_t^{-1}[A_1 - B_1].$$

Making use of equation (13) in the expressions A_1 and B_1, we write

$$A_1 = U_{n,0}U_{n+1,0}[U_{n+1,0} - U_{n-1,0} + U_{n+2,0} - U_{n,0}]t, \tag{14}$$
$$B_1 = U_{n,0}U_{n-1,0}[U_{n+1,0} - U_{n-1,0} + U_{n,0} - U_{n-2,0}]t. \tag{15}$$

and so

$$A_1 - B_1 = U_{n,0}\left[(U_{n+1,0} - U_{n-1,0})^2 - U_{n,0}(U_{n-1,0} + U_{n+1,0}) + U_{n+1,0}U_{n+2,0} + U_{n-2,0}U_{n-1,0}\right]t.$$

Hence

$$\begin{aligned} U_2(n,t) &= U_{n,0}\left\{\int_0^t (U_{n+1,0} - U_{n-1,0})^2 s\ ds - \int_0^t U_{n,0}(U_{n+1,0} + U_{n-1,0})s\ ds \right. \\ &\quad \left. + \int_0^t (U_{n+1,0}U_{n+2,0} + U_{n-2,0}U_{n-1,0})s\ ds\right\}, \\ U_2(n,t) &= U_{n,0}\left[(U_{n+1,0} - U_{n-1,0})^2 - U_{n,0}(U_{n+1,0} + U_{n-1,0}) \right. \\ &\quad \left. + (U_{n+1,0}U_{n+2,0} + U_{n-1,0}U_{n-2,0})\right]t^2/2. \end{aligned}$$

In a similar manner we compute $U_3(n,t), U_4(n,t), \ldots$ and so the infinite series (4) becomes

$$\begin{aligned} u(n,t) &= U_{n,0} + U_{n,0}[U_{n+1,0} - U_{n-1,0}]t + U_{n,0}\left[(U_{n+1,0} - U_{n-1,0})^2 - U_{n,0}(U_{n+1,0} + U_{n-1,0}) \right. \\ &\quad \left. + (U_{n+1,0}U_{n+2,0} + U_{n-2,0}U_{n-1,0})\right]t^2/2 + \cdots, \\ &= U_{n,0}\left\{1 + [U_{n+1,0} - U_{n-1,0}]t + \left[(U_{n+1,0} - U_{n-1,0})^2 - U_{n,0}(U_{n+1,0} + U_{n-1,0}) \right.\right. \\ &\quad \left.\left. + (U_{n+1,0}U_{n+2,0} + U_{n-2,0}U_{n-1,0})\right]t^2/2 + \cdots\right\} \end{aligned}$$

We below explain how to construct rational and one-soliton solutions separately.

2.1.1. *Rational solution.* We look for rational solution $u(n,t)$ of equation (1) satisfying the initial condition

$$u(n,0) = U_{n,0} = \frac{a_1n^2 + a_2n + a_3}{b_1n^2 + b_2n + b_3}, \tag{16}$$

where $a_1,\ a_2,\ a_3,\ b_1,\ b_2$ and b_3 are constants. Following the procedure outlined above, we computed the components $U_k, k \geq 1$ of the infinite series (4) and it leads to a closed form solution as

$$u(n,t) = \frac{(n+2t+1)(n+2t-2)}{(n+2t)(n+2t-1)}, \tag{17}$$

provided that the parameters satisfy
$a_1 = 1,\ a_2 = -1,\ a_3 = -2, b_1 = 1,\ b_2 = -1,\ b_3 = 0$. We would like to mention that more rational solutions with appropriate initial conditions can be constructed through ADM. For example a rational solution of (1)

$$u(n,t) = \frac{n+a}{1-2t}, \tag{18}$$

where a is arbitrary constant can be constructed with $u(n,0) = n + a$.

2.1.2. *One Soliton solution.* Next we look for a specific one soliton solution $u(n,t)$ of equation (1) satisfying the initial condition

$$u(n,0) = U_{n,0} = \frac{a_1 + a_2 e^{\kappa n} + a_3 e^{2\kappa n}}{b_1 + b_2 e^{\kappa n} + b_3 e^{2\kappa n}}, \tag{19}$$

where $\kappa,\ a_1,\ a_2,\ a_3,\ b_1,\ b_2$ and b_3 are constants. We then computed the elements of decomposition series (4) by solving (12) consistently. In this case, the infinite series (4) leads to a closed form solution as

$$u(n,t) = \frac{a_1(1+e^{\kappa(n+t+1)})(1+e^{\kappa(n+t-2)})}{(1+e^{\kappa(n+t)})(1+e^{\kappa(n+t-1)})}, \tag{20}$$

provided that the parameters satisfy

$$a_1 = \frac{\kappa}{e^{\kappa} - e^{-\kappa}},\ a_2 = a_1(e^{\kappa} + e^{-2\kappa}),\ a_3 = a_1 e^{-\kappa},$$

$$b_1 = 1,\ b_2 = 1 + e^{-\kappa},\ b_3 = e^{-\kappa}.$$

We would like to mention that two and N-soliton solutions can also be constructed through ADM by appropriately choosing the initial conditions. It is appropriate to mention that the decomposition series (4) not always leads to closed form solution. Therefore it is worthwhile to consider the approximate solution, that is truncated series (4) up to finite term. In order to verify numerically whether or not the approximate solution obtained through ADM leads to high accuracy we consider the solution (4) up to fifth term

$$u_{\text{appr}}(n,t) = \sum_{k=0}^{4} U_k(n,t) \tag{21}$$

with initial condition (20) at $t = 0$ and observed that it is good agreement with the exact solution (20) of ΔKdV (See tables 1 & 2 or figures 1a & 1b).

Comparison of approximate soliton solution (21) and one soliton solution (20)

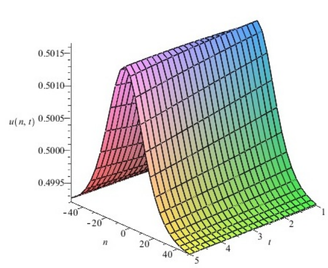

Figure 1a: Approximate soliton solution $u(n,t)$ of the ΔKdV with $k = 0.1$

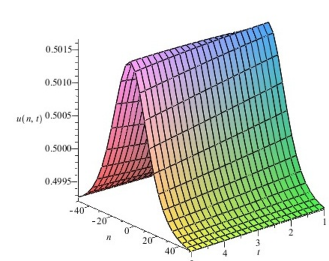

Figure 1b: Exact one soliton solution $u(n,t)$ of the ΔKdV with $k = 0.1$

n	Approximate solution	Exact solution
-25	0.4998688443	0.4998688444
-15	0.5006591678	0.5006591681
-5	0.5015172737	0.5015172740
0	0.5016671173	0.5016671174
5	0.5015172736	0.5015172740
15	0.5006591678	0.5006591677
25	0.4998688448	0.4998688444

Table 1. $k = 0.1$ and $t = 0.5$

n	Approximate solution	Exact solution
-25	0.4999303607	0.4999303608
-15	0.5007545186	0.5007545186
-5	0.5015698010	0.5015698032
0	0.5016608828	0.5016608829
5	0.5014551275	0.5014551251
15	0.5005653699	0.5005653701
25	0.4998113855	0.4998113855

Table 2. $k = 0.1$ and time $t = 1.5$

2.2. **ADM : Discrete Modified Korteweg-de Vries Equation.** Here we below provide a brief computational details of deriving rational and one soliton solution of $\Delta mKdV$ governed by

$$\frac{\partial u_n}{\partial t} = (1 + u_n^2)(u_{n+1} - u_{n-1}),\ n \in \mathbb{Z}, \tag{22}$$

which can be rewritten as

$$L_t u_n = u_{n+1} - u_{n-1} + F(u_n, u_{n+1}) - G(u_n, u_{n-1}), \tag{23}$$

where $L_t = \frac{\partial}{\partial t},\ F(u_n, u_{n+1}) = u_n^2 u_{n+1},\ G(u_n, u_{n-1}) = u_n^2 u_{n-1}$. Applying the inverse operator L_t^{-1} on both sides of (23) , we get

$$u(n,t) = u(n,0) + L_t^{-1}(u_{n+1} - u_{n-1}) + L_t^{-1}(F(u_n, u_{n+1}) - G(u_n, u_{n-1})). \tag{24}$$

The nonlinear terms $F(u_n, u_{n+1}),\ G(u_n, u_{n-1})$ can be expressed in terms of Adomian polynomials A_k's and B_k's as given in (6) and (7). Using decomposition series (4) along with (6) and (7) in (24), we obtain the following recursive relation

$$\begin{aligned} U_0(n,t) &= u(n,0), \\ U_{k+1}(n,t) &= L_t^{-1}(U_k(n+1,t) - U_k(n-1,t)) + L_t^{-1}[A_k - B_k],\ k \geq 0. \end{aligned}$$

Following the procedure outlined in the previous subsection with initial conditions $u(n,0) = \frac{i}{n}$ and $u(n,0) = sech(n)$ we find that the associated infinite series (4) results a rational and one soliton solution of (22) respectively as

$$u(n,t) = \frac{i}{(n+2t)}, \tag{25}$$

$$u(n,t) = sech(n+2t). \tag{26}$$

Thus it is clear that ADM provides an effective tool to derive both exact and approximate solutions to nonlinear PDΔEs.

3. ADM : Time Fractional Nonlinear PDΔEs

Before embarking into the details of ADM to time fractional nonlinear PDΔE, we would like to recall certain basic definitions and properties of fractional operators which are required for the remaining part of the article.

Definition 3.1. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of the function, $h \in L^1([a,b],\mathbb{R}_+)$ denoted by $I^{\alpha}_{a^+}$, is defined by [1–3]*

$$\begin{aligned} I^{\alpha}_{a^+}h(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\ ds,\ t > a, \\ I^{0}_{a^+}h(t) &= h(t), \end{aligned} \tag{27}$$

where Γ is the gamma function.

Definition 3.2. *The Caputo fractional differential operator of order $\alpha > 0$ of the function $h \in L^1([a,b],\mathbb{R}_+)$, denoted by $D^{\alpha}_{a^+}$, is defined by [1–3]*

$$\begin{aligned} D^{\alpha}_{a^+}h(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\ \frac{d^n h(s)}{ds^n}\ ds,\ t > a, \\ D^{0}_{a^+}h(t) &= h(t), \end{aligned} \tag{28}$$

where $n = [\alpha]+1$, the function $h(t)$ has absolutely continuous derivatives upto order $(n-1)$.

For simplicity we denote the operators $D^{\alpha}_{0^+}h(t)$ and $I^{\alpha}_{0^+}h(t)$ respectively as $D^{\alpha}h(t)$ and $I^{\alpha}h(t)$. Note that the above mentioned operators satisfy the following properties for the suitable functions $f(t)$ and $g(t)$

$$I^{\alpha}(f(t)+g(t)) = I^{\alpha}f(t) + I^{\alpha}g(t), \tag{29}$$

$$I^{\alpha}(D^{\alpha}f(t)) = f(t) - \sum_{r=0}^{n-1}\frac{f^{(r)}(0)}{r!}\ t^r,\ n-1 < \alpha \le n, \tag{30}$$

$$D^{\alpha}(I^{\alpha}f(t)) = f(t). \tag{31}$$

Obviously, the inverse operator for the Caputo fractional differential operator D^{α} is the Riemann-Liouville fractional integral operator I^{α} which is defined in (27).

Let us consider a time fractional nonlinear PDΔE with two independent variables and with prescribed initial conditions given by

$$L^{\alpha}_t u_n + R(u_{n-1}, u_n, u_{n+1}, ...) + N(u_{n-1}, u_n, u_{n+1}, ...) = s_n, \tag{32}$$

where $u_n = u(n,t)$, $u_{n-1} = u(n-1,t)$, $u_{n+1} = u(n+1,t)$, $n \in \mathbb{Z}$, $L^{\alpha}_t = D^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional differential operator of order $\alpha > 0$. Here $R(u_{n-1}, u_n, u_{n+1}, ...)$ refers the remaining part of the linear operator which may contain differential operator of order less than α while $N(u_{n-1}, u_n, u_{n+1}, ...)$ is the nonlinear term and s_n is the source term. Applying the operator I^{α} to both sides of (32) and using the property (30) along with the given initial conditions we get

$$u(n,t) = \sum_{r=0}^{n-1}\frac{u_n^{(r)}(0)}{r!}\ t^r - I^{\alpha}[R(u_{n-1}, u_n, u_{n+1}, ...) + N(u_{n-1}, u_n, u_{n+1}, ...)] + I^{\alpha}(s_n). \tag{33}$$

Let us assume that the nonlinear term $N(u_{n-1}, u_n, u_{n+1}, ...)$ be analytic and so it can be written as an infinite series of polynomials

$$N(u_{n-1}, u_n, u_{n+1}, ...) = \sum_{k=0}^{\infty} C_k, \tag{34}$$

where C_k are Adomian polynomials defined by the formula [9, 10]

$$C_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} N\left(\sum_{m=0}^{\infty} U_m(n,t)\lambda^m, \sum_{m=0}^{\infty} U_m(n+1,t)\lambda^m, \sum_{m=0}^{\infty} U_m(n-1,t)\lambda^m, ...\right)\right]_{\lambda=0}.$$

Substituting (4) along with (34) in (33), we get the required components of the decomposition series (4) as follows

$$\begin{aligned} U_0(n,t) &= \sum_{r=0}^{n-1} \frac{u_n^{(r)}(0)}{r!}\, t^r + I^\alpha(s_n), \\ U_{k+1}(n,t) &= -I^\alpha[R(U_k(n-1,t), U_k(n,t), U_k(n+1,t), ...) + C_k], \ k \geq 0, \end{aligned}$$

and solving them consistently give U_k's. Thus the series solution determined entirely. It is important to mention here that the obtained series may not lead to closed form solution always. However, for concrete problems, the m-term approximant ϕ_m solution defined by $\phi_m = \sum_{k=0}^{m-1} U_k(n,t), \ m \geq 1$ can be used for numerical computation.

4. Applications of ADM

4.1. **ADM: Time fractional ΔKdV.** Let us consider a time fractional ΔKdV given by

$$\frac{\partial^\alpha u_n}{\partial t^\alpha} = u_n(u_{n+1} - u_{n-1}), \ 0 < \alpha \leq 1, \ n \in \mathbb{Z}, \tag{35}$$

which can be written as

$$L_t^\alpha u_n = F(u_n, u_{n+1}) - G(u_n, u_{n-1}), \tag{36}$$

where $L_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}, \ F(u_n, u_{n+1}) = u_n u_{n+1}, \ G(u_n, u_{n-1}) = u_n u_{n-1}$. The nonlinear terms $F(u_n, u_{n+1})$ and $G(u_n, u_{n-1})$ can be expressed in terms of Adomian polynomials A_k's and B_k's as given in (6) and (7), (8) and (9). Applying the operator I^α on both sides of (36) along with the property(30) for $0 < \alpha \leq 1$, we obtain

$$u(n,t) = u(n,0) + I^\alpha(F(u_n, u_{n+1}) - G(u_n, u_{n-1})).$$

Substituting the decomposition series (4) along with equations (6), (7) in the above equation yields

$$\sum_{k=0}^{\infty} U_k(n,t) = u(n,0) + I^\alpha\left[\sum_{k=0}^{\infty}(A_k - B_k)\right],$$

which in turn gives

$$\begin{aligned} U_0(n,t) &= u(n,0), && (37) \\ U_{k+1}(n,t) &= I^\alpha[A_k - B_k], \ k \geq 0, && (38) \end{aligned}$$

and solving them consistently yields the components U_k of the decomposition series (4) when $\alpha \in (0,1]$. When $k = 0$, equation (38) reads

$$U_1(n,t) = I^\alpha[A_0 - B_0].$$

Using (27) along with the expressions A_0 and B_0, we write the above equation as

$$\begin{aligned} U_1(n,t) &= U(n,0)[U(n+1,0)-U(n-1,0)]\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds, \\ U_1(n,t) &= U(n,0)[U(n+1,0)-U(n-1,0)]\frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{39}$$

Now the second element in the decomposition series (4) reads

$$U_2(n,t) = I^\alpha[A_1 - B_1]. \tag{40}$$

Making use of the equation (39) in the expressions A_1 and B_1, we write

$$\begin{aligned} A_1 &= U_{n,0}U_{n+1,0}[U_{n+1,0}-U_{n-1,0}+U_{n+2,0}-U_{n,0}]\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ B_1 &= U_{n,0}U_{n-1,0}[U_{n+1,0}-U_{n-1,0}+U_{n,0}-U_{n-2,0}]\frac{t^\alpha}{\Gamma(\alpha+1)}, \end{aligned}$$

and so

$$\begin{aligned} A_1 - B_1 &= U_{n,0}\left[(U_{n+1,0}-U_{n-1,0})^2 - U_{n,0}(U_{n-1,0}+U_{n+1,0})\right. \\ &\quad \left. + U_{n+1,0}U_{n+2,0} + U_{n-2,0}U_{n-1,0}\right]\frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

For simplicity we write it as

$$A_1 - B_1 = \frac{f(n)t^\alpha}{\Gamma(\alpha+1)},$$

where

$$f(n) = U_{n,0}\left[(U_{n+1,0}-U_{n-1,0})^2 - U_{n,0}(U_{n-1,0}+U_{n+1,0}) + U_{n+1,0}U_{n+2,0} + U_{n-2,0}U_{n-1,0}\right],$$

and so equation (40) yields

$$\begin{aligned} U_2(n,t) &= I^\alpha\left(\tfrac{f(n)\ t^\alpha}{\Gamma(\alpha+1)}\right), \\ U_2(n,t) &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\tfrac{f(n)\ s^\alpha}{\Gamma(\alpha+1)}\ ds, \\ U_2(n,t) &= \tfrac{f(n)\ t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned}$$

In a similar manner we compute $U_3(n,t), U_4(n,t), \ldots$ and so the infinite series (4) becomes

$$u(n,t) = U_{n,0} + U_{n,0}[U_{n+1,0}-U_{n-1,0}]\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{f(n)\ t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots$$

which may not lead to closed form in general.

4.1.1. *Rational solution* $(0<\alpha<1)$. Here we explain how to derive rational solution of (35) satisfying the same initial condition (17)

$$U_{n,0} = u(n,0) = \frac{(n+1)(n-2)}{n(n-1)}.$$

Following the procedure outlined in the previous section with $\alpha = 0.5$, the approximate rational solution for (35) is given by

$$u(n,t) = \frac{(n+1)(n-2)}{n(n-1)} + \frac{4(2n-1)\ t^{\frac{1}{2}}}{n^2(n-1)^2\ \Gamma(\frac{3}{2})} - \frac{16(3n^2-3n+1)t}{n^3(n-1)^3} + \cdots \tag{41}$$

We would like to mention that the above series does not lead to closed form at the moment.

4.1.2. *One Soliton solution* $(0 < \alpha < 1)$. To derive approximate solution for time fractional ΔKdV expressed in terms of exponential functions, we consider the same initial condition (20)

$$U_{n,0} = u(n,0) = \frac{k(1 + e^{k(n-2)} + e^{k(n+1)} + e^{k(2n-1)})}{(e^k - e^{-k})(1 + e^{k(n)} + e^{k(n-1)} + e^{k(2n-1)})}. \tag{42}$$

The first component in the decomposition series is

$$\begin{aligned} U_{n,1} \quad = \quad & \frac{k^2\left(e^{k(n-2)} + 2e^{k(n-1)} + e^{k(3n-4)} + e^{k(n+2)} + 2e^{k(3n-1)} + e^{3kn}\right)t^\alpha}{(e^k - e^{-k})^2(1 + e^{kn})^2(1 + e^{k(n-1)})^2\Gamma(\alpha+1)} \\ & - \frac{k^2\left(e^{k(n+1)} + e^{k(n-3)} + 2e^{kn} + 2e^{k(3n-2)} + e^{k(3n+1)} + e^{3k(n-1)}\right)t^\alpha}{(e^k - e^{-k})^2(1 + e^{kn})^2(1 + e^{k(n-1)})^2\Gamma(\alpha+1)}. \end{aligned}$$

We would like to mention that the presence of the exponential terms in $U_{n,0}$ makes it complicated to evaluate the components $U_k(n,t), k \geq 2$ for a given $\alpha \in (0,1)$ analytically. Thus we have computed $U_k(n,t)$ with fixed values of α and k using *Maple or Mathematica* and plotted the obtained the approximate solution associated with one-soliton solution when $\alpha = 0.50$ and $\alpha = 0.80$ in Figure 2a and Figure 2b respectively.

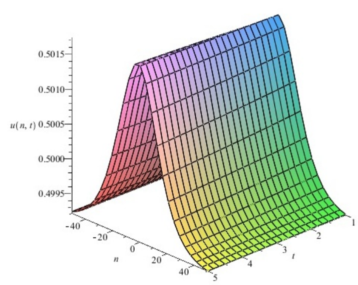

Figure 2a: Approximate soliton solution $u(n,t)$ of the time fractional ΔKdV with $k = 0.1$ and $\alpha = 0.5$

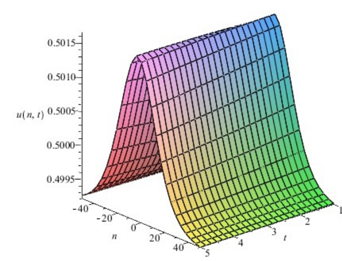

Figure 2b: Approximate soliton solution $u(n,t)$ of the time fractional ΔKdV with $k = 0.1$ and $\alpha = 0.8$

4.2. **ADM: Time fractional ΔmKdV equation.** We consider the time fractional ΔmKdV equation as

$$\frac{\partial^\alpha u_n}{\partial t^\alpha} = (1 + u_n^2)(u_{n+1} - u_{n-1}), \ 0 < \alpha \leq 1, \ n \in \mathbb{Z}, \tag{43}$$

which can be rewritten as

$$L_t^\alpha u_n = u_{n+1} - u_{n-1} + F(u_n, u_{n+1}) - G(u_n, u_{n-1}). \tag{44}$$

Applying the operator I^α on both sides of (44) and using the initial condition, we get

$$u(n,t) = u(n,0) + I^\alpha(u_{n+1} - u_{n-1}) + I^\alpha(F(u_n, u_{n+1}) - G(u_n, u_{n-1})). \tag{45}$$

The nonlinear terms $F(u_n, u_{n+1}),\ G(u_n, u_{n-1})$ can be expressed in terms of Adomian polynomials A_k's and B_k's as given in (6) and (7), (8) and (9). The required components of the decomposition series are obtained recursively as follows

$$\begin{aligned} U_0(n,t) &= u(n,0), && (46)\\ U_{k+1}(n,t) &= I^{\alpha}(U_k(n+1) - U_k(n-1)) + I^{\alpha}[A_k - B_k], k \geq 0. && (47)\end{aligned}$$

Using the above relation (47) along with initial condition $U_{n,0} = u(n,0) = \sinh(c)\ sech(cn)$ we compute the components $U_k, k > 0$ of (4). For example when $k = 0$ equation (47) becomes

$$U_1(n,t) = I^{\alpha}(U_{n+1,0} - U_{n-1,0}) + I^{\alpha}[A_0 - B_0].$$

Using (27) along with A_0 and B_0 we write the above equation as

$$\begin{aligned} U_1(n,t) &= (U_{n+1,0} - U_{n-1,0})(1 + U_{n,0}^2)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ ds,\\ U_1(n,t) &= (1 + U_{n,0}^2)\,(U_{n+1,0} - U_{n-1,0})\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}. && (48)\end{aligned}$$

Now the second element in the decomposition series (4) reads

$$U_2(n,t) = I^{\alpha}(U_{n+1,1} - U_{n-1,1}) + I^{\alpha}[A_1 - B_1]. \quad (49)$$

Making use of the equation (48) in the expressions A_1 and B_1, we write

$$\begin{aligned} A_1 - B_1 &= U_{n,0}\left[2(1+U_{n,0}^2)(U_{n+1,0} - U_{n-1,0})^2 + U_{n,0}(1+U_{n+1,0}^2)(U_{n+2,0} - U_{n,0})\right.\\ &\quad \left. -U_{n,0}(1+U_{n-1,0}^2)(U_{n,0} - U_{n-2,0})\right]\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

We write it as

$$A_1 - B_1 = \frac{h(n)\ t^{\alpha}}{\Gamma(\alpha+1)}, \quad (50)$$

where

$$\begin{aligned} h(n) &= U_{n,0}\left[2(1+U_{n,0}^2)(U_{n+1,0} - U_{n-1,0})^2 + U_{n,0}(1+U_{n+1,0}^2)(U_{n+2,0} - U_{n,0})\right.\\ &\quad \left. -U_{n,0}(1+U_{n-1,0}^2)(U_{n,0} - U_{n-2,0})\right].\end{aligned}$$

Similarly by using equation (48) we can write

$$U_{n+1,1} - U_{n-1,1} = \frac{g(n)\ t^{\alpha}}{\Gamma(\alpha+1)}, \quad (51)$$

where

$$g(n) = \left[(1+U_{n+1,0}^2)(U_{n+2,0} - U_{n,0}) - (1+U_{n-1,0}^2)(U_{n,0} - U_{n-2,0})\right].$$

Making use of the equations (50) and (51) in the above equation (49), we obtain

$$\begin{aligned} U_2(n,t) &= I^{\alpha}[g(n)\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}] + I^{\alpha}[h(n)\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}],\\ &= [g(n) + h(n)]\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}.\end{aligned}$$

In a similar manner we compute $U_3(n,t), U_4(n,t), \ldots$ and so the infinite series (4) becomes

$$u(n,t) = U_{n,0} + (1+U_{n,0}^2)\,(U_{n+1,0} - U_{n-1,0})\,\frac{t^{\alpha}}{\Gamma(\alpha+1)} + [g(n)+h(n)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots$$

Here also we have computed $U_k(n,t)$ and obtained approximate solution associated with both rational and one-soliton for a given $\alpha \in (0,1)$. Figures 3a and 3b represent one-soliton solution with the initial condition $u(n,0) = \sinh(c)\ sech(cn)$ when $c = 0.10,\ \alpha = 0.5$ and $c = 0.10,\ \alpha = 0.8$ respectively.

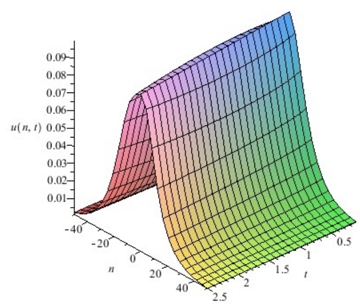

Figure 3a: Approximate soliton solution $u(n,t)$ of the time fractional ΔmKdV with $c = 0.1$ and $\alpha = 0.5$

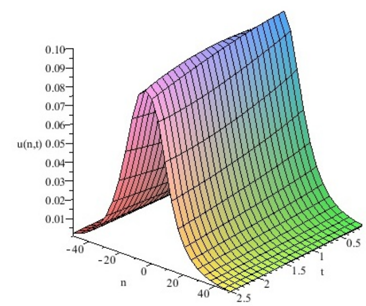

Figure 3b: Approximate soliton solution $u(n,t)$ of the time fractional ΔmKdV with $c = 0.1$ and $\alpha = 0.8$

4.3. **ADM: Time fractional Toda lattice equation.** In this subsection an attempt is made to extend ADM to two coupled time fractional PDΔE through the well known Toda lattice equation governed by

$$\frac{\partial^\alpha u_n}{\partial t^\alpha} = u_n(v_n - v_{n-1}), \tag{52}$$

$$\frac{\partial^\alpha v_n}{\partial t^\alpha} = v_n(u_{n+1} - u_n),\ 0 < \alpha \leq 1,\ n \in \mathbb{Z}, \tag{53}$$

which can be rewritten as

$$L_t^\alpha u_n = F(u_n, v_{n-1}, v_n), \tag{54}$$

$$L_t^\alpha v_n = G(u_n, u_{n+1}, v_n). \tag{55}$$

In addition to the infinite series solution (4), we assume the solution $v(n,t)$ of the form

$$v(n,t) = \sum_{k=0}^{\infty} V_k(n,t). \tag{56}$$

As before the nonlinear terms $F(u_n, v_{n-1}, v_n),\ G(u_n, u_{n+1}, v_n)$ can be written as an infinite series of Adomian polynomials, that is

$$F(u_n, v_{n-1}, v_n) = \sum_{k=0}^{\infty} A_k, \tag{57}$$

$$G(u_n, u_{n+1}, v_n) = \sum_{k=0}^{\infty} B_k. \tag{58}$$

The explicit expressions for A_k and B_k can be determined from the following formulae

$$A_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} F\left(\sum_{m=0}^{\infty} U_m(n,t)\lambda^m,\ \sum_{m=0}^{\infty} V_m(n-1,t)\lambda^m,\ \sum_{m=0}^{\infty} V_m(n,t)\lambda^m\right)\right]_{\lambda=0},$$

$$B_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} G\left(\sum_{m=0}^{\infty} U_m(n,t)\lambda^m,\ \sum_{m=0}^{\infty} U_m(n+1,t)\lambda^m,\ \sum_{m=0}^{\infty} V_m(n,t)\lambda^m\right)\right]_{\lambda=0}.$$

Applying the operator I^α on both sides of equations (54) and (55) along with the property (30) for $0 < \alpha \leq 1$, we obtain

$$u(n,t) = u(n,0) + I^\alpha(F(u_n, v_{n-1}, v_n)), \tag{59}$$

$$v(n,t) = v(n,0) + I^\alpha(G(u_n, u_{n+1}, v_n)). \tag{60}$$

Substituting the equations (4) and (56) along with (57) and (58) into the above equations (59) and (60) yield

$$\begin{aligned} U_{n,0} &= u(n,0), \\ V_{n,0} &= v(n,0), \\ U_{k+1}(n,t) &= I^\alpha[A_k], \ k \geq 0, \\ V_{k+1}(n,t) &= I^\alpha[B_k], \ k \geq 0. \end{aligned}$$

Following the procedure outlined in the previous subsection, one can compute the components U_k and V_k, $k \geq 0$ along with the initial condition [25]

$$\begin{aligned} U_{n,0} &= u(n,0) = -\coth(d)\ c + c\tanh(dn), \\ V_{n,0} &= v(n,0) = -\coth(d)\ c - c\tanh(dn), \end{aligned}$$

where c and d are constants. Proceeding further we obtain

$$\begin{aligned} U_1(n,t) &= U_{n,0}(V_{n,0} - V_{n-1,0})\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ V_1(n,t) &= V_{n,0}(U_{n+1,0} - U_{n,0})\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ U_2(n,t) &= \frac{f_1(n)\ t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ V_2(n,t) &= \frac{f_2(n)\ t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned}$$

where

$$\begin{aligned} f_1(n) &= U_{n,0}\left[(V_{n,0} - V_{n-1,0})^2 + V_{n,0}(U_{n+1,0} - U_{n,0}) - V_{n-1,0}(U_{n,0} - U_{n-1,0})\right], \\ f_2(n) &= V_{n,0}\left[(U_{n+1,0} - U_{n,0})^2 + U_{n+1,0}(V_{n+1,0} - V_{n,0}) - U_{n,0}(V_{n,0} - V_{n-1,0})\right]. \end{aligned}$$

In a similar manner we compute $U_k(n,t)$ and $V_k(n,t)$ for $k \geq 3$ and so the infinite series solutions (4) and (56) become

$$\begin{aligned} u(n,t) &= U_{n,0} + U_{n,0}(V_{n,0} - V_{n-1,0})\tfrac{t^\alpha}{\Gamma(\alpha+1)} + \tfrac{f_1(n)\ t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots, \\ v(n,t) &= V_{n,0} + V_{n,0}(U_{n+1,0} - U_{n,0})\tfrac{t^\alpha}{\Gamma(\alpha+1)} + \tfrac{f_2(n)\ t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \end{aligned}$$

It is straightforward to derive approximate solution for (52) and (53) associated with one-soliton solution satisfying the initial conditions. Here again we computed the approximate solutions $u(n,t)$ and $v(n,t)$ satisfying the initial conditions given above and displayed in Figure 4a and Figure 4b respectively.

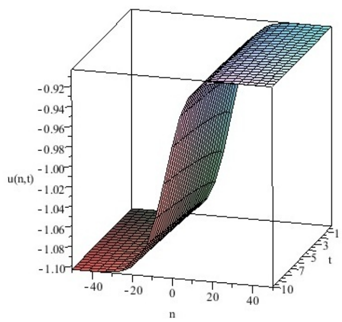

Figure 4a: Approximate solution $u(n,t)$ of the time fractional Toda lattice equation with $d = 0.1, c = 0.1$ and $\alpha = 0.5$

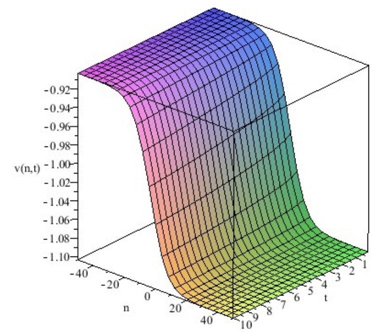

Figure 4b: Approximate solution $v(n,t)$ of the time fractional Toda lattice equation with $d = 0.1, c = 0.1$ and $\alpha = 0.5$

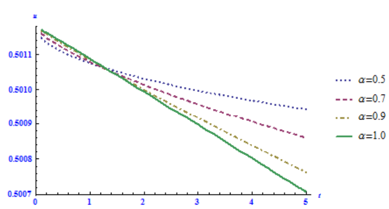

Figure 5a: Effect of α on the solution $u(n,t)$ of the time fractional ΔKdV with $k = 0.1$ and $n = 10$

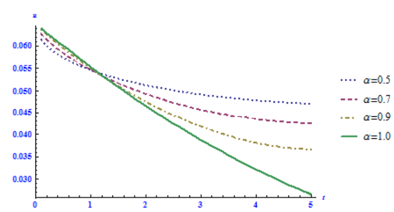

Figure 5b: Effect of α on the solution $u(n,t)$ of the time fractional ΔmKdV with $c = 0.1$ and $n = 10$

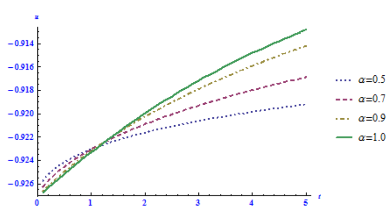

Figure 5c: Effect of α on the solution $u(n,t)$ of the time fractional Toda lattice equation with $d = 0.1$, $c = 0.1$ and $n = 10$

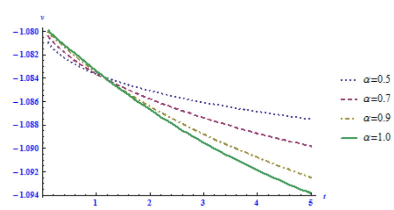

Figure 5d: Effect of α on the solution $v(n,t)$ of the time fractional Toda lattice equation with $d = 0.1$, $c = 0.1$ and $n = 10$

5. Summary and Discussion

In this article, an attempt is made to extend the ADM to time fractional nonlinear PDΔEs in general and time fractional ΔKdV equation, time fractional ΔmKdV equation and time fractional Toda lattice equation in particular and derived their approximate solutions. We would like to mention that the computations were performed by using Maple and Mathematica softwares. The obtained approximate solutions and the effect of time fractional order α are shown graphically.

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