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GENERALIZATIONS OF SOME QI TYPE INEQUALITIES USING FRACTIONAL INTEGRAL

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ABSTRACT. In this paper, we give some generalization to some Feng Qi type inequalities using Fractional Integral.

1. INTRODUCTION

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term fractional is a misnomer, but it is retained following the prevailing use. The fractional calculus may be considered an old and yet novel topic. According to the Riemann-Liouville approach to fractional calculus the notion of fractional integral of order α ($\alpha > 0$) is a natural consequence of the well known formula (usually attributed to Cauchy), that reduces the calculation of the n-fold primitive of a function f(t) to a single integral of convolution type. In our notation the Cauchy formula reads

$$J^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau,\ t > 0, n \in \mathbb{N}.$$

In a natural way one is led to extend the above formula from positive integer values of the index to any positive real values by using the Gamma function. Indeed, noting that (n - 1)! $= \Gamma(n)$, and introducing the arbitrary positive real number α , one defines the Fractional Integral of order $\alpha > 0$:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau,\ t > 0, \alpha \in \mathbb{R}^+.$$

In literature, fractional integration and fractional differentiation have been studied extensively by several researchers either in classical analysis or in the quantum one [2, 5, 7, 9].

In [10], Qi proposed the following problem, which has attracted much attention from some mathematicians [1, 2, 4, 5, 6, 8, 7, 11]:
Under what conditions does the inequality:

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$$\int_a^b [f(x)]^t dx \geq [\int_a^b f(x)dx]^{t-1} \tag{1}$$

hold for $t > 1$?

In this paper we establish generalization of some Qi type inequalities using fractional integral.

This paper is organized as follow: In section 2, we present some basic definitions that will be used later. In section 3, we establish some inequality using fractional integral.

2. Basic Definitions

Definition 1. *A real valued function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(]0, \infty[)$.*

Definition 2. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} f(\tau) d\tau; \quad \alpha > 0, \quad t > 0. \tag{2}$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.
For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t); \quad \alpha \geq 0, \quad \beta \geq 0, \tag{3}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{4}$$

3. Main results

Theorem 1. *Let $p \geq 1$ be a real number and f be a function defined on $[0, b]$ satisfying*

$$f(0) \geq 0 \quad \text{and} \quad f^{'}(x) \geq \frac{p b^{(\alpha-1)(p-1)}}{\alpha^{(p-1)}} (b-x)^{(\alpha-1)}; \quad x \in [0, b], \quad \alpha \geq 1. \tag{5}$$

Then we have

$$J^\alpha \left(f^{p+2}\right)(b) \geq \frac{[\Gamma(\alpha)]^p}{b^{(p-1)}} \left[J^\alpha(f)(b)\right]^{p+1}. \tag{6}$$

Proof. Put $g(x) = \displaystyle\int_0^x (b-t)^{(\alpha-1)} f(t) dt$ and

$$F(x) = \int_0^x (b-t)^{(\alpha-1)} f^{p+2}(t) dt - \frac{1}{b^{(p-1)}} \left[\int_0^x (b-t)^{(\alpha-1)} f(t) dt\right]^{p+1}, \quad x \in [0, b].$$

We have

$$\begin{aligned} F^{'}(x) &= (b-x)^{(\alpha-1)}f^{p+2}(x) - \frac{(p+1)}{b^{(p-1)}}g^{'}(x)g^{p}(x) \\ &= (b-x)^{(\alpha-1)}f^{p+2}(x) - \frac{(p+1)}{b^{(p-1)}}(b-x)^{(\alpha-1)}f(x)g^{p}(x) \\ &= (b-x)^{(\alpha-1)}f(x)\left[f^{p+1}(x) - \frac{(p+1)}{b^{(p-1)}}g^{p}(x)\right] \\ &= (b-x)^{(\alpha-1)}f(x)h(x), \end{aligned}$$

where $h(x) = f^{p+1}(x) - \dfrac{(p+1)}{b^{(p-1)}}g^{p}(x)$.
On the other hand, we have

$$\begin{aligned} h^{'}(x) &= (p+1)f^{'}(x)f^{p}(x) - \frac{p(p+1)}{b^{(p-1)}}g^{'}(x)\,[g(x)]^{p-1} \\ &= (p+1)f^{'}(x)f^{p}(x) - \frac{p(p+1)}{b^{(p-1)}}(b-x)^{(\alpha-1)}f(x)\,[g(x)]^{p-1} \\ &= (p+1)f(x)[f^{'}(x)f^{p-1}(x) - \frac{p}{b^{(p-1)}}(b-x)^{(\alpha-1)}\,[g(x)]^{p-1}\,. \end{aligned}$$

Since f increases, then for $x \in [0,b]$,

$$g(x) = \int_0^x (b-t)^{(\alpha-1)}f(t)dt \le f(x)\int_0^x (b-t)^{(\alpha-1)}dt \le f(x)\frac{b^{\alpha}}{\alpha}.$$

Therefore,

$$h^{'}(x) \ge (p+1)f^{p}(x)\left[f^{'}(x) - \frac{pb^{\alpha(p-1)}}{\alpha^{(p-1)}b^{(p-1)}}(b-x)^{(\alpha-1)}\right].$$

We deduce, from (5), that h increases on $[0,b]$.
Finally, since $h(0) = f^{p+1}(0) \ge 0$, then F increase and $F(b) \ge F(0) \ge 0$.
Which completes the proof. ■

Theorem 2. *Let f be a function defined on $[0,b]$ satisfying*

$$f(0) \ge 0 \quad and \quad f^{'}(x) \ge \frac{2}{p+1}(b-x)^{(\alpha-1)} \quad for \quad x \in [0,b], \quad \alpha \ge 1. \tag{7}$$

Then for all $p > 0$, we have

$$J^{\alpha}\left(f^{2p+1}\right)(b) \ge \Gamma(\alpha)\left[J^{\alpha}(f^{p})(b)\right]^{2}. \tag{8}$$

Proof. For $x \in [0,b]$, we define

$$H(x) = \int_0^x (b-t)^{(\alpha-1)}f^{2p+1}(t)dt - \left(\int_0^x (b-t)^{(\alpha-1)}f^{p}(t)dt\right)^2 and$$

$$g(x) = \int_0^x (b-t)^{(\alpha-1)}f^{p}(t)dt.$$

Clearly $H(0) = 0$ and
$H^{'}(x) = (b-x)^{(\alpha-1)}f^{2p+1}(x) - 2(b-x)^{(\alpha-1)}f^{p}(x)g(x) = (b-x)^{(\alpha-1)}f^{p}(x)\left[f^{p+1}(x) - 2g(x)\right].$
Setting
$G(x) = f^{p+1}(x) - 2g(x)$.

Then we have $G(0) = f^{p+1}(0) \geq 0$ and

$$G^{'}(x) = (p+1)f^{'}(x)f^{p}(x) - 2(b-x)^{(\alpha-1)}f^{p}(x) = f^{p}(x)\left[(p+1)f^{'}(x) - 2(b-x)^{(\alpha-1)}\right].$$

From the condition (3), we have $G^{'}(x) \geq 0$
and $G(0) \geq 0$, so we get $G(x) > 0$.
On the other hand $H(0) = 0$ and $H^{'}(x) \geq 0$ for all $x \in [0, b]$.
In particular

$$H(x) = \int_0^x (b-t)^{(\alpha-1)} f^{2p+1}(t)dt - \left(\int_0^x (b-t)^{(\alpha-1)} f^{p}(t)dt\right)^2 \geq 0,$$

and Theorem 1 complete the proof. ■

Corollary 1. *let $\beta > 0$ and f be a function defined on $[0, b]$ satisfying*

$$f(0) \geq 0 \quad and \quad f^{'}(x) \geq \frac{2}{p+1}(b-x)^{(\alpha-1)} \quad for \quad x \in [0, b], \quad \alpha \geq 1.$$

Then for all positive integer m, we have

$$J^{\alpha}\left(f^{(\beta+1)2^{m}-1}\right)(b) \geq \Gamma^{2^{m}-1}(\alpha)\left[J^{\alpha}(f^{\beta})(b)\right]^{2^{m}}. \tag{9}$$

Proof. We suggest here a proof by induction. For this purpose, We note

$$p_m(\beta) = (\beta+1)2^m - 1.$$

We have

$$p_m(\beta) > 0 \quad and \quad p_{m+1}(\beta) = 2p_m(\beta) + 1. \tag{10}$$

From theorem 2, we deduce that the inequality (9) is true for $m = 1$.
Suppose that (9) holds for an integer m and let us prove it for $m + 1$.
By using the relation (10) and theorem 2, we obtain

$$J^{\alpha}\left(f^{(\beta+1)2^{m+1}-1}\right)(b) \geq \Gamma(\alpha)\left[J^{\alpha}(f^{(\beta+1)2^{m}-1})(b)\right]^2. \tag{11}$$

And by assumption, we have

$$J^{\alpha}\left(f^{(\beta+1)2^{m}-1}\right)(b) \geq \Gamma^{2^{m}-1}(\alpha)\left[J^{\alpha}(f^{\beta})(b)\right]^{2^{m}}. \tag{12}$$

Finally, the relation (11) and (12) imply that the inequality (9) is true for $m + 1$.
This completes the proof. ■

Corollary 2. *Let f be a function defined on $[0, b]$ satisfying*

$$f(0) \geq 0 \quad and \quad f^{'}(x) \geq \frac{2}{p+1}(b-x)^{(\alpha-1)} \quad for \quad x \in [0, b], \quad \alpha \geq 1,$$

and $\beta > 0$. For $m \in \mathbb{N}$, we have

$$[J^{\alpha}(f^{(\beta+1)2^{m+1}-1})(b)]^{\frac{1}{2^{m+1}}} \geq [\Gamma(\alpha)]^{\frac{1}{2^{m+1}}}\left[J^{\alpha}(f^{(\beta+1)2^{m}-1})(b)\right]^{\frac{1}{2^{m}}}. \tag{13}$$

Proof. Since, from theorem 2,

$$J^\alpha\left(f^{(\beta+1)2^{m+1}-1}\right)(b) \geq \Gamma(\alpha)\left[J^\alpha(f^{(\beta+1)2^m-1})(b)\right]^2, \tag{14}$$

then

$$\left[J^\alpha\left(f^{(\beta+1)2^{m+1}-1}\right)(b)\right]^{\frac{1}{2^{m+1}}} \geq [\Gamma(\alpha)]^{\frac{1}{2^{m+1}}}\left[J^\alpha(f^{(\beta+1)2^m-1})(b)\right]^{\frac{1}{2^m}}. \tag{15}$$

■

Corollary 3. *Let f be a function defined on $[0,b]$ satisfying*

$$f(0) \geq 0 \quad and \quad f^{'}(x) \geq \frac{2}{p+1}(b-x)^{(\alpha-1)} \quad for \quad x \in [0,b], \quad \alpha \geq 1.$$

For all integer $m \geq 2$ we have

$$J^\alpha\left(f^{2^{m+1}-1}\right)(b) \geq [\Gamma(\alpha)]^{2^m-1}\left[J^\alpha f(b)\right]^{2^m}. \tag{16}$$

Proof. By using corollary 1 for $\beta = 1$, we obtain the result. ■

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