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AN APPROXIMATE ANALYTICAL SOLUTION OF COUPLED NONLINEAR FRACTIONAL DIFFUSION EQUATIONS

K. M. SAAD

Abstract. In recent years, fractional reaction-diffusion models have been studied due to their usefulness and importance in many areas of mathematics, statistics, physics, and chemistry. In a fractional diffusion equation, the second derivative in the spatial variable is replaced by a fractional derivative. The resulting solutions spread faster than classical solutions and may exhibit asymmetry, depending on the fractional derivative used. In this paper, a fractional exponential operator is considered as a general approach for solving partial fractional differential equations. We develop an approach for solving coupled nonlinear fractional diffusion equations with nonlinear source terms. These solutions will be evaluated numerically based on approximation analytical solutions. Comparisons between the approximate analytical solution and numerical solutions are shown.

1. Introduction

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in many areas of physics and engineering [33]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by differential equations of fractional order [5, 15, 22, 30]. Exact solutions of most fractional differential equations cannot be easily found. Thus analytical and numerical methods must be used. Some of the numerical methods for solving fractional differential equations were presented in [8, 11, 12, 13]. Recently, several mathematical methods including the Adomian decomposition method [14, 17], the variational iteration method [24, 35], the homotopy analysis method [10, 9, 13] and the fractional method [15] have been developed to obtain exact and approximate analytic solutions.

Numerical methods for solving variable order fractional differential equations with various kinds of the variable order fractional derivative have been proposed [4, 6, 26, 28, 27, 29]

The book by Oldham and Spanier [25] has played a key role in the development of the subject. Some fundamental results related to solving fractional differential equations may be found in [22, 15, 16]. Recently, several authors, for example [20,

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19, 31, 7, 18], have investigated the fractional diffusion/wave equation and its special properties. The fractional diffusion and wave equations have important applications to mathematical physics. Fractional diffusion equation describes diffusion in special types of porous media [23]. It is also used to model anomalous diffusion in plasma transport.

Merkin and Needham [21] considered the reaction-diffusion travelling waves that can develop in a coupled system involving simple isothermal autocatalysis kinetics. They assumed that reactions took place in two separate and parallel regions, with, in I, the reaction being given by quadratic autocatalysis

$$A + B \to 2B(rate\ k_1 ab), \tag{1}$$

together with a linear decay step

$$B \to C(rate\ k_2 b), \tag{2}$$

where a and b are the concentrations of reactant A and autocatalyst B, the $k_i(i = 1, 2)$ are the rate constants and C is some inert product of reaction. The reaction in region II was the quadratic autocatalytic step (1) only. The two regions were assumed to be coupled via a linear diffusive interchange of the autocatalytic species B. We shall consider a similar system as I, but with cubic autocatalysis

$$A + 2B \to 3B(rate\ k_3 ab^2) \tag{3}$$

together with a linear decay step

$$B \to C(rate\ k_4 b). \tag{4}$$

This leads to the system of equations (5)–(8).

Outline: Section 2 of this paper is devoted to the formulation of the approximate analytical solution for solving coupled nonlinear fractional diffusion equations with nonlinear source terms. We state several results that allow us to calculate the approximate analytical solution. In section 3, we will use the approximate analytic solution. In this section the governing equations are presented and Picard iteration is used to obtain the approximate analytical solution. Also, comparisons between the approximate analytical solution and numerical solutions of the governing partial differential equations (PDEs) are shown. Finally, we will evaluate the solution of the nonlinear fractional diffusion equations. Conclusions will be presented in Section 4.

2. Analytical framework

The following boundary value problem on $0 < x < \infty$ and $t > 0$ for the dimensionless concentrations (α_1, β_1) in region I and (α_2, β_2) in region II of species A and B is considered

$$\frac{\partial \alpha_1}{\partial t} = \frac{\partial^2 \alpha_1}{\partial x^2} - \alpha_1 \beta_1^2, \tag{5}$$

$$\frac{\partial \beta_1}{\partial t} = \frac{\partial^2 \beta_1}{\partial x^2} + \alpha_1 \beta_1^2 - k\beta_1 + \gamma(\beta_2 - \beta_1), \tag{6}$$

$$\frac{\partial \alpha_2}{\partial t} = \frac{\partial^2 \alpha_2}{\partial x^2} - \alpha_2 \beta_2^2, \tag{7}$$

$$\frac{\partial \beta_2}{\partial t} = \frac{\partial^2 \beta_2}{\partial x^2} + \alpha_2 \beta_2^2 + \gamma(\beta_1 - \beta_2), \tag{8}$$

with the boundary conditions

$$\alpha_i(0, t) = \alpha_i(L, t) = 1,\ \beta_i(0, t) = \beta_i(L, t) = 0. \tag{9}$$

The dimensionless constants k and γ represent the strength of the autocatalyst decay and the coupling between the two regions respectively. For convenience, we make the transformation $\alpha_i = 1 - u_i$, $i = 1, 2$ in (5)–(8), to obtain

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2} + (1-u_1)\beta_1^2, \tag{10}$$

$$\frac{\partial \beta_1}{\partial t} = \frac{\partial^2 \beta_1}{\partial x^2} + (1-u_1)\beta_1^2 - k\beta_1 + \gamma(\beta_2 - \beta_1), \tag{11}$$

$$\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} + (1-u_2)\beta_2^2, \tag{12}$$

$$\frac{\partial \beta_2}{\partial t} = \frac{\partial^2 \beta_2}{\partial x^2} + (1-u_2)\beta_2^2 + \gamma(\beta_1 - \beta_2). \tag{13}$$

The boundary conditions become

$$u_i(0,t) = u_i(L,t) = 0,\ \beta_i(0,t) = \beta_i(L,t) = 0. \tag{14}$$

In this work, we will develop the approach of [2, 1, 3] to find an analytical approximate solution of the following set of equations

$$\frac{\partial u_1}{\partial t} = \frac{\partial^\sigma u_1}{\partial x^\sigma} + (1-u_1)\beta_1^2, \tag{15}$$

$$\frac{\partial \beta_1}{\partial t} = \frac{\partial^\sigma \beta_1}{\partial x^\sigma} + (1-u_1)\beta_1^2 - k\beta_1 + \gamma(\beta_2 - \beta_1), \tag{16}$$

$$\frac{\partial u_2}{\partial t} = \frac{\partial^\sigma u_2}{\partial x^\sigma} + (1-u_2)\beta_2^2, \tag{17}$$

$$\frac{\partial \beta_2}{\partial t} = \frac{\partial^\sigma \beta_2}{\partial x^\sigma} + (1-u_2)\beta_2^2 + \gamma(\beta_1 - \beta_2). \tag{18}$$

We assume the Dirichlet boundary conditions are

$$u_i(0,t) = u_i(L,t) = 0,\ \beta_i(0,t) = \beta_i(L,t) = 0,\ 1 \le \sigma \le 2. \tag{19}$$

These equations are obtained from the original system (10)–(13) by replacing the second order space derivative by a fractional derivative. We propose a method to find the formal solution of (15)–(18) that satisfies the boundary conditions (19). This Picard iteration scheme generates a sequence of approximate solutions. This sequence of iterations is then truncated at the first approximation. We write the system of equations (15)–(18) as

$$\mathbf{U}_t = \hat{\mathbf{M}}\ \mathbf{U} + \mathbf{S}(\mathbf{U}), \tag{20}$$

$$\hat{\mathbf{M}} = \begin{pmatrix} \partial_x^\sigma & 0 & 0 & 0 \\ 0 & \partial_x^\sigma - k - \gamma & 0 & \gamma \\ 0 & 0 & \partial_x^\sigma & 0 \\ 0 & \gamma & 0 & \partial_x^\sigma - \gamma \end{pmatrix}, \tag{21}$$

$$\mathbf{S}(\mathbf{U}) = \begin{pmatrix} (1-u_1)\beta_1^2 & (1-u_1)\beta_1^2 & (1-u_2)\beta_2^2 & (1-u_2)\beta_2^2 \end{pmatrix}^T. \tag{22}$$

In (20) we consider the vector $\mathbf{S}$ as a source term. The method of integrating factors enables the first order differential equation (20) to be written as

$$\frac{d}{dt}\{e^{-t\hat{\mathbf{M}}}\mathbf{U}\} = e^{-t\hat{\mathbf{M}}}\mathbf{S}(\mathbf{U}). \tag{23}$$

Integrating from 0 to t then gives

$$\mathbf{U}(x,t) = \mathbf{U}_L(x,t) + \int_0^t e^{(t-\tau)\hat{\mathbf{M}}}\ \mathbf{S}(\mathbf{U}(x,\tau))d\tau, \mathbf{U}_L(x,t) = e^{t\hat{\mathbf{M}}}\mathbf{U}(x,0). \tag{24}$$

We require that $\mathbf{U}_L$ satisfies the boundary conditions $u_i(x=0,t) = u_i(x=L,t) = 0$ and $\beta_i(x=0,t) = \beta_i(x=L,t) = 0$. We now construct the Picard iteration sequence

$$\mathbf{U}^{(n)}(x,t) = \mathbf{U}^{(0)}(x,t) + \int_0^t e^{(t-\tau)\hat{\mathbf{M}}}\ \mathbf{S}_{n-1}(\mathbf{U}^{(n-1)}(x,\tau))d\tau, \quad n = 1,2,3,\ldots \tag{25}$$

The solution of the linear problem, the initial iterate, is given by

$$\mathbf{U}^{(0)}(x,t) = e^{t\hat{\mathbf{M}}}\mathbf{U}(x,0). \tag{26}$$

We take $\mathbf{U}^{(0)}(x,0)$ to satisfy the boundary conditions, namely

$$\mathbf{U}(x,0) = \sum_{n=1}^{\infty} \begin{pmatrix} a_n & b_n & c_n & d_n \end{pmatrix}^T (\sin(\bar{n}x) + \sin(\bar{n}(L-x))), \bar{n} = \frac{n\pi}{L}. \tag{27}$$

Thus the solution of equation (26) is

$$\mathbf{U}^{(0)}(x,t) = \sum_{n=1}^{\infty} e^{t\mathbf{M}} \begin{pmatrix} a_n & b_n & c_n & d_n \end{pmatrix}^T (\sin(\bar{n}x) + \sin(\bar{n}(L-x))), \tag{28}$$

$$\mathbf{M} = \begin{pmatrix} -\bar{n}^{\sigma} & 0 & 0 & 0 \\ 0 & -\bar{n}^{\sigma} - k - \gamma & 0 & \gamma \\ 0 & 0 & -\bar{n}^{\sigma} & 0 \\ 0 & \gamma & 0 & -\bar{n}^{\sigma} - \gamma \end{pmatrix}. \tag{29}$$

Calculating the matrix exponential e^{tM} using Mathematica gives

$$\mathbf{U}^{(0)}(x,t) = \sum_{0}^{\infty} \Psi_n(t)(\sin(\bar{n}x) + \sin(\bar{n}(L-x)))e^{-t(\bar{n})^{\sigma}}, \tag{30}$$

$$\Psi_n(t) = \begin{pmatrix} a_n \\ \frac{1}{2\mu}(b_n(k\lambda_- + \mu\lambda_+) - 2\gamma d_n\lambda_-) \\ c_n \\ \frac{1}{2\mu}(d_n(-k\lambda_- + \mu\lambda_+) - 2\gamma b_n\lambda_-) \end{pmatrix}, \tag{31}$$

$$\lambda_{\pm}(t) = (1 \pm e^{\mu t}), \quad \mu = \sqrt{k^2 + 4\gamma^2}. \tag{32}$$

The arbitrary constants a_n, b_n, c_n, and d_n are determined by the initial conditions namely on $[0, L]$. This is the same solution as that obtained in the Appendix using an eigenfunction expansion and in reference [34].

3. Approximate Analytical Solutions

Now, in the original variables, we have for the concentration $\alpha_i^{(0)}$, $\beta_i^{(0)}$

$$\alpha_1^{(0)}(x,t) = 1 - \sum_{n=1}^{\infty} a_n \sin(\frac{\bar{n}}{2}L)\cos\bar{n}(\frac{L}{2} - x)e^{-t(\bar{n})^{\sigma}}, \tag{33}$$

$$\beta_1^{(0)}(x,t) = \sum_{n=1}^{\infty} \frac{1}{2\mu}(b_n(k\lambda_- + \mu\lambda_+) - 2\gamma d_n\lambda_-)\sin(\frac{\bar{n}}{2}L)\cos\bar{n}(\frac{L}{2} - x)e^{-t(\bar{n})^{\sigma}}, \tag{34}$$

$$\alpha_2^{(0)}(x,t) = 1 - \sum_{n=1}^{\infty} c_n \sin(\frac{\bar{n}}{2}L)\cos\bar{n}(\frac{L}{2}-x)e^{-t(\bar{n})^{\sigma}}, \tag{35}$$

$$\beta_2^{(0)}(x,t) = \sum_{n=1}^{\infty} \frac{1}{2\mu}(d_n(-k\lambda_- + \mu\lambda_+) - 2\gamma b_n\lambda_-)\sin(\frac{\bar{n}}{2}L)\cos\bar{n}(\frac{L}{2}-x)e^{-t(\bar{n})^{\sigma}}. \tag{36}$$

By using the Picard sequence of solutions given by (25), the first approximation is given by

$$\mathbf{U}^{(1)}(x,t) = \mathbf{U}^{(0)}(x,t) + \int_0^t e^{(t-\tau)\hat{\mathbf{M}}}\ \mathbf{S}_0(\mathbf{U}^{(0)}(x,\tau))d\tau, \tag{37}$$

where

$$\mathbf{S}_0(\mathbf{U}^{(0)}(x,\tau)) = \begin{pmatrix} (1-u_1^{(0)})\beta_1^{(0)2} & (1-u_1^{(0))})\beta_1^{(0)2} & (1-u_2^{(0)})\beta_2^{(0)2} & (1-u_2^{(0)})\beta_2^{(0)2} \end{pmatrix}. \tag{38}$$

As $u_1^{(0)}, u_2^{(0)}, \beta_1^{(0)}$, and $\beta_2^{(0)}$ have Fourier series, so do $(1-u_1^{(0)})\beta_1^{(0)2}$ and $(1-u_2^{(0)})\beta_2^{(0)2}$. By using Fourier sine series to satisfy the boundary conditions, the first approximation is

$$\mathbf{U}^{(1)}(x,t) = \mathbf{U}^{(0)}(x,t) + \int_0^t \sum_{r=1}^{\infty} \Phi_r \sin(\bar{r})e^{-(t-\tau)(\bar{r})^{\sigma}}\ \mathbf{S}_0(\mathbf{U}^{(0)}(x,\tau))d\tau \tag{39}$$

and

$$\Phi_r = \frac{2}{L}\int_0^L \Psi_n(t-\tau)\mathbf{S}_0(\mathbf{U}^{(0)}(x,\tau))\sin(\bar{r}x)dx, \quad \bar{r} = \frac{r\pi}{L}. \tag{40}$$

4. Numerical result and discussion

In this section, we implement the proposed method to solve the coupled nonlinear fractional diffusion equations. The numerical and approximate analytical solutions of (5)–(9) for the special case $\sigma = 2$ are plotted against x through Figures 1–3 for various values of t and $\gamma = 0.1$, $k = 0.2$, $a_n = 0.1$, $b_n = 0.001$, $c_n = 0.2$, $d_n = 0.001$, and $L = 100$.

It is seen that the solutions are symmetric about the mid-plane $x = L/2$, as expected and evolve towards a parabola, as shown by [32]. We note from Figures 1–3 that as t increases, α_1 and α_2 tend to 1 and β_1 and β_2 tend to 0, which are the steady states for these quantities. These steady state solutions have been studied in reference [21].

It can be seen from Figures 1–3 that the absolute error obtained by the present method decreases as t increases. This is illustrated in Table 1. Therefore, the proposed method is an efficient and accurate method that can be used to provide approximate analytical solutions of coupled nonlinear diffusion equations. The approximate analytical solution of (15)–(18) then approaches the approximate analytical solution of the original equations (5)–(8) as $\sigma \to 2$. The analytical approximate results for $t = 30$, 50, 100 when $\sigma = 1.2, 1.5, 1.7, 2$ for the same parameter values as for Figures 1–3 are shown in Figures 4–6. The comparisons show that as $\sigma \to 2$, the analytical approximate solutions of equations (15)–(19) tend to the analytic approximate solutions of equations (5)–(9). In constrast to the solutions shown in Figures 1–3, the solutions in Figures 4–6 show that α_1 and α_2 tend to 1 and β_1 and β_2 tend to 0 as t increases. Furthermore, this evolution towards the steady states becomes faster as σ decreases.

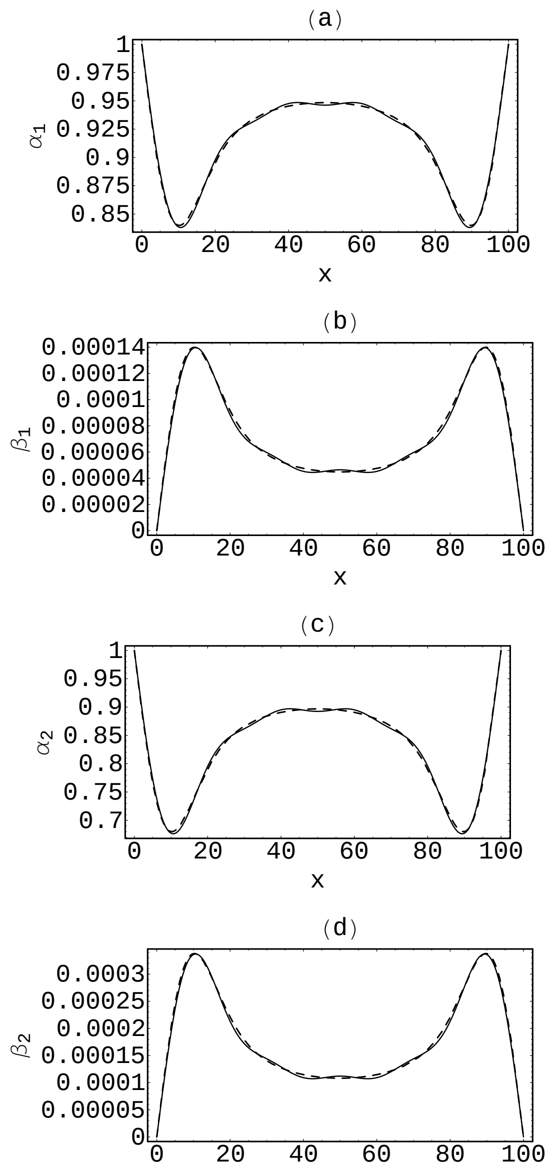

Figure 1. Comparison of numerical solutions with approximate analytic solution at $t = 30$ for (5-8) with $\sigma = 2$, $\gamma = 0.1, k = 0.2, a_n = 0.1, b_n = 0.001, c_n = 0.2, d_n = 0.001$, and $L = 100$.

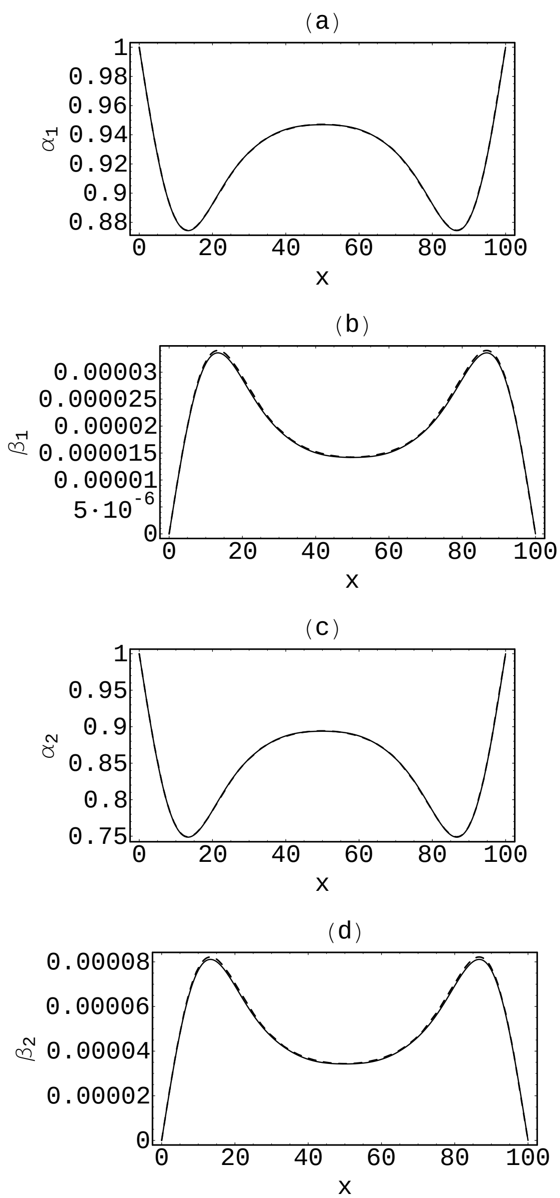

FIGURE 2. Comparison of numerical solutions with approximate analytic solution at $t = 50$ for (5-8) with $\sigma = 2$, $\gamma = 0.1, k = 0.2, a_n = 0.1, b_n = 0.001, c_n = 0.2, d_n = 0.001$, and $L = 100$.

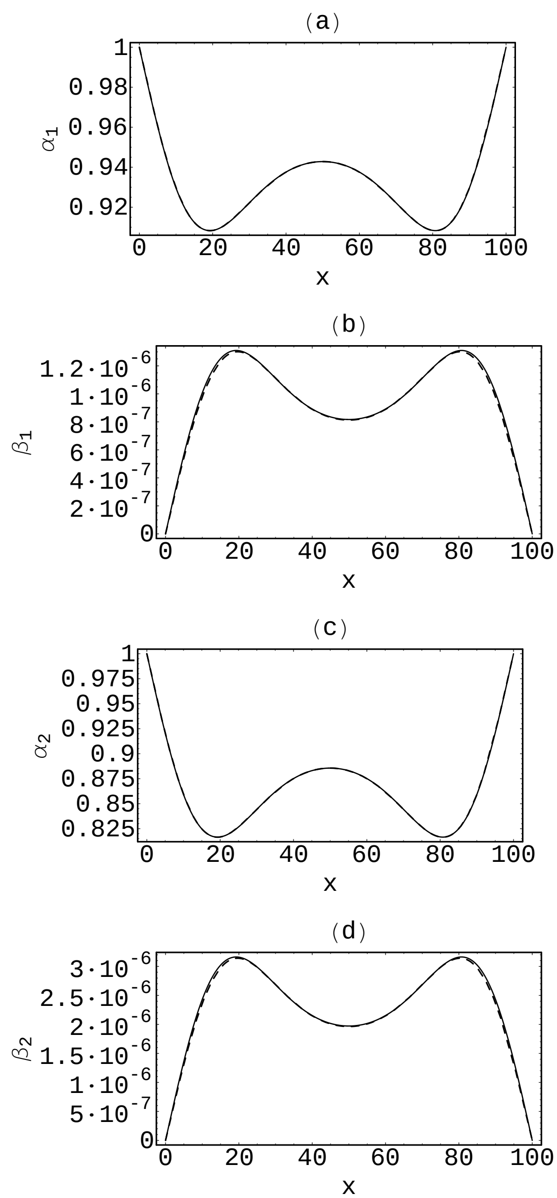

FIGURE 3. Comparison of numerical solutions with approximate analytic solution at $t = 100$ for (5-8) with $\sigma = 2$, $\gamma = 0.1, k = 0.2, a_n = 0.1, b_n = 0.001, c_n = 0.2, d_n = 0.001$, and $L = 100$.

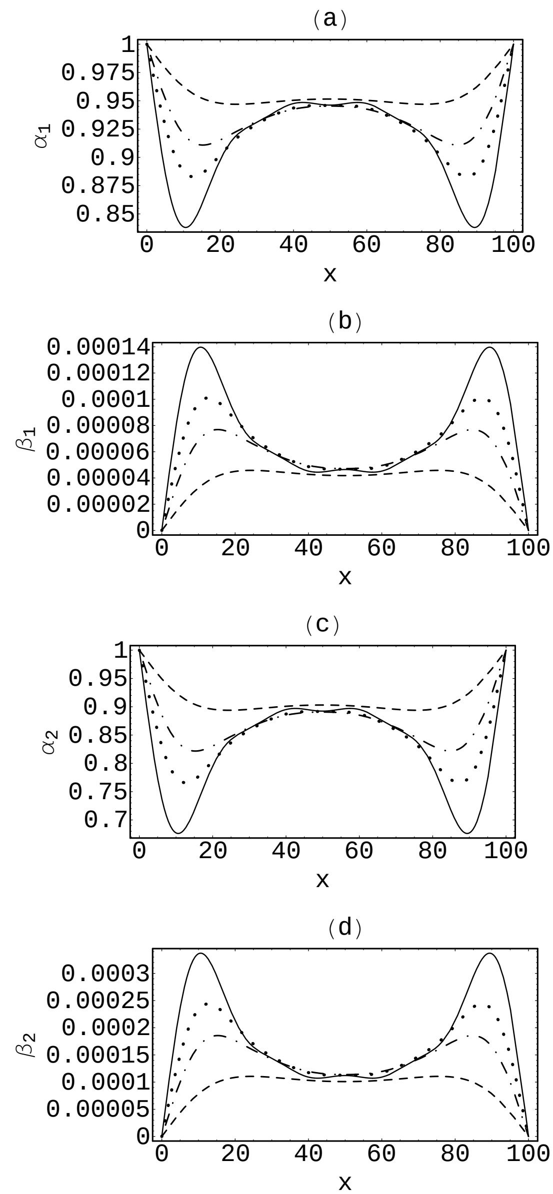

FIGURE 4. The approximate analytical solution for (15-19) with different values of $\sigma = 1.2(- - -), 1.5(- \cdot -), 1.7(\cdots), 2(-)$ when $t = 30$, $\gamma = 0.1, k = 0.2, a_n = 0.1, b_n = 0.001, c_n = 0.2, d_n = 0.001$, $L = 100$.

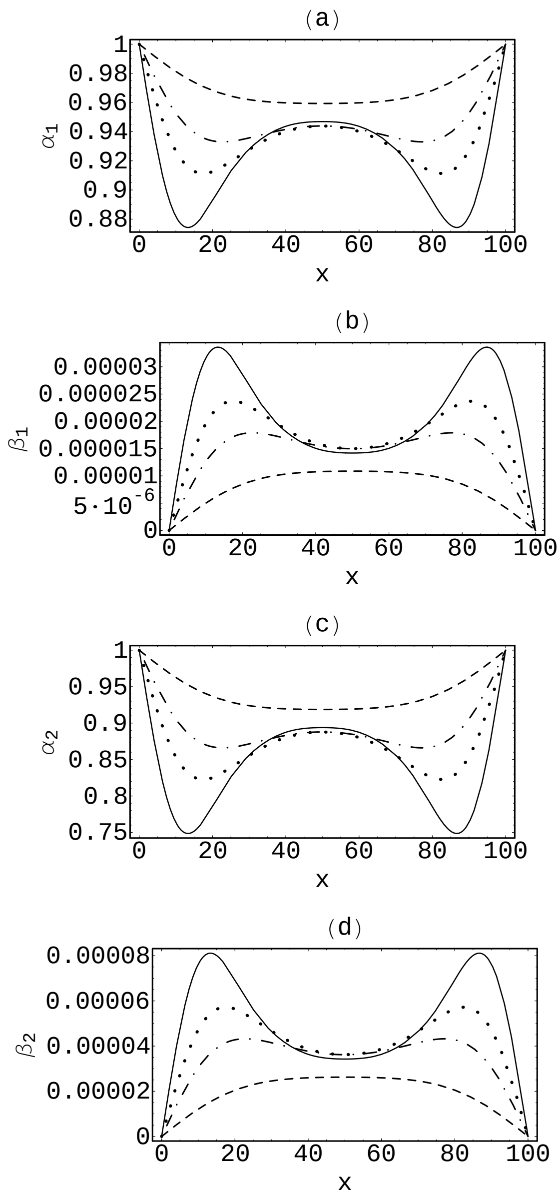

FIGURE 5. The approximate analytical solution for (15-19) with different values of $\sigma = 1.2(- - -), 1.5(- \cdot -), 1.7(\cdot\cdot\cdot), 2(-)$ when $t = 50$, $\gamma = 0.1, k = 0.2, a_n = 0.1, b_n = 0.001, c_n = 0.2, d_n = 0.001$, $L = 100$.

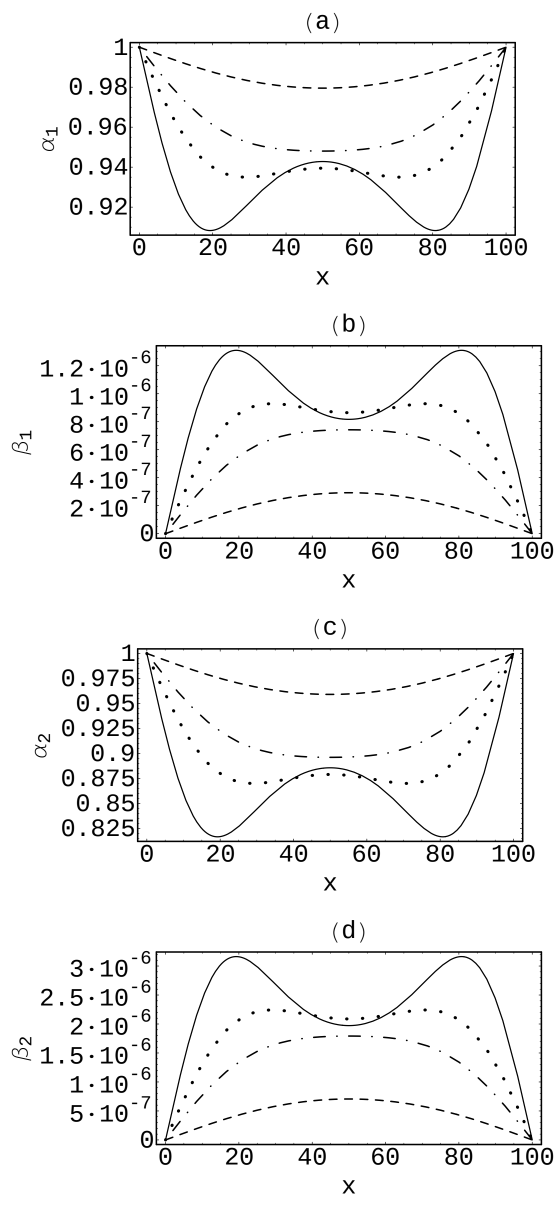

FIGURE 6. The approximate analytical solution for (15-19) with different values of $\sigma = 1.2(---), 1.5(-\cdot-), 1.7(\cdots), 2(-)$ when $t = 100,\ \gamma = 0.1, k = 0.2, a_n = 0.1, b_n = 0.001, c_n = 0.2, d_n = 0.001\ , L = 100$.

t	E_{α_1}	E_{β_1}	E_{α_2}	E_{β_2}
30	2.3×10^{-3}	3.5×10^{-2}	4.9×10^{-3}	3.4×10^{-2}
50	2.4×10^{-4}	3.4×10^{-3}	5.1×10^{-4}	3.6×10^{-2}
100	1.3×10^{-6}	4.4×10^{-3}	3.8×10^{-6}	4.47×10^{-3}

Table 1. The absolute error between numerical solutions with approximate analytic solutions at $x = 50$ for (5-8) with $\sigma = 2$, $\gamma = 0.1, k = 0.2, a_n = 0.1, b_n = 0.001, c_n = 0.2, d_n = 0.001$, and $L = 100$.

The absolute error of the approximate solutions as compared with the numerical solutions is summarised in Table 1. This table shows the error at $x = 50$ for $\sigma = 2$, $\gamma = 0.1$, $k = 0.2$, $a_n = 0.1$, $b_n = 0.001$, $c_n = 0.2$, $d_n = 0.001$ and $L = 100$. In the Table, E_{α_1}, E_{β_1}, E_{α_2} and E_{β_2} denote the absolute error for α_1, β_1, α_2 and β_2, respectively. A noted above, the error decreases as t increases, with the error in α_1 and α_2 decreasing more rapidly than that of β_1 and β_2. All the results were calculated by using the symbolic computation software Mathematica.

5. Conclusion

In this paper, the exponential operator and fractional exponential operator were applied to solve the systems of equations (5)–(9) and (15)–(19). Comparisons are made between approximate analytical solutions and numerical solutions for the system of equations (5)–(9) in order to illustrate the validity of the technique. The absolute error of the present approximate method decreases as t increases. The approximate analytical solutions for various value of σ were found. It was shown that these solutions approach the solutions of the original system (5)–(9) as $\sigma \to 2$.

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7. Appendix

We shall derive the solution of the first order differential equation (20) with homogeneous Dirichlet boundary conditions using an eigenfunction expansion. To this end we have the definition

Definition $\partial^2\varphi/\partial x^2$ has a complete set of orthonormal eigenfunctions φ_n with corresponding eigenvalues $\bar{n}^2$ on the bounded domain D. Hence

$$\frac{\partial^2\varphi}{\partial x^2} = -\bar{n}^2\varphi \tag{41}$$

on D with $B(\varphi) = 0$ on ∂D, where $B(\varphi)$ is one of the standard three homogeneous boundary conditions. Let

$$\mathcal{F}_\rho = \{\sum_{n=1}^{\infty} f = c_n\varphi_n, c_n = \langle f, \varphi_n\rangle, \sum_{n=1}^{\infty} |c_n|^2|\bar{n}|_n^\rho < \infty,\ \rho = max(\sigma, 0)\}. \tag{42}$$

Then for any $f \in \mathcal{F}_\rho$ is defined by

$$\frac{\partial^\sigma}{\partial\, x^\sigma} f = \sum_{n=1}^{\infty} -c_n (\bar{n})^\sigma \varphi_n. \tag{43}$$

Let us now set the solution of (20) in the form

$$\mathbf{U(x,t)} = \sum_{n=1}^{\infty} \mathbf{V}(t)(\sin(\bar{n}x)+\sin(\bar{n}(L-x))), \mathbf{V}(t) = \begin{pmatrix} v_1(t) & v_2(t) & v_3(t) & v_4(t) \end{pmatrix}^T. \tag{44}$$

This solution automatically staisfies the boundary conditions $u_i = 0$ at $x = 0, L$ and $\beta_i = 0$ at $x = 0, L$. Substituting this solution form into the differential equation (20) results in

$$\{\frac{d\mathbf{V}(t)}{dt} - \mathbf{MV}(t)\}(\sin(\bar{n}x) + \sin(\bar{n}(L-x))) = 0. \tag{45}$$

Here M is the matrix (29). V is the solution of

$$\frac{d\mathbf{V}(t)}{dt} - M\mathbf{V}(t) = 0 \tag{46}$$

which gives

$$\mathbf{V}(t) = e^{tM}\mathbf{V}(0), \tag{47}$$

$\mathbf{V}(0)$ is then obtained from the initial condition

$$\mathbf{U(x)} = \sum_{n=1}^{\infty} \mathbf{V}(0)(\sin(\bar{n}x) + \sin(\bar{n}(L-x))) \tag{48}$$

giving

$$\mathbf{V}(0) = \frac{2}{L}\int_0^L \mathbf{U(y)}(\sin(\bar{n}y) + \sin(\bar{n}(L-y)))dy = \begin{pmatrix} a_n & b_n & c_n & d_n \end{pmatrix}^T. \tag{49}$$

The solution of the differential equation (20) is finally

$$\mathbf{U}(x,t) = \sum_{n=1}^{\infty} e^{tM} \begin{pmatrix} a_n & b_n & c_n & d_n \end{pmatrix}^T (\sin(\bar{n}x) + \sin(\bar{n}(L-x))) \tag{50}$$

The matrix exponential e^{tM} is expanded using Mathematica to give

$$\mathbf{U}(x,t) = \sum_{n=1}^{\infty} \Psi_n(t)(\sin(\bar{n}x) + \sin(\bar{n}(L-x)))e^{-t(\bar{n})^\sigma}, \tag{51}$$

where $\Psi_n(t)$ are given by (31).

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K. M. Saad
Faculty of Arts and Science, Najran University, Najran, Saudi Arabia. On leave from Taiz University , Yemen
E-mail address: `khaledma_sd@hotmail.com`