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EXISTENCE OF POSITIVE SOLUTIONS FOR SEMI-POSITONE FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we present some results about the existence of positive solutions for nonlinear semipositone fractional boundary value problems by using Krasnoselskii's fixed point theorem.

1. Introduction

In the last few years, fractional differential equations have been studied extensively because modeling capabilities in engineering, science, economy, and other fields; see [3, 6, 7] for a good overview.

Many researchers are interested by the subject of the existence, uniqueness and non existence of solutions for fractional boundary value problems, see [1, 2, 5, 10, 12] and references therein.

In 2007, M. El-Shahed considered the fractional boundary value problem (see [1])

$$\begin{cases} D_{0^+}^{\alpha}u(t) + \lambda a(t)f(u(t)) = 0, \ 0 < t < 1, \ 2 < \alpha < 3, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative. Existence and non existence of positive solutions are obtained by means of Krasnoselskii's fixed point theorem.

Zhou *et al.* [12] studied the following fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t) = p(t)f(t, u(t)) - q(t), \ 0 < t < 1, \ 2 < \alpha < 3, \\ u(0) = u(1) = u'(1) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative. By using the Krasnoselskii's fixed point theorem, results on multiplicity of positive solutions are presented. Also we note that there is a current interest in questions of positive

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properties of Green's function and the existence of positive solutions of semipositone boundary value problems, one may see [8]-[12] and references therein.

Motivated by the works above, in this paper we study the following boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0,\ 0<t<1,\ 2<\alpha<3, \\ u(0)=u'(0)=u'(1)=0, \end{cases} \tag{1}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, and the nonlinear continuous function $f:[0,1]\times\mathbb{R}^{+}\longrightarrow\mathbb{R}$ is *semipositone*; i.e., the nonlinerity $f(t,u)$ may change sign. We prove some new existence results by using Krasnoselskii's fixed-point theorem.

The paper is organized as follows: In Section 2, we present the necessary definitions and we give some preliminary results that will be used in the proof of the main result. In Section 3, we establish the existence of the positive solutions for the boundary value problem (1) via Krasnoselskii's fixed point theorem, while some extensions of these results are given in Section 4. At the end, in Section 5, we give an example to illustrate our main result.

2. Preliminaries

For the reader's convenience, we present some necessary definitions from fractional calculus theory and lemmas. They can be found in [3, 6, 7].

Definition 2.1. *The Riemann-Liouville fractional integral of order q of a function $g\in L^{1}((0,1),\mathbb{R})$ is defined as*

$$I_{0+}^{q}g(t)=\frac{1}{\Gamma(q)}\int_{0}^{t}\frac{g(s)}{(t-s)^{1-q}}ds,\ q>0,$$

where Γ is the Gamma function.

Definition 2.2. *For a continuous function $g:(0,+\infty)\to\mathbb{R}$, the Riemann-Liouville derivative of fractional order q is defined as*

$$D_{0+}^{q}g(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-s)^{n-q-1}g(s)ds,\ n-1<q<n,\ n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q, provided the right-hand side is point-wise defined on $(0,+\infty)$.

Lemma 2.3. *(see [3]) Let $q>0$, if we assume $x\in C(0,1)\cap L^{1}(0,1)$, then the fractional differential equation $D_{0+}^{q}x(t)=0$ has*

$$x(t)=c_{1}t^{q-1}+c_{2}t^{q-2}+\ldots+c_{N}t^{q-N},\quad c_{i}\in\mathbb{R},\ i=1,2,\ldots,N,$$

as unique solutions, where N is the smallest integer greater than or equal to q.

In view of Lemma 2.3, it follows that

Lemma 2.4. *(see [3]) Assume that $x\in C(0,1)\cap L^{1}(0,1)$ with a fractional derivative of order $q>0$ that belongs to $C(0,1)\cap L^{1}(0,1)$. Then*

$$I_{0+}^{q}\ D_{0+}^{q}x(t)=x(t)+c_{1}t^{q-1}+c_{2}t^{q-2}+\ldots+c_{N}t^{q-N} \tag{2}$$

for some $c_{i}\in\mathbb{R},\ i=0,1,2,\ldots,N$, where N is the smallest integer greater than or equal to q.

The following two lemmas was proved in [1].

Lemma 2.5. *Let $g : [0,1] \to \mathbb{R}$ be a given continuous function. Then a unique solution of the boundary value problem*

$$\begin{cases} D^{\alpha}_{0^+} x(t) + g(t) = 0, \ 0 < t < 1, \ 2 < \alpha < 3, \\ x(0) = x'(0) = x'(1) = 0, \end{cases} \tag{3}$$

is given by

$$x(t) = \int_0^1 G(t,s) g(s) ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-2} t^{\alpha-1}, & \text{if } \ 0 \le t \le s \le 1, \\ (1-s)^{\alpha-2} t^{\alpha-1} - (t-s)^{\alpha-1}, & \text{if } \ 0 \le s \le t \le 1. \end{cases} \tag{4}$$

Lemma 2.6. $G(t,s) \ge q(t) G(1,s)$, *where* $q(t) = t^{\alpha-1}$ *for* $0 \le t, s \le 1$.

It is obvious that

$$G(t,s) \ge 0, \qquad G(1,s) \ge G(t,s), \qquad 0 \le t, \ s \le 1.$$

To prove the main result, we need the following well-known fixed point theorem of cone expansion and compression of norm type due to Krasnoselskii. Before state it, we give the following definition.

Definition 2.7. *Let E be a real Banach space. A nonempty closed set $K \subset E$ is said to be a cone provided that*
$au + bv \in K$ *for all* $u, v \in K$ *and all* $a \ge 0, \ b \ge 0,$ *and*
$u, \ -u \in K$ *implies* $u = 0$.

Theorem 2.8. [4]. *Let E be a Banach space and let $K \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open bounded subsets of E, with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let*

$$F : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$$

be completely continuous operator. In addition suppose that either

(I): $\|Fu\| \le \|u\|, \forall u \in K \cap \partial\Omega_1$ *and* $\|Fu\| \ge \|u\|, \forall u \in K \cap \partial\Omega_2$, *or*
(II): $\|Fu\| \le \|u\|, \forall u \in K \cap \partial\Omega_2$ *and* $\|Fu\| \ge \|u\|, \forall u \in K \cap \partial\Omega_1$

holds. Then F has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

3. Main Result

Consider the Banach space $E = C([0,1], \mathbb{R})$ endowed with the norm

$$\|u\| = \max_{t \in [0,1]} |u(t)|.$$

We define the cone K in the Banach space E by

$$K = \{u \in E, \ u(t) \ge q(t)\|u\|, \ t \in [0,1]\}, \ \text{where} \ q(t) = t^{\alpha-1}.$$

We need in the sequel the following lemma.

Lemma 3.1. *Let $x_1(t)$ be the unique solution of the following boundary value problem boundary value problem*

$$\begin{cases} D^{\alpha}_{0^+}x(t)+1=0,\ 0<t<1,\ 2<\alpha<3, \\ x(0)=x'(0)=x'(1)=0. \end{cases} \tag{5}$$

Then

$$x_1(t)\le L\cdot q(t), \qquad t\in[0,1],$$

where $q(t)=t^{\alpha-1}$ and $L=\dfrac{1}{(\alpha-1)\Gamma(\alpha)}$.

Proof. Using the Green function (4) by Lemma 2.5 we have

$$\begin{aligned} x_1(t) &= \int_0^1 G(t,s)ds \\ &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}ds-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\ &\le \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}ds \\ &= \frac{1}{(\alpha-1)\Gamma(\alpha)}t^{\alpha-1}. \end{aligned}$$

The proof is complete. □

Our main result in this section is

Theorem 3.2. *Let $f:[0,1]\times\mathbb{R}^+\longrightarrow\mathbb{R}$ be a continuous function. Assume that:*

(C1) $\lim\limits_{x\to\infty}\dfrac{f(t,x)}{x}=\infty$ *uniformly on $t\in[\sigma,1-\sigma]$ for $\sigma\in\left(0,\frac{1}{2}\right)$;*

(C2) *there exists $M>0$ such that $f(t,x)\ge -M$, for all $t\in[0,1]$, and all $x\ge 0$.*

If there exist $\lambda>0, r>0$ such that

$$0<\lambda\le\min\left\{\frac{r}{\widehat{f}_r\|x_1\|},\frac{r}{LM}\right\}, \qquad \widehat{f}_r=\sup_{t\in[0,1],0\le x\le r}[f(t,x)+M], \tag{6}$$

where x_1 is the unique solution of the boundary value problem (5), then the boundary value problem (1) has a positive solution.

Proof. Let

$$x(t)=\lambda M x_1(t),$$

where x_1 is the unique solution of the boundary value problem (5). We shall show that the following approximately boundary value problem

$$\begin{cases} D^{\alpha}_{0^+}u(t)+\lambda\widetilde{f}(t,u(t)-x(t))=0,\ 0<t<1,\ 2<\alpha<3, \\ u(0)=u'(0)=u'(1)=0, \end{cases} \tag{7}$$

where

$$\widetilde{f}(t,z)=\begin{cases} f(t,z)+M,\ z\ge 0, \\ f(t,0)+M,\ z\le 0, \end{cases}$$

has a positive solution u^*.

In view of Lemma 2.5 we define the operator $\widetilde{F} : E \to E$ by

$$\widetilde{F}u(t) = \int_0^1 \lambda G(t,s)\widetilde{f}(s, u(s) - x(s))ds. \tag{8}$$

We shall prove that $\widetilde{F}$ has a fixed point in our cone K. Firstly we prove that $\widetilde{F} : K \longrightarrow K$. For any $u \in K$, we note that $\widetilde{F}u(t)$ is continuous on $[0,1]$, and since $G(t,s) \geq 0$ we have $G(1,s) \geq 0$. Then, by Lemma 2.5 we obtain

$$\begin{aligned}
\widetilde{F}u(t) &= \int_0^1 \lambda G(t,s)\widetilde{f}(s, u(s) - x(s))ds \\
&\geq t^{\alpha-1} \int_0^1 \lambda G(1,s)\widetilde{f}(s, u(s) - x(s))ds \\
&\geq t^{\alpha-1} \max_{t\in[0,1]} \int_0^1 \lambda G(t,s)\widetilde{f}(s, u(s) - x(s))ds \\
&= q(t)\|\widetilde{F}u\|, \quad \forall t, s \in [0,1].
\end{aligned}$$

Thus $\widetilde{F}(K) \subset K$. Then from the definition of $\widetilde{F}$, it is easy to prove that $\widetilde{F}$ is a completely continuous operator. The continuity of $\widetilde{F}$ is obvious by the continuity of the nonlinear function f. By using the Arzelà-Ascoli theorem, we can prove that the operator $\widetilde{F}$ is compact. Then, $\widetilde{F} : K \to K$ is a completely continuous, and each fixed point of $\widetilde{F}$ in K is a solution of boundary value problem (7).

We define the ball Ω_1 in the Banach space E by

$$\Omega_1 = \{u \in E, \ \ \|u\| < r\}.$$

For $u \in K \cap \partial\Omega_1$, we have $0 \leq u(t) \leq \|u\| = r$ for $t \in [0,1]$ and by (6) we have

$$\begin{aligned}
\widetilde{F}u(t) &= \int_0^1 \lambda G(t,s)\widetilde{f}(s, u(s) - x(s))ds \\
&\leq \lambda \widehat{f}_r \int_0^1 G(t,s)ds \\
&= \lambda \widehat{f}_r x_1(t) \\
&\leq \lambda \widehat{f}_r \|x_1\| \\
&\leq r.
\end{aligned}$$

Therefore, we get $\|\widetilde{F}u\| \leq r = \|u\|$ for $u \in K \cap \partial\Omega_1$.

For $\sigma \in \left(0, \frac{1}{2}\right)$ fixed. Let k be a positive real number such that

$$\frac{1}{2}\lambda k B \left(\inf_{t\in[\sigma,1-\sigma]} q(t)\right) > 1, \tag{9}$$

where

$$B = \frac{1}{(\alpha-1)\Gamma(\alpha+1)}. \tag{10}$$

In view of $(C1)$, there exists $A > 0$ such that for all $z \geq A$ and $t \in [\sigma, 1-\sigma]$

$$\widetilde{f}(t,z) = f(t,z) + M \geq kz. \tag{11}$$

Now, set

$$R = r + \max\left\{2\lambda ML, 2A\left(\inf_{t\in[\sigma,1-\sigma]} q(t)\right)^{-1}\right\}. \tag{12}$$

Let us define the ball Ω_2 in the Banach space E by

$$\Omega_2 = \{u \in E,\ \ \|u\| < R\}.$$

We shall prove that $\|\widetilde{F}u\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$. Let $u \in K \cap \partial\Omega_2$. Then, $\|u\| = R$. Using Lemma 3.1 and the fact that $u \in K$ we get for $t \in [0,1]$

$$x(t) = \lambda M x_1(t) \leq \lambda MLq(t) \leq \lambda ML\frac{u(t)}{R}.$$

Thus, for $t \in [0,1]$

$$u(t) - x(t) \geq \left(1 - \lambda ML\frac{1}{R}\right)u(t) \geq \left(1 - \lambda ML\frac{1}{R}\right)Rq(t),$$

and, by (12), it follows that for $t \in [\sigma, 1-\sigma]$

$$u(t) - x(t) \geq \left(1 - \lambda ML\frac{1}{R}\right)R\left(\inf_{t\in[\sigma,1-\sigma]} q(t)\right) \geq \frac{1}{2}R\left(\inf_{t\in[\sigma,1-\sigma]} q(t)\right) \geq A.$$

Hence, by (11), we see that for $t \in [\sigma, 1-\sigma]$

$$\widetilde{f}(t, u(t) - x(t)) \geq k(u(t) - x(t)) \geq \frac{1}{2}kR\left(\inf_{t\in[\sigma,1-\sigma]} q(t)\right).$$

Then, by Lemma 2.6 and (9), we find

$$\begin{aligned}
\|\widetilde{F}u\| &= \max_{t\in[0,1]}\int_0^1 \lambda G(t,s)\widetilde{f}(s, u(s) - x(s))ds \\
&\geq \frac{1}{2}kR\left(\inf_{t\in[\sigma,1-\sigma]} q(t)\right)\lambda \max_{t\in[0,1]}\int_0^1 q(t)G(1,s)ds \\
&\geq \left[\frac{1}{2}\lambda kB\left(\inf_{t\in[\sigma,1-\sigma]} q(t)\right)\right]R \\
&\geq R.
\end{aligned}$$

Then, we get $\|\widetilde{F}u\| \geq R = \|u\|$ for $u \in K \cap \partial\Omega_2$. Therefore assertion (I) of Theorem 2.8 is satisfied. Then $\widetilde{F}$ has a fixed point $u^* \in K$ which satisfies $r \leq \|u^*\| \leq R$.

Furthermore, using (6) and Lemma 3.1, we get for $t \in [0,1]$,

$$u^*(t) \geq q(t)\|u^*\| \geq rq(t) \geq \lambda MLq(t) \geq \lambda M x_1(t) = x(t).$$

Therefore, for $t \in [0,1]$ we have

$$v^*(t) := u^*(t) - x(t) \geq 0.$$

Now we shall prove that v^* is in fact a positive solution of our problem (1). To see this we have for $t \in [0,1]$, u^* is a fixed point of the operator $\widetilde{F}$. Then

$$\begin{aligned}
u^*(t) &= \widetilde{F}u^*(t) \\
&= \int_0^1 \lambda G(t,s)\widetilde{f}(s, u^*(s) - x(s))ds \\
&= \int_0^1 \lambda G(t,s)[f(s, u^*(s) - x(s)) + M]ds
\end{aligned}$$

$$\begin{aligned} &= \int_0^1 \lambda G(t,s) f(s, u^*(s) - x(s)) ds + \lambda M \int_0^1 G(t,s) ds \\ &= \int_0^1 \lambda G(t,s) f(s, u^*(s) - x(s)) ds + \lambda M x_1(t) \\ &= \int_0^1 \lambda G(t,s) f(s, u^*(s) - x(s)) ds + x(t). \end{aligned}$$

This implies that

$$\begin{aligned} v^*(t) &= u^*(t) - x(t) \\ &= \int_0^1 \lambda G(t,s) f(s, u^*(s) - x(s)) ds \\ &= \int_0^1 \lambda G(t,s) f(s, v^*(s)) ds. \end{aligned}$$

Consequently, by Lemma 2.5, it is easy to see that v^* is a positive solution of our boundary value problem (1). This completes the proof. □

4. Some Extensions

In this section we give some extensions of the result proved in the previous section. We need the following lemma:

Lemma 4.1. *Let $x_2(t)$ be the unique solution of the following boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} x(t) + \lambda M(t) = 0, \ 0 < t < 1, \ 2 < \alpha < 3, \\ x(0) = x'(0) = x'(1) = 0. \end{cases} \tag{13}$$

Then

$$x_2(t) \le \lambda\theta \cdot q(t), \qquad t \in [0,1],$$

where $q(t) = t^{\alpha-1}$ and

$$\theta = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} M(s) ds. \tag{14}$$

Proof. Using the Green function (4) by Lemma 2.5 we have

$$\begin{aligned} x_2(t) &= \int_0^1 \lambda G(t,s) M(s) ds \\ &= \int_0^1 \lambda \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} M(s) ds - \int_0^t \lambda \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} M(s) ds \\ &\le \lambda t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} M(s) ds \\ &= \lambda\theta \cdot q(t). \end{aligned}$$

The proof is complete. □

Theorem 4.2. *Let $f : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function satisfying (C1). Moreover we assume that:*

(A1) *For any* $(t,x) \in [0,1] \times \mathbb{R}_+^*$, $f(t,x)$ *satisfies*

$$-M(t) \le f(t,x) \le p(t)\psi(x),$$

with $M, p \in C([0,1], \mathbb{R}_+^*)$ *and* $\psi \in C(\mathbb{R}_+, \mathbb{R}_+^*)$.

If there exists $\lambda > 0$ *such that*

$$0 < \lambda \le \min\left\{1, \frac{\theta}{\widehat{\psi}_\theta \int_0^1 G(1,s)[p(s)+M(s)]ds}\right\}, \quad \widehat{\psi}_\theta = \max_{0\le\tau\le\theta}\{1, \psi(\tau)\}, \tag{15}$$

where θ *is represented in (14), then the boundary value problem (1) has a positive solution.*

Proof. To prove this result, we consider the following approximately nonlinear boundary value problem

$$\begin{cases} D_{0^+}^\alpha u(t) + \lambda[f(t,u^*(t)) + M(t)] = 0, \ 0 < t < 1, \ 2 < \alpha < 3, \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{16}$$

where

$$u^*(t) = \max\{u(t) - x_2(t), 0\},$$

and $x_2(t)$ is the unique solution of the boundary value problem (13). We shall show that the approximately boundary value problem (16) has a positive solution v.

In view of Lemma 2.5 we define the operator $S : E \to E$ by

$$Su(t) = \int_0^1 \lambda G(t,s) f^*(s,u(s))ds, \tag{17}$$

where

$$f^*(t,u(t)) = f(t,u^*(t)) + M(t).$$

We shall prove that S has a fixed point in our cone K. Firstly we prove that $S : K \longrightarrow K$. For any $u \in K$, we note that $Su(t)$ is continuous on $[0,1]$, and since $G(t,s) \ge 0$ we have $G(1,s) \ge 0$, then by Lemma 2.5 we obtain

$$\begin{aligned} Su(t) &= \int_0^1 \lambda G(t,s) f^*(s,u(s))ds \\ &\ge t^{\alpha-1} \int_0^1 \lambda G(1,s) f^*(s,u(s))ds \\ &\ge t^{\alpha-1} \max_{t\in[0,1]} \int_0^1 \lambda G(t,s) f^*(s,u(s))ds \\ &= q(t)\|Su\|, \quad \forall t,s \in [0,1]. \end{aligned}$$

Thus $S(K) \subset K$. Then from the definition of S, it is easy to prove that S is a completely continuous operator. The continuity of S is obvious by the continuity of the nonlinear function f. By using the Arzelà-Ascoli theorem, we can prove that the operator S is compact. Then, $S : K \to K$ is a completely continuous, and each fixed point of S in K is a solution of boundary value problem (16).

We define the ball P_1 in the Banach space E by

$$P_1 = \{u \in E, \ \|u\| < \theta\}.$$

For $u \in K \cap \partial P_1$, we have $0 \leq u^*(t) \leq u(t) \leq \|u\| = \theta$ for $t \in [0,1]$ and by $(A1)$ and (15) we have

$$\begin{aligned} Su(t) &= \int_0^1 \lambda G(t,s) f^*(s,u(s))ds \\ &= \lambda \int_0^1 G(t,s)[f(s,u^*(s)) + M(s)]ds \\ &\leq \lambda \int_0^1 G(1,s)[p(s)\psi(u^*(s)) + M(s)]ds \\ &\leq \lambda \left(\max_{0\leq\tau\leq\theta} \psi(\tau) \right) \int_0^1 G(1,s)[p(s)+M(s)]ds \\ &\leq \theta. \end{aligned}$$

Therefore, we get $\|Su\| \leq \theta = \|u\|$ for $u \in K \cap \partial P_1$.
For $\sigma \in \left(0, \frac{1}{2}\right)$ fixed. Let k be a positive real number such that

$$\frac{1}{2}\lambda k B \left(\inf_{t\in[\sigma,1-\sigma]} q(t) \right) > 1, \tag{18}$$

where

$$B = \frac{1}{(\alpha-1)\Gamma(\alpha+1)}.$$

In view of $(C1)$, there exists $A > 0$ such that for all $z \geq A$ and all $t \in [\sigma, 1-\sigma]$

$$f(t,z) \geq kz. \tag{19}$$

Now, set

$$R = \theta + \max\left\{ 2\lambda\theta, 2A \left(\inf_{t\in[\sigma,1-\sigma]} q(t) \right)^{-1} \right\}. \tag{20}$$

Let us define the ball P_2 in the Banach space E by

$$P_2 = \{u \in E, \ \ \|u\| < R\}.$$

We shall prove that $\|\widetilde{F}u\| \geq \|u\|$ for $u \in K \cap \partial P_2$. Let $u \in K \cap \partial P_2$. Then, $\|u\| = R$. Using Lemma 4.1 and the fact that $u \in K$ we get for $t \in [0,1]$

$$x_2(t) \leq \lambda\theta q(t) \leq \lambda\theta \frac{u(t)}{R}.$$

Thus implies for $t \in [0,1]$

$$u(t) - x_2(t) \geq \left(1 - \lambda\theta\frac{1}{R}\right) u(t) \geq \left(1 - \lambda\theta\frac{1}{R}\right) R q(t),$$

and, noting (20), it follow for $t \in [\sigma, 1-\sigma]$

$$u(t) - x_2(t) \geq \left(1 - \lambda\theta\frac{1}{R}\right) R \left(\inf_{t\in[\sigma,1-\sigma]} q(t) \right) \geq \frac{1}{2} R \left(\inf_{t\in[\sigma,1-\sigma]} q(t) \right) \geq A.$$

Hence, by (19), we see that for $t \in [\sigma, 1-\sigma]$

$$f(t, u^*(t)) \geq k u^*(t) \geq \frac{1}{2} k R \left(\inf_{t\in[\sigma,1-\sigma]} q(t) \right).$$

Then, by Lemma 2.6 and (18), we find

$$\begin{aligned}
\|Su\| &= \max_{t\in[0,1]}\int_0^1 \lambda G(t,s)f^*(s,u(s))ds \\
&= \max_{t\in[0,1]}\int_0^1 \lambda G(t,s)[f(s,u^*(s))+M(s)]ds \\
&\geq \frac{1}{2}kR\left(\inf_{t\in[\sigma,1-\sigma]} q(t)\right)\lambda \max_{t\in[0,1]}\int_0^1 q(t)G(1,s)ds + \lambda \max_{t\in[0,1]}\int_0^1 q(t)G(1,s)M(s)ds \\
&\geq \left[\frac{1}{2}\lambda kB\left(\inf_{t\in[\sigma,1-\sigma]} q(t)\right)\right]R \\
&\geq R.
\end{aligned}$$

Then, we get $\|Su\| \geq R = \|u\|$ for $u \in K\cap\partial P_2$. Therefore assertion (I) of Theorem 2.8 is satisfied. Then S has a fixed point $v \in K$ which satisfies $\theta \leq \|v\| \leq R$.

Furthermore, using (15) and Lemma 4.1, we get for $t \in [0,1]$,

$$v(t) \geq q(t)\|v\| \geq \theta q(t) \geq \lambda\theta q(t) \geq x_2(t).$$

Therefore, for $t \in [0,1]$ we have

$$v^*(t) := v(t) - x_2(t) \geq 0.$$

Now we shall prove that v^* is in fact a positive solution of our problem (1). To see this we have for $t \in [0,1]$, v is a fixed point of operator S. Then

$$\begin{aligned}
v(t) &= Sv(t) \\
&= \int_0^1 \lambda G(t,s)f^*(s,v(s))ds \\
&= \int_0^1 \lambda G(t,s)[f(s,v^*(s))+M(s)]ds \\
&= \int_0^1 \lambda G(t,s)f(s,v(s)-x_2(s))ds + \int_0^1 \lambda G(t,s)M(s)ds \\
&= \int_0^1 \lambda G(t,s)f(s,v(s)-x_2(s))ds + x_2(t).
\end{aligned}$$

Thus

$$\begin{aligned}
v^*(t) &= v(t) - x_2(t) \\
&= \int_0^1 \lambda G(t,s)f(s,v(s)-x_2(s))ds \\
&= \int_0^1 \lambda G(t,s)f(s,v^*(s))ds.
\end{aligned}$$

Consequently, by Lemma 2.5, it is easy to see that v^* is a positive solution of our boundary value problem (1). The proof is completed. □

Theorem 4.3. *Let $f : [0,1]\times\mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function satisfying (C1). Moreover we assume that:*

(H1) *For any $(t,x) \in [0,1]\times\mathbb{R}_+^*$, $f(t,x)$ satisfies $f(t,x) \geq -M(t)$, with $M \in C([0,1],\mathbb{R}_+^*)$.*

If there exists $\lambda > 0$ such that

$$0 < \lambda \leq \min\left\{1, \frac{\theta}{B\widehat{f_\theta}},\right\}, \quad \widehat{f_\theta} = \sup_{t\in[0,1],0\leq\tau\leq\theta}[f(t,\tau) + M(t)], \tag{21}$$

where θ is represented in (14), then the boundary value problem (1) has a positive solution.

Proof. The proof is similar to that of Theorem 4.2. We omit the details. $\square$

5. An Example

Example 5.1. *We consider the following boundary value problem for fractional order*

$$\begin{cases} D_{0^+}^{5/2}u(t) + \lambda\dfrac{\sin t}{1+t^2}u^2(t) = 0, \ 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{22}$$

Here $\alpha = \dfrac{5}{2}$, $f(t,u) = \dfrac{\sin t}{1+t^2}u^2$. Let $M = 1$ and $r > 0$. Then by Theorem 3.2, if

$$0 < \lambda \leq \min\left\{\frac{8r}{15\sqrt{\pi}(r^2+1)}, \frac{15\sqrt{\pi}}{8}r\right\},$$

the boundary value problem (22) has a positive solution.

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