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AN EFFICIENT METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS USING BERNSTEIN POLYNOMIALS

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Abstract. In this paper we propose an efficient numerical technique for solving fractional initial value problems. It is based on the Bernstein polynomials. We derive an explicit form for the Bernstein operational matrix of fractional order integration. Numerical results are presented. In order to show the efficiency of the presented method, we compare our results with some operational matrix techniques.

1. Introduction

Fractional calculus is a branch of mathematics that deals with a generalization of the well-known operations of differentiations and integrations to arbitrary fractional orders. Fractional derivative provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with the classical integer-order models in which such effects are in fact neglected. Fractional calculus found many applications in various fields of physical sciences such as electrochemical process [1-2], dielectric polarization [3], earthquakes [4], fluid-dynamic traffic model [5], solid mechanics [6], bioengineering[7-9] and economics[10]. Fractional derivatives and integrals also appear in theory of control of dynamical systems, when the controlled system and the controller are described by a fractional differential equation.
In recent years, a number of methods have been proposed and applied successfully to approximate various types of fractional differential equations such as Adomian decomposition method [11-13] and [45], Variational iteration method [14-15] and [40-42], Homotopy perturbation method [16-17] and [43], Homotopy analysis method [18], fractional differential transform method [19-23], power series method [24], and other methods [25-29], [38-39], and [46-48].

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Recently, wavelets operational matrix are used to find the solution of the fractional differential equations. Li et al. [30] derived the Haar wavelets operational matrix of fractional order integration with the Block pluse functions. Li [31] used chebyshev wavelet operational matrix to approximate the solution of the same problem. Saadatmandi and Dehghan [32] used the Legendre operational matrix of differentiation to solve such problems.
Bernstein polynomials have been used for solving partial differential equations [33]. More recently, we used Bernstein's approximation to find the stable solution of the problem of Abel inversion [34-35]. Then we studied Abel'a integral equation arising in classical theory of elasticity [36].
In this paper we present an efficient numerical method for solving linear and nonlinear fractional differential equations. Bernstein operational matrix of fractional order integration is developed and is applied for solving fractional differential equations. Some illustrative examples are given to demonstrate the validity and the effectiveness of the proposed method. Finally, we compare our results with some operational matrix methods such as chebyshev wavelet and Haar wavelets methods.

2. Bernstein polynomials and function approximation

2.1. Bernstein polynomials. A Bernstein polynomial is a linear combination of Bernstein basis polynomials. The Bernstein basis polynomials of degree n are defined by

$$B_{i,n}(t) = \begin{pmatrix} n \\ i \end{pmatrix} t^i(1-t)^{n-i}, \qquad for\ i = 0, 1, 2, \cdots, n. \tag{1}$$

Let V_n be the linear space that is consisting of all polynomials of degree less than or equal to n in $\Re[t]$-the ring of polynomials over the field $\Re$. Then,

$$\{B_{i,n}(t) : i = 0, 1, 2, \cdots, n\}$$

is a basis for V_n. For simplicity, we assume that $B_{i,n} = 0$ if $i < 0$ or $i > n$. Thus, any polynomial $P(t)$ in V_n can be written as

$$P(t) = \sum_{i=0}^{n} c_i B_{i,n}(t). \tag{2}$$

In this case, $P(t)$ is called a polynomial in Bernstein form and the coefficients c_i are called Bernstein coefficients. It is easy to verify the following properties:

(1) $B_{i,n}(0) = \delta_{i0}$ and $B_{i,n}(1) = \delta_{in}$, where δ is the Kronecker delta function.
(2) $B_{i,n}(t)$ has one root, each of multiplicity i and $n-i$, at $t = 0$ and $t = 1$ respectively.
(3) $B_{i,n}(t) \geq 0$ for $t \in [0, 1]$ and $B_{i,n}(1-t) = B_{n-i,n}(t)$.
(4) For $i \neq 0$, $B_{i,n}$ has a unique local maximum in $[0, 1]$ at $t = i/n$ and the maximum value is $i^i n^{-n}(n-i)^{n-i} \begin{pmatrix} n \\ i \end{pmatrix}$.
(5) $\sum_{i=0}^{n} B_{i,n}(t) = 1$.
(6) $B_{i,n-1}(t) = \left(\frac{n-i}{n}\right) B_{i,n}(t) + \left(\frac{i+1}{n}\right) B_{i+1,n}(t)$.
(7) Let $f(t) \in C[0, 1]$, then $B_n(f)(t) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{i,n}(t)$ converges to $f(t)$ uniformly on $[0, 1]$ as $n \to \infty$

(8) Let $f(t) \in C^{(k)}[0, 1]$, then

$$\left\|B_n(f)^{(k)}\right\|_\infty \leq \frac{(n)_k}{n^k} \left\|f^{(k)}\right\|_\infty \; and \; \left\|f^{(k)} - B_n(f)^{(k)}\right\|_\infty \to 0$$

as $k \to \infty$, where$\| \, . \, \|_{\infty}$ is the supremum norm and

$$\frac{(n)_k}{n^k} = \left(1 - \frac{0}{n}\right) \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)$$

is an eigenvalue of B_n. For more details, see [44].

2.2. **Function approximation.** Using Gram-Schmidt orthonormalization process, we can normalize the Bernstein basis polynomials. The new set of orthonormal polynomials is denoted by $\{b_{i,\,n}(t) \, : \, i = 0, 1, 2, \cdots, n\}$. Any function f in $L^2[0, 1]$ can be in terms of $\{b_{i,\,n}(t) \, : \, i = 0, 1, 2, \cdots, n\}$ as

$$f(t) = \lim_{n \to \infty} \sum_{i=0}^{n} c_{i\,n} \, b_{i,\,n}(t), \tag{3}$$

where, $c_{i\,n} = \langle f, \, b_{i\,,n} \rangle = \int\limits_0^1 f(t) b_{i\,,n}(t) dt$.

If the series (2.3) is truncated at $n = m - 1$, then we have

$$f\ (t) \cong \sum_{i=0}^{n} c_{in} \, b_{i\,,n} = C^T \psi(t), \tag{4}$$

where C and $\psi(t)$ are $m \times 1$ matrices which are given by

$$C = [c_{0n}, \, c_{1n}, \, \cdots, \, c_{n\,n}]^T \tag{5}$$

and

$$\psi(t) = [b_{0\,,n}(t), \, b_{1,\,n}(t), \cdots, \, b_{n,n}(t)]^T. \tag{6}$$

If the domain is $[0, T]$ where $T > 1$, we use define

$$\psi(t) = [h_{0\,,n}(t), \, h_{1,\,n}(t), \cdots, \, h_{n,n}(t)]^T$$

where $h_{i,\,n}(\bar{t}) = \frac{b_{i,\,n}(\bar{t}/T)}{\sqrt{T}}$.

3. Bernstein operational matrix of fractional order integration

3.1. ***Fractional integral and derivative.*** In this section, we review the definition and some preliminary results of the fractional derivatives.

Definition 1. *The Rimann-Liouville fractional integral operator I^{α} of order $\alpha > 0$ on the usual Lebesgue space $L_1[0, 1]$ is given by*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{f(t)}{(t-\tau)^{1-\alpha}} d\tau, \tag{7}$$

$$I^{0} f(t) = f(t), \tag{8}$$

where $\Gamma(\alpha) = \int_0^{\infty} \nu^{\ \alpha-1} e^{-\nu} d\upsilon$ *is the Euler gamma function.*

In the next definition we define the Caputo fractional derivative of order α.

Definition 2. *The Caputo fractional derivative of order* α *is defined by*

$$D^{\ \alpha} f(t) = I^{\ n-\alpha} D^{\ n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{9}$$

provided that the integral exists, where $n = [\alpha] + 1$, $[\alpha]$ *is the integer part of the positive real number* $\alpha, t > 0$.

The following properties hold:

$$(I^{\ \alpha} t^{\vartheta}) = \frac{\Gamma(\vartheta+1)}{\Gamma(\alpha+\vartheta+1)} t^{\vartheta+\alpha} \tag{10}$$

and

$$I^{\ \alpha} D^{\ \alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \tag{11}$$

for $f \in L_1[0,1]$. From now on all fractional derivatives are in Caputo sense.

3.2. ***Block Pulse Functions and operational matrix of fractional integration.*** A set of m Block Pulse Functions (BPF) on $[0,1)$ are defined as follows:

$$b_i\ (t) = \begin{cases} 1, & \frac{i}{m} \le t < \frac{i+1}{m} \\ 0, & otherwise \end{cases}, \tag{12}$$

where $i = 0, 1, \ldots m-1$. These functions are disjoint and orthogonal, i.e.,

$$b_i\,(t)\, b_j\,(t) = \begin{cases} 0 & i \ne j \\ b_i\ (t) & i = j \end{cases}, \tag{13}$$

and

$$\int_0^1 b_i\,(t)\, b_j\,(t) dt = \begin{cases} 0 & i \ne j \\ \frac{1}{m} & i = j \end{cases}. \tag{14}$$

Kilicman and Al Zhour [37] have obtained the Block Pulse operational matrix of the fractional order integration F^{α} as follows:

$$(I^{\ \alpha} B_m\ )\,(t) = F^{\alpha} B_m\ (t) \tag{15}$$

where $B_m\ (t) = [b_0\,(t)\,, b_1\,(t)\,,\ \ldots,\ b_{m-1}(t)]^T$, $\varepsilon_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$ and

$$F^{\alpha} = \frac{1}{m^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \ldots & \varepsilon_{m-1} \\ 0 & 1 & \varepsilon_1. & \ldots & \varepsilon_{m-2} \\ 0 & 0 & 1 & \ldots & \varepsilon_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

If the domain of the solution in the fractional differential equation is $[0,T]$ where $T > 1$, we can use the same definition for $b_i(t)$.

3.3. ***Bernstein operational matrix of the fractional integration.*** In this section, we derive the Bernstein polynomials operational matrix of the fractional order integration. First, we rewrite the Riemann–Liouville fractional order integration as follows:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \tag{16}$$

where $\alpha > 0$ and $t^{\alpha-1} * f(t)$ denotes the convolution product of the functions $t^{\alpha-1}$ and $f(t)$. The operational matrix of integration of $\psi(t)$, which is defined in equation (2.6), can be obtained as

$$\int_0^t \psi(\tau)\, d\tau = P\, \psi(t) \tag{17}$$

where P is $m \times m$ matrix. Orthonormal Bernstein polynomials can be written in terms of the Block Pulse functions as

$$\psi_m(t) = \Phi_{m\times m} B_m(t) \tag{18}$$

where

$$B_m(t) = [b_0(t), b_1(t), \ldots, b_{m-1}(t)]^T.$$

Let $P^{\alpha}_{m\times m}$ be the Bernstein polynomials operational matrix of the fractional order integration. Then

$$I^{\alpha}\psi_m(t) = P^{\alpha}_{m\times m}\psi_m(t). \tag{19}$$

Equations (3.9) and (3.12) imply that

$$I^{\alpha}\psi_m(t) = I^{\alpha}\Phi_{m\times m}B_m(t) = \Phi_{m\times m}\, I^{\alpha}B_m(t) = \Phi_{m\times m}F^{\alpha}B_m(t). \tag{20}$$

From equations (3.12), (3.13) and (3.14) we get

$$P^{\alpha}_{m\times m}\psi_m(t) = P^{\alpha}_{m\times m}\Phi_{m\times m}B_m(t) = \Phi_{m\times m}F^{\alpha}B_m(t). \tag{21}$$

Then, the Bernstein polynomials operational matrix of the fractional order integration $P^{\alpha}_{m\times m}$ is given by

$$P^{\alpha}_{m\times m} = \Phi_{m\times m}F^{\alpha}{\Phi_{m\times m}}^{-1}. \tag{22}$$

4. Results and discussions

In this section we consider six examples to demonstrate the performance and efficiency of the present method. Comparison with Haar wavelet operational matrix method (HWOM method) and Chebyshev wavelet operational matrix method (CWOM method).

Examples 4.1 Consider the linear fractional differential equation, [25],

$$D^{\alpha}y(t) = -y(t), \quad 0 < \alpha \leq 2, \tag{23}$$

subject to

$$y(0) = 1,\ y'(0) = 0.$$

The exact solution of the above problem is

$$y(t) = E_{\alpha}(-t^{\alpha})$$

where

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}. \tag{24}$$

It is easy to see that for $\alpha = 1$, the exact solution is $y(t) = e^{-t}$. If $0 < \alpha \leq 1$, we use only the condition $y(0) = 1$. For $1 < \alpha \leq 2$, we use both initial conditions.

Let $D^{\alpha} y(t) = K_m^T \psi_m(t)$, then

$$y(t) = I^{\alpha} K_m^T \psi_m(t) + 1 = K_m^T P_{m\times m}^{\alpha} \psi_m(t) + 1. \tag{25}$$

Using equation (3.12), we have

$$y(t) = K_m^T P_{m\times m}^{\alpha} \Phi_{m\times m} B_m(t) + 1. \tag{26}$$

From Equation (4.1) and Equation (4.3), we have

$$K_m^T \Phi_{m\times m} B_m(t) + K_m^T P_{m\times m}^{\alpha} \Phi_{Bm\times m} B_m(t) + 1 = 0$$

or

$$K_m^T \Phi_{m\times m} B_m(t) + K_m^T P_{m\times m}^{\alpha} \Phi_{Bm\times m} B_m(t) + 1 = 0 \tag{27}$$

For solving Equation (4.5), we use the Matlab function fsolve. Figure 1 represents the graphs of the exact solution and the approximate solutions using the proposed method for different values of α which are $\alpha = 0.5$ (red), 0.75 (green), 0.95 (blue), and 1 (yellow) for $m = 24$. Figure 2 represents the graphs of the exact solution and the approximate solutions using the proposed method, HWOM method, and HWOM method for $\alpha = 1$ and $m = 24$. It is worth mention that the graphs of the proposed method, HWOM method, and HWOM method for $\alpha = 1$ are coincide. Figure 3 represents the graphs of the exact solution and the approximate solutions using the proposed method for different values of α which are $\alpha = 1.25$ (blue), 1.5 (green), 1.75 (red), and 1.95 (yellow) for $m = 24$. Figure 4 represents the graphs of the absolute error of the proposed method for $\alpha = 1.25$ and $m = 24$.

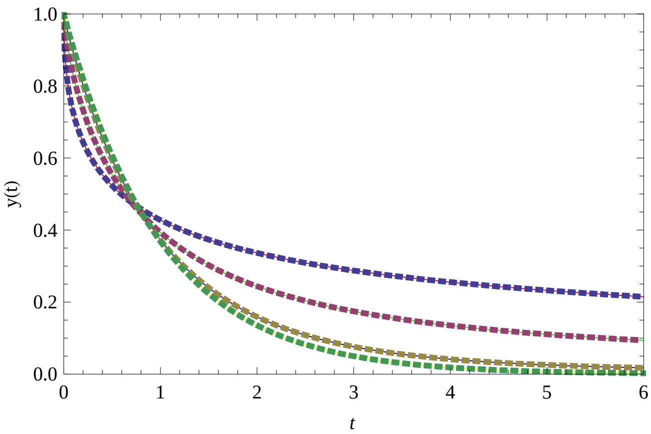

Figure 1: Proposed solution (–) and Exact solution (-o-o-)

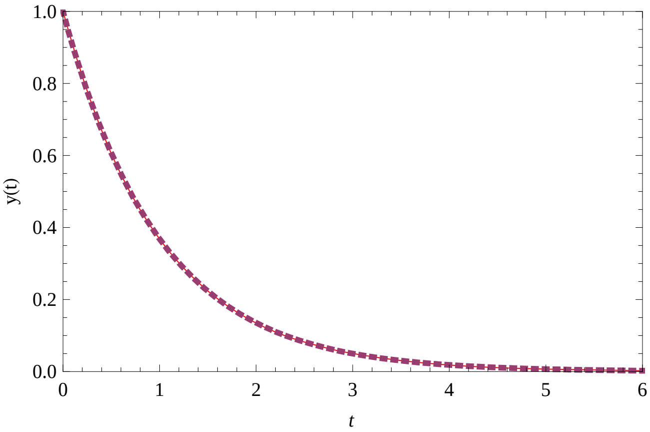

Figure 2: Proposed solution (–) with exact one (-o-o-)

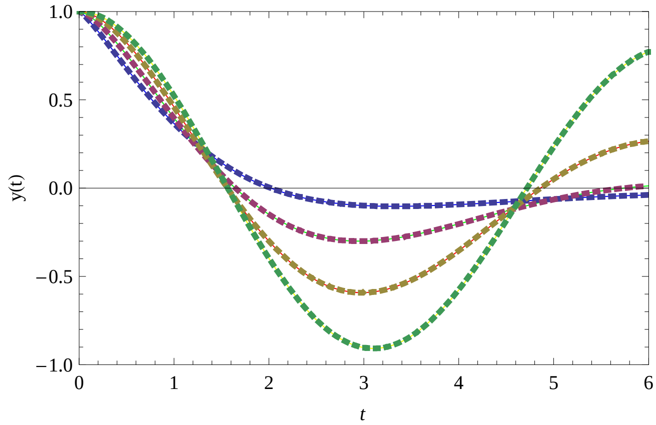

Figure 3: Proposed solution (–) and Exact solution (-o-o-)

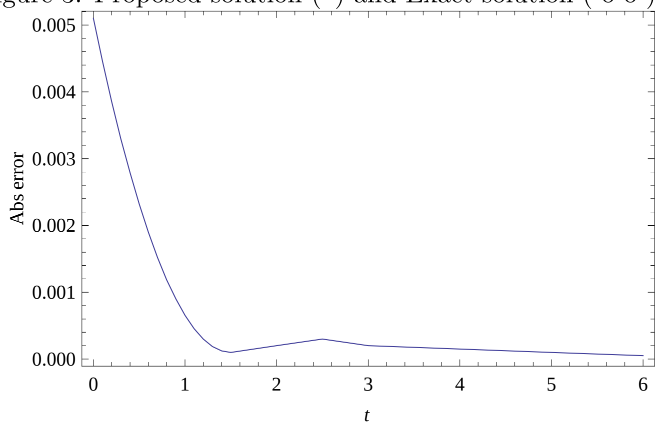

Figure 4: Absolute error of the proposed method for $m = 24$

Example 4.2 Consider the nonlinear fractional differential equation, [31],

$$a\ D^2 y\left(t\right) + b\ D^{\ \alpha_2}\ y\left(t\right) + \ c\ D^{\ \alpha_1} y(t) + \ e\ y^3(t) = f\left(t\right), \quad 0 < \alpha_1 \leq 1,\ 1 < \alpha_2 \leq 2 \tag{28}$$

subject to

$$y(0) = y'(0) = 0$$

where

$$f(t) = \frac{2a}{\Gamma(2)}\, t + \frac{2b}{\Gamma(4-\alpha_2)} t^{3-\alpha_2} + \frac{2c}{\Gamma(4-\alpha_1)} t^{3-\alpha_1} + \frac{et^9}{27}.$$

The exact solution of Problem (4.6) is $y(t) = \frac{1}{3}t^3$. Let

$$\begin{aligned} D^2 y(t) &= K_m^T \quad \psi_m\ (t), \\ D^{\alpha_2} y(t) &= K_m^T\ P_{m\times m}^{2-\alpha_2}\ \psi_m\ (t), \\ D^{\alpha_1} y(t) &= K_m^T\ P_{m\times m}^{2-\alpha_1}\ \psi_m\ (t), \end{aligned}$$

then

$$y(t) = I^{\,2} K_m^T \quad \psi_m\ (t) = K_m^T \quad P_{m\times m}^{\,2} \psi_m\ (t) = K_m^T \quad P_{m\times m}^{\,2} \Phi_{m\times m} B_m\ (t)\,. \tag{29}$$

Similarly, $f(t)$ can be expanded in terms of the orthonormal Bernstein polynomials as follows

$$f(t) = f_m^T \psi_m\ (t) \tag{30}$$

or

$$f(t) = f_m^T \Phi_{m\times m} B_m\ (t)\,. \tag{31}$$

Assume that

$$K_m^T\ P_{m\times m}^2\ \Phi_{m\times m} = [a_1, a_2, \ldots\ a_m]\,. \tag{32}$$

Equation (3.7) implies that

$$y^3(t) = \left[a_1^3\ , \quad a_2^3\ , \ldots a_m^3\right] B_m(t)\,. \tag{33}$$

Equations (3.12) and (4.6)-(4.11) give us

$$a\ K_m^T \Phi_{m\times m} B_m\ (t) + b\ K_m^T P_{m\times m}^{2-\alpha_2} \Phi_{m\times m} B_m\ (t) + \tag{34}$$

$$c\ K_m^T P_{m\times m}^{2-\alpha_1}\ \Phi_{m\times m}\ B_m\ (t)\ + e\ \left[a_1^3\ ,\ a_2^3, \ldots\quad a_m^3\right] B_m\ (t) - f_m^T \Phi_{m\times m} B_m\ (t) = 0$$

In this example, we chose $a = 1, b = 1, c = 1, e = 1, \alpha_1 = 0.333$, and $\alpha_2 = 1.234$. For solving Equation (4.12), we use the Matlab function fsolve. Figure 5 represents

the graphs of the exact solution (Red) and the approximate solutions using the proposed method (green), HWOM method(blue), and HWOM method (yellow) for $m = 24$. It is worth mention that the graphs of the proposed method, HWOM method, and HWOM method for $\alpha = 1$ are coincide.

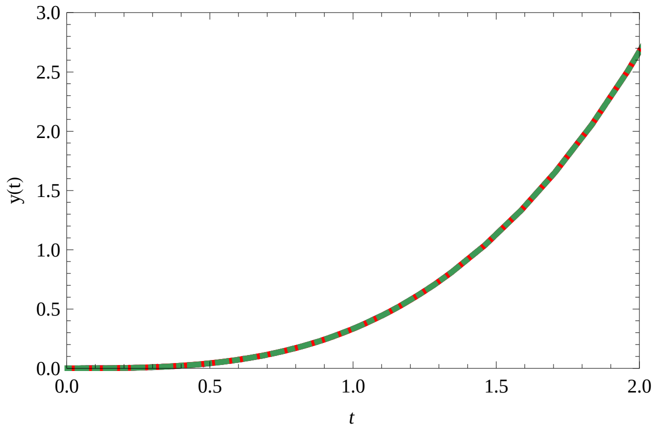

Figure 5: Exact solution, proposed method, HWOM method, and HWOM method
In Table 1 we compare the absolute error of our results with the absolute error of the results of Li [31]. It is worth mention that the proposed method gives better results that Li's results with fewer number of Bernstein polynomials.

Table (1)

t	**Proposed method (m=16)**	**Li[31] (m=24)**
0.1	1.53E-4	8.195E-5
0.2	2.93 E-4	2.052E-4
0.3	4.23 E-4	2.951E-4
0.4	5.43 E-4	3.054E-4
0.5	6.55 E-4	5.080E-4
0.6	7.60 E-4	4.296E-4
0.7	8.90E-4	6.385E-4
0.8	9.50 E-4	7.118E-4
0.9	1.03 E-3	6.027E-4

Example 4.3 Consider the nonlinear fractional differential equation, [30],

$$aD^{2.2}y(t)+bD^{\alpha_2}\ y(t)+\ cD^{\alpha_1}y(t)+\ e\quad y^3(t)\ =f(t),\quad 0<\alpha_1\leq 1,\ 1<\alpha_2\leq 2 \tag{35}$$

subject to

$$y(0)=y'(0)=y''(0)=0$$

where

$$f(t)=\frac{2a}{\Gamma(1.8)}t^{0.8}+\frac{2b}{\Gamma(4-\alpha_2)}t^{3-\alpha_2}+\quad\frac{2c}{\Gamma(4-\alpha_1)}t^{3-\alpha_1}\ +e\ \frac{t^9}{27}.$$

The Exact solution of problem (4.13) is $y(t)=\frac{1}{3}t^3$. Let

$$\begin{aligned}
D^{2.2}y(t) &= K_m^T\quad\psi_m(t),\\
D^{\alpha_2}y(t) &= K_m^TP_{m\times m}^{2.2-\alpha_2}\psi_m\ (t),\\
D^{\alpha_1}y(t) &= K_m^TP_{m\times m}^{2.2-\alpha_1}\psi_m\ (t),
\end{aligned}$$

then

$$y(t)=\ K_m^TP_{m\times m}^{2.2}\Phi_{Bm\times m}B_m\ (t).$$

Using the same procedure as in Example 4.2, we get

$$K_m^T \Phi_{Bm\times m} B_m\ (t) + K_m^T P_{m\times m}^{2.2-\alpha_2} \Phi_{Bm\times m} B_m\ (t) + \tag{36}$$

$$K_m^T P_{m\times m}^{2.2-\alpha_1} \Phi_{Bm\times m} B_m\ (t) + \ \left[a_1^3\ ,\quad a_2^3\ ..., a_m^3\right] B_m\ (t) - f_m^T \Phi_{Bm\times m} B_m\ (t) = 0.$$

In this example, we chose $a = 1, b = 1, c = 1, e = 1, \alpha_1 = 0.75, \alpha_2 = 1.25$. For solving Equation (4.14), we use Matlab function fsolve. Figure 6 represents the graphs of the exact solution (red) and the approximate solutions using the proposed method (green), HWOM method(blue) , and CWOM method (yellow) for $m = 24$.

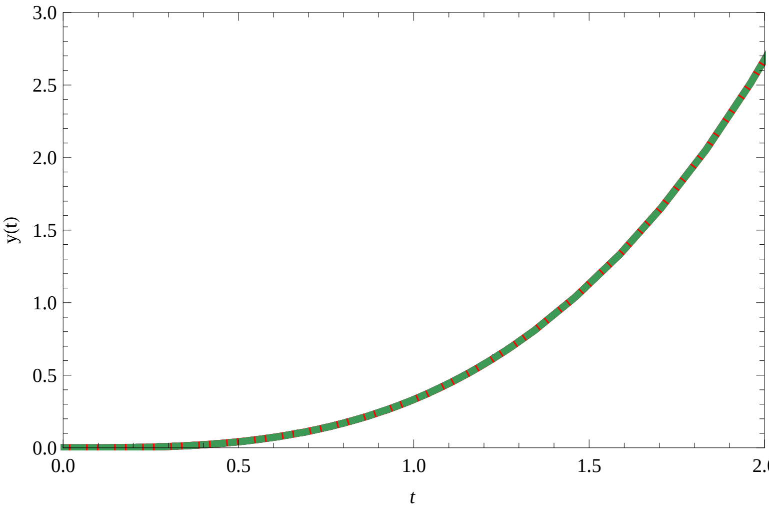

Figure 6: Exact solution, proposed method, HWOM method, and CWOM method

In Table 2 we compare the absolute error of our results with absolute error of the results of Li [30].

Table 2

t	**Proposed method (m=12)**	**Li [30] (m=16)**	**Proposed method (m=16)**	**Li [30] (m=32)**
0.1	3.25E-4	2.0 E-4	1.87E-4	6.96E-5
0.2	6.58E-4	5.0E-4	3.8 E-4	1.17 E-4
0.3	9.72E-4	8.0 E-4	5.6 E-4	1.75 E-4
0.4	1.26E-3	9.0 E-4	7.3 E-4	2.74 E-4
0.5	1.52E-3	1.4 E-3	8.74 E-4	3.52 E-4
0.6	1.76E-3	1.2 E-3	1.01 E-4	3.87 E-4
0.7	1.97E-3	1.7 E-3	1.13 E-4	3.58 E-4
0.8	2.16E-3	1.9 E-3	1.24 E-4	3.96 E-4
0.9	2.33E-3	1.6 E-3	1.34 E-4	5.36 E-4

Example 4.4 Consider the nonlinear fractional differential equation, [30],

$$aD^{2.0}y\,(t) + b\,(t)\,D^{\alpha_2}\ y\,(t) + c\,(t)\,Dy\,(t) +\ e\,(t)\,D^{\alpha_1}y(t) +\ k\,(t)\,y\,(t) = f\,(t)\,,$$

subject to

$$y\,(0) = 2,\quad y'\,(0) = 0$$

where

$$\begin{aligned}
f(t) &= a - \frac{b(t)}{\Gamma(3-\alpha_2)} t^{2-\alpha_2} - c(t)\,t + \frac{e(t)}{\Gamma(3-\alpha_1)} t^{2-\alpha_1} + k(t)\left(2 - \frac{1}{2}t^2\right),\\
b(t) &= t^{0.5},\\
c(t) &= t^{1/3},\\
e(t) &= t^{1/4},\\
k(t) &= t^{1/5},
\end{aligned}$$

and $0 < \alpha_1 \leq 1,\ 1 < \alpha_2 \leq 2$. The Exact solution of problem (4.15) is $y(t) = 2 - \frac{1}{2}t^2$. Let

$$\begin{aligned}
D^2 y(t) &= K^T \psi_m(t),\\
D^{\alpha_2} y(t) &= K^T P^{2-\alpha_2}_{m\times m} \psi_m(t),\\
Dy(t) &= K^T P^{1}_{m\times m} \psi_m(t)\\
D^{\alpha_1} y(t) &= K^T P^{2-\alpha_1}_{m\times m} \psi_m(t),
\end{aligned}$$

then

$$y(t) = K^T P^{2}_{m\times m} \psi_m(t) + 2. \tag{37}$$

Using the same procedure as in Example 4.2, we get

$$\begin{aligned}
&K^T \psi_m(t) + b(t) K^T P^{2-\alpha_2}_{m\times m} \psi_m(t) + c(t) K^T P^{1}_{m\times m} \psi_m(t) + \qquad (38)\\
&e(t) K^T P^{2-\alpha_1}_{m\times m} \psi_m(t) + k(t)\left[K^T P^{2}_{m\times m} \psi_m(t) + 2\right] = f^T_m \psi_m(t).
\end{aligned}$$

In this example, we chose $a = 1$, $\alpha_1 = 0.333$, and $\alpha_2 = 1.234$. For solving Equation (4.17), we use Matlab function fsolve. Figure 7 represents the graphs of the exact solution (red) and the approximate solutions using the proposed method green), HWOM method (blue), and CWOM method (yellow) for $m = 24$. Figure 8 represents the graphs of the absolute error of the proposed method for $m = 24$.

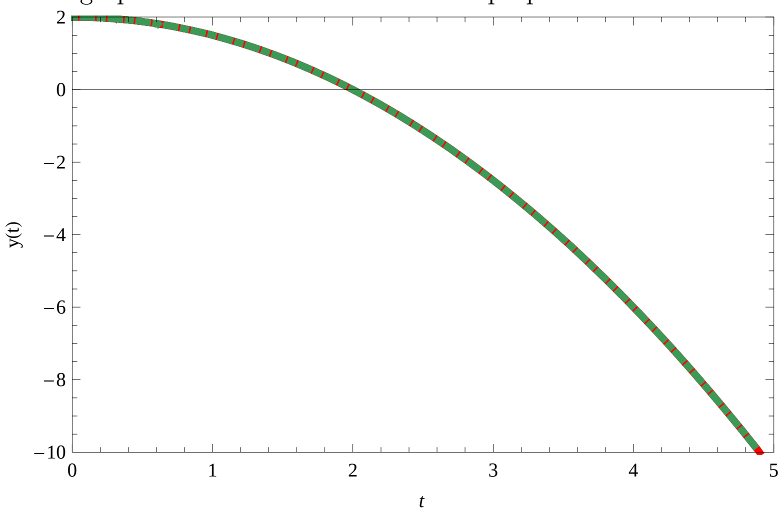

Figure 7: Exact solution, proposed method, HWOM method, and CWOM method

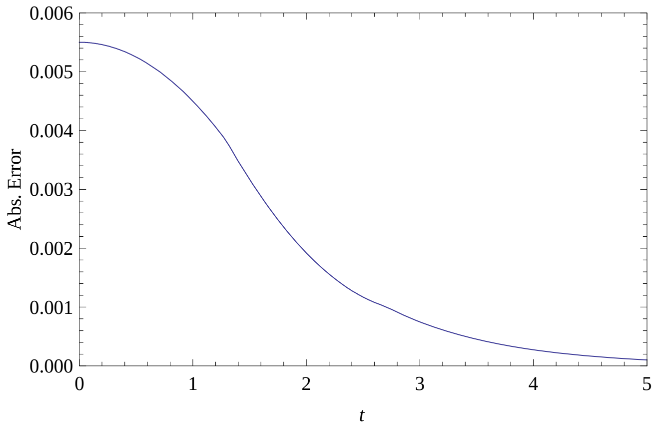

Figure 8: Absolute error of the proposed method for $m = 24$

Example 4.5 Consider the nonlinear fractional differential equation, [25],

$$D^{\alpha} y(t) = f(t) - \ y^{\frac{3}{2}}(t) \ , 0 < \alpha_2 \leq 2 \tag{39}$$

subject to

$$y(0) = 0, \quad y'(0) = 0$$

where

$$f(t) = f(t) = \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - \ \frac{\Gamma\left(5+\frac{\alpha}{2}\right)}{\Gamma\left(5-\frac{\alpha}{2}\right)} t^{4-\frac{\alpha}{2}} \ + \frac{9}{4}\Gamma(\alpha+1) + \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^3.$$

The Exact solution of problem (4.18) is $y(t) = \ t^8 - \ 3t^{4+\frac{\alpha}{2}} \ + \frac{9}{4}t^{\alpha}$. Let

$$\begin{aligned}
D^{\alpha} y(t) &= K^T \psi_m\ (t), \\
D^{\alpha_2} y(t) &= K^T P_{m\times m}^{2-\alpha_2} \psi_m\ (t), \\
Dy(t) &= K^T P_{m\times m}^{1} \psi_m\ (t) \\
D^{\alpha_1} y(t) &= K^T P_{m\times m}^{2-\alpha_1} \psi_m\ (t),
\end{aligned}$$

then

$$y(t) = \ K^T P_{m\times m}^{\alpha} \psi_m\ (t), \ f(t) = f_m^T \psi_m\ (t). \tag{40}$$

Using the same procedure as in Example 4.2, we generate a system of nonlinear equations which can be solve by Matlab. Figure 9 represents the graphs of the exact solution and the approximate solutions using the proposed method for different values of α which are $\alpha = 0.5$ (red), 0.75 (green), 1.25 (blue), 1.5 (yellow), and 1.75 (black) for $m = 24$. It is worth mention that the absolute error of the proposed method for $\alpha = 0.5,\ 0.75,\ 1.25,\ 1.5,\ 1.75$ and $m = 24$ is less that $2 * 10^{-3}$.

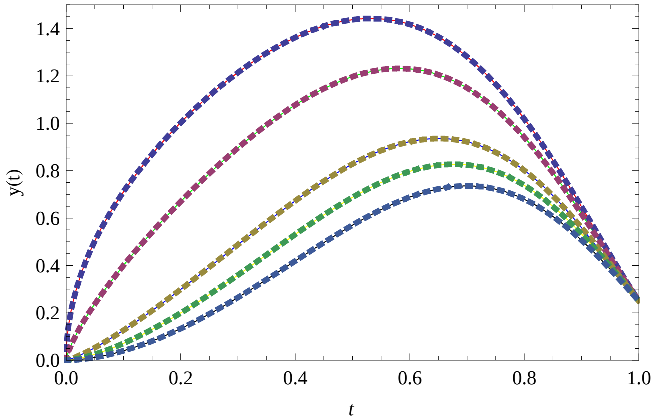

Figure 9: Proposed solution (–) and Exact solution (-o-o-)

Example 4.6 Consider the nonlinear fractional differential equation, [27],

$$D^{0.5}y(t) = -y(t) + t^2 + \frac{2}{\Gamma(2.5)}t^{1.5}, \tag{41}$$

subject to

$$y(0) = 0.$$

The Exact solution of problem (4.18) is $y(t) = t^2$. Using the same procedure as in Example 4.2, we generate a system of linear equations which can be solve by Matlab. Figure 10 represents the graphs of the exact solution and the approximate solutions using the proposed method for $m = 24$. Figure 11 represents the graphs of the absolute error of the proposed method for $m = 24$.

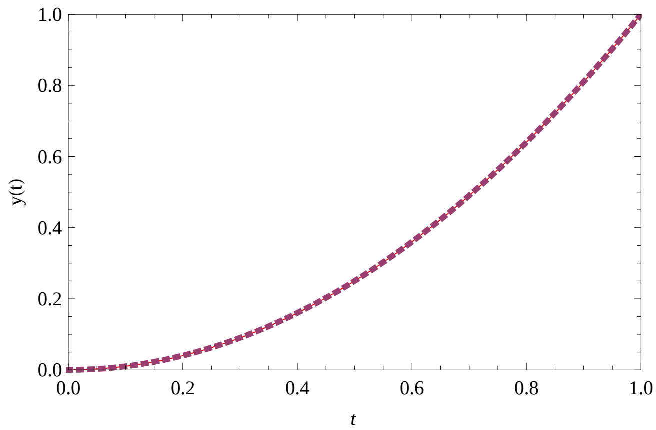

Figure 10: Proposed solution (–) and Exact solution (-o-o-)

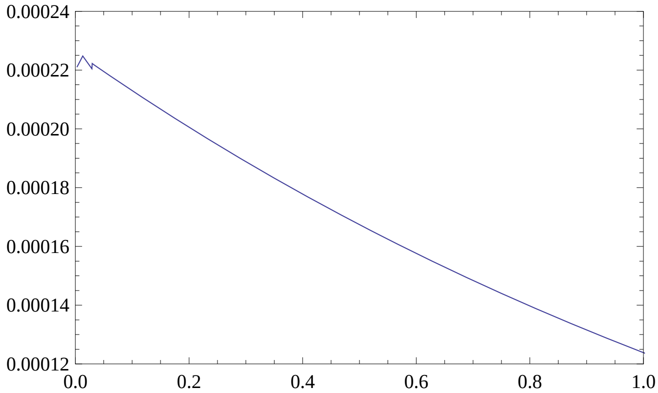

Figure 11: Absolute error of the proposed method for $m = 24$

In Table 3 we compare the absolute error of our results with absolute error of the results of Ford [27].

Table 3

Time t	Error obtained by Diethelm & Ford [27]		Error obtained using proposed method
	h=0.1	h=0.04	
5	0.010995	0.002819	0.000426
10	0.012018	0.003067	0.000312

5. Conclusion

In this paper we present an efficient numerical method for solving linear and nonlinear fractional differential equations. We develop and apply the Bernstein operational matrix of fractional order integration for solving fractional differential equations. We present six numerical examples to demonstrate the validity and the effectiveness of the proposed method. In addition, we compare our results with Ford [27], HWOM method [30], and CWOM method [31], see Figures (5)-(7). Also, we compare our results with the exact solution of the fractional initial value problems which are presented in Examples (1)-(6), see Figures (1)-(3), (5)-(7),(9), (10). From Figures (4)-(8) and (11), we see that the absolute error of the proposed method is within 10^{-3}. The main advantage of the proposed method is small size of the Bernstein operational matrix of fractional order integration produces high accuracy, see tables (1)-(3). Also, the complexity of the proposed method is small comparing with the complexity of the CWOM method and HWOM method. This method is computer oriented and it is easy to program it. Finally, one can generalize these techniques to system of fractional differential equations. We will leave this for the future work.

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